(In)effienciency in Mean Field Games

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Mean Field Games (MFG) are Nash equilibria in

- nonatomic games = infinitely many agents having individually a negligible influence on the global system (as in Schmeidler (1973), or Mas-Colell (1983, 1984))
- in a optimal control framework = each agent acts on his state which evolves in continuous time and has a payoff depending on the other's position (stochastic optimal control)

Pioneering works :

— Models invented by Lasry-Lions (2006) and Caines-Huang-Malhamé (2006)

— Similar models in the economic literature : heterogeneous agent models (Aiyagari ('94), Bewley ('86), Krusell-Smith ('98),...)

Outline

Description of the MFG equilibrium

The MFG system is a Nash equilibrium of a continuous game where the payoff is of optimal control type.

- Each tiny agent knows the initial density m_0 of all agents and forecasts that this probability density will be $(m(t,\cdot))_{t\geq 0}$ in the future.
- He solves his optimal control problem accordingly.
- When all the agents play optimality, their probability density $(\tilde{m}(t,\cdot))_{t\geq 0}$ evolves in time.
- Nash equilibrium means that the original forecast was correct : $\tilde{m}(t,\cdot) = m(t,\cdot)$ for all t.

Notation

- T is the finite horizon of the game, the state space is $\mathbb{R}^d$ $(d \in \mathbb{N}\backslash\{0\})$,
- The initial population density at time $t_0 = 0$ is m_0,
- $L = L(x, \alpha, m)$ is the running cost, $g = g(x, m)$ is the terminal cost,
- If $u = u(t, x)$ is a map, $Du(t, x)$ is its first order space derivative.

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The problem for a small player.

- A small agent is at initial time t_0 at a position x_0. He acts through his control (α_t) on his state (X_t), which evolves according to the SDE

$$dX_t = \alpha_t dt + \sqrt{2} dB_t, \qquad X_{t_0} = x_0,$$

where $(B_t)_{t \in [0,T]}$ is a standard Brownian Motion.

- Forecasting the evolution of the population density $(m(t))_{t \in [0,T]}$, the agent solves the optimal control problem

$$u(t_0, x_0) := \inf_{\alpha} \mathbf{E} \left[\int_{t_0}^{T} L(X_s, \alpha_s, m(s))\, ds + g(X_T, m(T)) \right].$$

- **The value function** u is characterized by the Hamilton-Jacobi equation

$$\left\{ \begin{array}{l} -\partial_t u(t,x) - \Delta u(t,x) + H(x, Du(t,x), m(t)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d \\ u(T,x) = g(x, m(T)) \qquad \text{in } \mathbb{R}^d \end{array} \right.$$

where $H(x,p,m) = \sup_{\alpha \in \mathbb{R}^d} -\alpha \cdot p - L(x, \alpha, m)$.

- **The optimal feedback** of the agent is then given by $\alpha^*(t,x) := -D_p H(x, Du(t,x), m(t))$.

Evolution of the population density.

- If all agents implement their optimal control and if their BM are independent, **the mean field theory** says that the population density $\tilde{m}$ actually evolves according to the Kolmogorov equation

$$\left\{ \begin{array}{l} \partial_t \tilde{m}(t,x) - \Delta m(t,x) - \mathrm{div}\big(\tilde{m}(t,x) D_p H(x, Du(t,x), m(t))\big) = 0 \text{ in } (0,T)\times \mathbb{R}^d \\ \tilde{m}(0,\cdot) = m_0 \text{ in } \mathbb{R}^d. \end{array} \right.$$

At equilibrium,

- one must have $\tilde{m} = m$.

This heuristic yields to the **MFG system** :

$$(MFG) \quad \left\{ \begin{array}{lll} (i) & -\partial_t u - \Delta u + H(x, Du, m(t)) = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ (ii) & \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du, m)) = 0 & \text{in } [0,T]\times\mathbb{R}^d \\ (iii) & m(0,\cdot) = m_0, \ u(T,x) = g(x, m(T)) & \text{in } \mathbb{R}^d \end{array} \right.$$

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Basic results of the MFG system

For the **MFG equilibrium system :**

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- **Existence of solutions :** holds under general conditions (Lasry-Lions)
- **Uniqueness** cannot be expected in general, but holds
  - for small couplings or in a short time horizon (Huang-Caines-Malhamé, Lasry-Lions)
  - under a monotonicity conditions (Lasry-Lions) : if $H = H(x,p) - f(x,m)$ and

$$\int_{\mathbb{R}^d} (f(x,m) - f(x,m'))d(m-m') \geq 0, \ \int_{\mathbb{R}^d} (g(x,m) - g(x,m'))d(m-m') \geq 0.$$

- **Link with differential games** with finitely many players.
  - from the MFG system to the $N-$player differential games
    Many contributions (Huang-Caines-Malahmé, Carmona-Delarue, ...)
  - from Nash equilibria of $N-$player differential games to the MFG system.
    - LQ differential games (Bardi, Bardi-Priuli)
    - Open loop NE (Fischer, Lacker),
    - Closed loop NE (C.-Delarue-Lasry-Lions, Lacker).

Outline

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1. The MFG equilibrium

2. Efficiency of MFG equilibria
   - (A) definition of efficiency
   - Characterization of efficiency
   - Quantifying the inefficiency

3. Application to a mean field limit
   - Lacker's convergence result
   - Example of an ergodic cost

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The question of efficiency

Questions :

- Does the global cost of a Nash equilibrium differ from the optimal social cost an optimal planner can achieve ?
- If different, how far ?

Some references :

- Dubey ('86) = equilibria are generically inefficient.
- "Price of anarchy"
  - Koutsoupias-Papadimitriou ('99)
    Noncooperative games in which agents share a common resource
  - Roughgarden-Tardos ('02), Johari-Tsitsiklis ('03)
    In the framework of selfish routing games and congestion games
  - ...
- For differential and MFG problems :
  - Başar-Zhu ('11) : price of anarchy in LQ differential games
  - Balandat-Tomlin ('13) : numerical computations and Braess's paradox
  - Carmona-Graves-Tan (in preparation) : LQ MFG and MFG on finite Markov chains.

Problem for the global planner ($N-$player problem)

The optimal control problem : Agent $i \in \{1, \dots, N\}$ controls her dynamics :

$$\begin{cases} dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i, & t \in [0, T], \\ X_0^i = x_0^i \end{cases}$$

and wants to minimize the individual cost

$$J^i(\alpha^1, \dots, \alpha^N) = \mathbf{E}\left[\int_0^T L(X_t^i, \alpha_t^i, m_{\mathbf{X}_t}^N)\, dt + g(X_T^i, m_{\mathbf{X}_T}^N)\right],$$

where $m_{\mathbf{X}_t}^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$ is the empirical measure of the players.

The cost for the global planner is

$$J(\alpha^1, \dots, \alpha^N) = \frac{1}{N}\sum_{i=1}^N J^i(\alpha^1, \dots, \alpha^N).$$

Problem for the global planner (mean field limit)

Lacker '17 proves that

$$\lim_{N\to+\infty} \inf_{\alpha^1,\dots,\alpha^N} J(\alpha^1,\dots,\alpha^N) = \mathcal{C}^*,$$

where $\mathcal{C}^*$ is the cost associated with the McKean-Vlasov control problem :

$$\mathcal{C}^* := \inf_{(m,\alpha)} \int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x),m(t))m(t,x)\ dxdt + \int_{\mathbb{R}^d} g(x,m(T))m(T,x)dx$$

subject to

$$\partial_t m - \Delta m + \mathrm{div}(m\alpha) = 0, \qquad m(0,x) = m_0(x).$$

Global cost associated with an MFG Nash equilibrium

Given a Nash MFG equilibrium, i.e., a solution $(\bar{u}, \bar{m})$ to

$$(MFG) \quad \left\{ \begin{array}{lll} (i) & -\partial_t \bar{u} - \Delta \bar{u} + H(x, D\bar{u}, \bar{m}(t)) = 0 & \text{in } [0,T] \times \mathbb{R}^d \\ (ii) & \partial_t \bar{m} - \Delta \bar{m} - \mathrm{div}(\bar{m} D_p H(x, D\bar{u}, \bar{m})) = 0 & \text{in } [0,T] \times \mathbb{R}^d \\ (iii) & \bar{m}(0,\cdot) = m_0, \ \bar{u}(T,x) = g(x, \bar{m}(T)) & \text{in } \mathbb{R}^d \end{array} \right.$$

recall that $\alpha^*(t,x) := -D_p H(x, D\bar{u}(t,x), \bar{m}(t))$ is the optimal feedback.

The social cost associated with $(\bar{u}, \bar{m})$ is defined by

$$\mathcal{C}(\bar{u}, \bar{m}) := \int_0^T \int_{\mathbb{R}^d} L(x, \alpha^*(t,x), \bar{m}(t)) \bar{m}(t,x) dx dt + \int_{\mathbb{R}^d} g(x, \bar{m}(T)) \bar{m}(T,x) dx.$$

Efficiency in MFG systems

Given an MFG Nash equilibrium $(\bar{u}, \bar{m})$, compare

$$\mathcal{C}(\bar{u}, \bar{m}) := \int_0^T \int_{\mathbb{R}^d} L(x, \alpha^*(t,x), \bar{m}(t))\bar{m}(t,x)dxdt + \int_{\mathbb{R}^d} g(x, \bar{m}(T))\bar{m}(T,x)dx.$$

with the optimal cost of a global planner :

$$\mathcal{C}^* := \inf_{(m,\alpha)} \int_0^T \int_{\mathbb{R}^d} L(x, \alpha(t,x), m(t))m(t,x)\; dxdt + \int_{\mathbb{R}^d} g(x, m(T))m(T,x)dx$$

subject to

$$\partial_t m - \Delta m + \mathrm{div}(m\alpha) = 0, \qquad m(0,x) = m_0(x).$$

Definition

An MFG equilibrium $(\bar{u}, \bar{m})$ is efficient if $\mathcal{C}(\bar{u}, \bar{m}) = \mathcal{C}^*$.

The MFG system is globally efficient if, for any initial condition (t_0, m_0), there exists an MFG equilibrium starting from (t_0, m_0) which is efficient.

Outline

Derivatives in the space of measures

We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed for the Monge-Kantorovich distance

$$\mathbf{d}_1(m, m') = \sup_{\phi} \int_{\mathbb{T}^d} \phi(y)\, d(m - m')(y),$$

where the supremum is taken over all Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$ with a Lipschitz constant bounded by 1.

Derivatives

A map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y) d(m' - m)(y) ds.$$

We set

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y).$$

- Note that $\frac{\delta U}{\delta m}$ is defined up to an additive constant. We adopt the normalization convention

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m,y)dm(y) = 0.$$

- An example : if

$$U(m) = \int_{\mathbb{T}^d} \phi(x)dm(x),$$

then

$$\frac{\delta U}{\delta m}(m,y) = \phi(y) - \int_{\mathbb{T}^d} \phi(z)dm(z), \qquad D_mU(m,y) = D\phi(y).$$

- D_mU controls the Lipschitz norm of U :

$$|U(m_1) - U(m_2)| \leq \|D_mU\|_\infty \mathbf{d}_1(m_1, m_2) \qquad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d).$$

Assumptions

- The Lagrangian $L = L(x, \alpha, m) : \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is of class C^2 with respect to all variables and satisfies
$$C^{-1} I_d \leq D^2_{pp} L(x, \alpha, m) \leq C I_d,$$
$$\left|\frac{\delta L}{\delta m}(x, \alpha, m, y)\right| + \left|\frac{\delta^2 L}{\delta m^2}(x, \alpha, m, y, z)\right| \leq C(1 + |\alpha|^2),$$
$$\left|D_\alpha \frac{\delta L}{\delta m}(x, \alpha, m, y)\right| \leq C(1 + |\alpha|), \qquad \left|D^2_\alpha L(x, p, m)\right| \leq C.$$
We define the convex conjugate H of L as
$$H(x, p, m) = \sup_{\alpha \in \mathbb{R}^d} \{-p \cdot \alpha - L(x, \alpha, m)\},$$
and we assume that H is of class C^2 as well.
- The coupling function $g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is globally Lipschitz continuous with space derivatives $\partial_{x_i} g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ also Lipschitz continuous. We also assume that the map g is C^2 with respect to m and that its derivatives $\frac{\delta g}{\delta m} : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ and $\frac{\delta^2 g}{\delta m^2} : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ are Lipschitz continuous.

Optimality condition for the planner's cost

The planner's problem :

$$\mathcal{C}^* := \inf_{(m,\alpha)} \int_0^T \int_{\mathbb{R}^d} L(x,\alpha(t,x),m(t))m(t,x)\; dxdt + \int_{\mathbb{R}^d} g(x,m(T))m(T,x)dx$$

subject to

$$\partial_t m - \Delta m + \mathrm{div}(m\alpha) = 0, \qquad m(0,x) = m_0(x).$$

Proposition (Lasry-Lions '06, C.-Rainer)

Under our standing assumptions, the planner's problem has at least one solution.

For any solution $(\hat{m},\hat{w})$ of the planer's problem, there exists $\hat{u}$ such that $(\hat{u},\hat{m})$ solves

$$\left\{\begin{array}{l} -\partial_t \hat{u} - \Delta \hat{u} + H(x, D\hat{u}, \hat{m}(t)) = \displaystyle\int_{\mathbb{R}^d} \frac{\delta L}{\delta m}(y,\hat{\alpha}(t,y),x,\hat{m}(t))\hat{m}(t,y)dy \text{ in } (0,T)\times\mathbb{T}^d \\ \partial_t \hat{m} - \Delta \hat{m} - \mathrm{div}(\hat{m} D_pH(x,D\hat{u}(t,x),\hat{m}(t))) = 0 \qquad \text{in } (0,T)\times\mathbb{T}^d \\ \hat{m}(0,x) = m_0(x),\;\; \hat{u}(T,x) = \dfrac{\delta \widehat{\mathcal{G}}}{\delta m}(\hat{m}(T),x) \qquad \text{in } \mathbb{T}^d \\ \hat{\alpha}(t,x) = D_pH(x,D\hat{u}(t,x),\hat{m}(t)) \qquad \text{in } (0,T)\times\mathbb{T}^d, \end{array}\right.$$

where $\widehat{\mathcal{G}}(m) := \int_{\mathbb{T}^d} g(x,m)m(dx)$.

A necessary condition for efficiency

Proposition

Let $(\bar{u}, \bar{m})$ be an MFG Nash equilibrium. If $(\bar{u}, \bar{m})$ is efficient, then, for any $(t,x) \in [0,T] \times \mathbb{T}^d$,

$$\int_{\mathbb{T}^d} \frac{\delta L}{\delta m}(y, \alpha^*(t,y), x, \bar{m}(t))\bar{m}(t,y)dy = 0 \qquad \text{and} \qquad \int_{\mathbb{T}^d} \frac{\delta g}{\delta m}(t, \bar{m}(T), x)\bar{m}(T,y)dy = 0,$$

where $\alpha^*(t,x) := -D_pH(x, D\bar{u}(t,x), \bar{m}(t))$.

Proof.

- As $\mathcal{C}(\bar{u}, \bar{m}) = \mathcal{C}^*$ holds, $(\bar{m}, \alpha^*)$ minimizes $\mathcal{C}^*$.
- Hence there exists v such that $(v, \bar{m})$ solves

$$\begin{cases} -\partial_t v - \Delta v + H(x, Dv, \bar{m}(t)) = \int_{\mathbb{R}^d} \frac{\delta L}{\delta m}(y, \hat{\alpha}(t,y), x, \bar{m}(t))\bar{m}(t,y)dy \\ \partial_t \bar{m} - \Delta \bar{m} - \mathrm{div}(\bar{m} D_p H(x, Dv(t,x), \bar{m}(t))) = 0 \\ \bar{m}(0,x) = m_0(x), \ v(T,x) = \frac{\delta \widehat{\mathcal{G}}}{\delta m}(\bar{m}(T), x) \\ \hat{\alpha}(t,x) = D_p H(x, Dv(t,x), \bar{m}(t)). \end{cases}$$

  with $-D_p H(x, Dv(t,x), m(t)) = \alpha^*(t,x) = -D_p H(x, D\bar{u}(t,x), m(t))$.
- Hence $D\bar{u} = Dv$.
- This implies that $\bar{u}(t,x) = v(t,x) + c(t)$ (for some $c(t) \in \mathbb{R}$).
- Compare the equations satisfied by $\bar{u}$ and v :

$$-c'(t) = \int_{\mathbb{R}^d} \frac{\delta L}{\delta m}(y, \alpha^*(t,y), x, m(t))m(t,y)dy.$$

- Integrate against $m(t)$:

$$-c'(t) = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{\delta L}{\delta m}(y, \alpha^*(t,y), x, m(t))m(t,y)m(t,x)dydx = 0.$$

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- Thus

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$$\int_{\mathbb{R}^d} \frac{\delta L}{\delta m}(y, \alpha^*(t,y), x, \bar m(t))\bar m(t,y)dy = 0 \qquad \forall (t,x) \in [0,T]\times\mathbb{T}^d.$$

Proof.

- As $\mathcal{C}(\bar u, \bar m) = \mathcal{C}^*$ holds, $(\bar m, \alpha^*)$ minimizes $\mathcal{C}^*$.
- Hence there exists v such that $(v, \bar m)$ solves

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Characterization of global efficiency

Here we assume that H is of separate form :

$$H(x, p, m) = H(x, m) - f(x, m).$$

Theorem

The MFG system is globally efficient IFF

$$(*) \qquad \int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(y, m, x)m(dy) = 0, \int_{\mathbb{T}^d} \frac{\delta g}{\delta m}(y, m, x)m(dy) = 0, \qquad \forall (x, m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d),$$

Remarks :

- The condition is independent of H.
- Note that, in the separate setting,

$$\frac{\delta L}{\delta m}(x, \alpha, y, \bar{m}) = \frac{\delta f}{\delta m}(x, m, y).$$

Remarks (continued) :

- Condition $(*)$ is equivalent to the existence of C^2 maps $\mathcal{F} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that

$$f(x,m) = \mathcal{F}(m) + \frac{\delta \mathcal{F}}{\delta m}(m,x), \;\; g(x,m) = \mathcal{G}(m) + \frac{\delta \mathcal{G}}{\delta m}(m,x).$$

- Moreover, if

$$f(x,m) = \mathcal{F}(m) + \frac{\delta \mathcal{F}}{\delta m}(m,x)$$

and $\mathcal{F}$ is not affine, then f genuinely depends on m.

(Counter-)Examples

Assume that H is of separate form and $g \equiv 0$. Recall that the MFG system is globally efficient IFF $\int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(y, m, x)m(dy) = 0$.

1. If $f = f(m)$ does not depend on x, then the MFG system if globally efficient if only if f is constant.

   Proof. Indeed
   $$\int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(m, y)m(dx) = \frac{\delta f}{\delta m}(m, y).$$
   Hence the MFG system if globally efficient IFF $\frac{\delta f}{\delta m} \equiv 0$, which means f constant.

2. We now assume that f derives from a potential : There exists a C^1 map $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that $f = \delta\Phi/\delta m$. Then the MFG system is globally efficient IFF $f \equiv 0$.

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Outline

1. The MFG equilibrium
2. Efficiency of MFG equilibria
   - (A) definition of efficiency
   - Characterization of efficiency
   - Quantifying the inefficiency
3. Application to a mean field limit
   - Lacker's convergence result
   - Example of an ergodic cost

A lower bound

Theorem

Let $(\bar{u}, \bar{m})$ an MFG equilibrium. Then, for any $\varepsilon > 0$,

$$\mathcal{C}(u,m) - \mathcal{C}^* \geq \; C^{-1}\varepsilon^2 \Big(\int_0^{T-\varepsilon} \int_{\mathbb{T}^d} \Big[\int_{\mathbb{T}^d} \frac{\delta L}{\delta m}(x, \alpha^*(t,x), y, \bar{m}(t)) \bar{m}(t,x) dx \Big]^2 dydt \Big)^2$$
$$+ C^{-1} \Big(\int_{\mathbb{T}^d} \Big[\int_{\mathbb{T}^d} \frac{\delta g}{\delta m}(x, \bar{m}(T), y) \bar{m}(T,x) dx \Big]^2 dy \Big)^4,$$

where $\alpha^*(t,x) = -D_pH(x, D\bar{u}(t,x), \bar{m}(t))$ and the constants $C \geq 1$ depends on the regularity of H, g and on m_0 and where $C \geq 1$.

Remark If g does not depend on m, one can replace $\int_0^{T-\varepsilon}$ by $\int_0^T$.

Upper bound

Assume that $H = H(x,p) - f(x,m)$ is of separated form and set

$$\hat{\mathcal{F}}(m) := \int_{\mathbb{T}^d} f(x,m)m(dx), \qquad \hat{\mathcal{G}} := \int_{\mathbb{T}^d} g(x,m)m(dx).$$

Theorem

Assume in addition that the maps $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ are convex on $\mathcal{P}(\mathbb{T}^d)$. If $(\bar{u},\bar{m})$ is an MFG equilibrium, then

$$\mathcal{C}(\bar{u},\bar{m}) - \mathcal{C}^* \leq C\Big(\int_{t_0}^T \int_{\mathbb{T}^d} \Big[\int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(x,y,\bar{m}(t))\bar{m}(t,x)dx\Big]^2 dydt + \int_{\mathbb{T}^d} \Big[\int_{\mathbb{T}^d} \frac{\delta g}{\delta m}(x,\bar{m}(T),y)\bar{m}(T,x)dx\Big]^2 dy\Big)^{1/2},$$

where the constants $C \geq 1$ depends on the regularity of H, f, g and on m_0.

Examples

We always assume that H is of separate form and $g \equiv 0$.

1. If $f = f(m)$ does not depend on x, then

$$\mathcal{C}(\bar{u}, \bar{m}) - \mathcal{C}^* \geq C_\varepsilon^{-1} \left\{ \sup_{t_1 \neq t_2} \frac{|f(\bar{m}(t_2)) - f(\bar{m}(t_1))|}{(t_2 - t_1)^{1/2}} \right\}^4 .$$

   where the supremum is taken over $t_1, t_2 \in [\varepsilon, T - \varepsilon]$.

2. We now assume that f derives from a potential : $f = \delta\Phi/\delta m$. Then

$$C^{-1}\varepsilon^{-2} \left(\int_\varepsilon^{T-\varepsilon} \int_{\mathbb{T}^d} [f(y, \bar{m}(t))]^2 \, dydt \right)^2 \leq \mathcal{C}(\bar{u}, \bar{m}) - \mathcal{C}^*$$

$$\leq C \left(\int_0^T \int_{\mathbb{T}^d} [f(y, \bar{m}(t))]^2 \, dydt \right)^{1/2} .$$

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The $N-$player game

- $N-$small players
- Players now play path dependent strategies :

$$\mathcal{A}^i = \left\{ \alpha^i : [0,T] \times (C^0([0,T],\mathbb{R}^d))^N \to \mathbb{R}^d \text{ measurable, bounded, nonanticipative} \right\}.$$

- Dynamics : $dX^i_t = \alpha^i_t(\mathbf{X}_\cdot)dt + dB^i_t$,
(where the B^i are i.i.d. B.M. and α^i is the control of Player i)

- Goal of the players : **to minimize** over α^i the cost

$$J^i(\alpha^1,\dots,\alpha^N) = \mathbf{E}\left[\int_0^T L(X^i_t,\alpha^i_t,m^{N,i}_{\mathbf{X}_t})dt + G(X^i_T,m^{N,i}_{\mathbf{X}_T})\right],$$

where $m^{N,i}_{\mathbf{x}} = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}$ if $\mathbf{x} = (x_1,\dots,x_N)$.

- Nash equilibrium : $(\bar\alpha^1,\dots,\bar\alpha^N)$ s.t., for any $i\in\{1,\dots,N\}$, $\bar\alpha^i$ minimizes $\alpha^i \to J^i(\alpha^i,(\bar\alpha^j)_{j\neq i})$.

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Theorem (Lacker '18)

If $\bar{\alpha}^N = (\bar{\alpha}^{N,1}, \dots, \bar{\alpha}^{N,N})$ is a Nash equilibrium in the $N-$player game, then the empirical measure flow

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{\bar{\mathbf{X}}_t^N}$$

is tight in $C^0([0,T], \mathcal{P}(\mathbb{R}^d))$ and every limit point is a weak MFG equilibrium.

By weak MFG equilibrium μ^*, one means that there exists a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a filtration $(\mathcal{F}_t)$, a B.M. B, $\alpha^* = \alpha^*(t, x, m.)$ semi-Markov and X^* such that

- (μ_t^*) is $(\mathcal{F}_t)-$adapted,
- $dX_t^* = \alpha^*(t, X_t^*, \mu_\cdot^*)dt + dB_t$,
- α^* is minimizes
$$\alpha \to \mathbf{E}\left[\int_0^T L(X_t, \alpha_t, \mu_t^*)dt + G(X_T, \mu_T^*)\right],$$
- the consistence holds : $\mu_t^* = \mathbf{P}\left[X_t^* \in \cdot | \mathcal{F}_t^{\mu^*}\right]$.

Remarks

- Actually holds in a broader framework (relaxed controls, approximate equilibria,...).
- Here no monotonicity assumption.
  Under the monotonicity condition, weak MFG equilibria coincide with classical ones : Lacker's result extends C.-Delarue-Lasry-Lions without requiring regularity.
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Outline

The Nash equilibrium payoffs

We assume here that players observe each other passed trajectory (but *not* the control).

$$\mathcal{A}^i = \left\{ \alpha^i : [0,+\infty) \times (C^0([0,+\infty),\mathbb{T}^d))^N \to \mathbb{R}^d \text{ measurable, bounded, nonanticipative} \right\}.$$

The ergodic costs : for player $i \in \{1,\dots,N\}$,

$$J^i(\mathbf{x}_0,\alpha) = \limsup_{T\to+\infty} \frac{1}{T}\mathbf{E}\left[\int_0^T L(\alpha^i_t, X^i_t) + F(m^{N,i}_{\mathbf{X}_t})\, dt\right],$$

where $\mathbf{x}_0 = (x_0^1,\dots,x_0^N)$, $\alpha = (\alpha^1,\dots,\alpha^N)$ and

$$dX^i_t = \alpha^i_t((\mathbf{X}_s)_{s\le t})dt + dB^i_t,\ X^i_0 = x^i_0, \qquad m^{N,i}_{\mathbf{X}_t} = \frac{1}{N-1}\sum_{j\neq i}\delta_{X^j_t}.$$

Definition

A $N-$tuple $\mathbf{e} = (e^1,\dots,e^N) \in \mathbb{R}^N$ is a Nash equilibrium payoff if, for any $\varepsilon > 0$, there exists $(\bar\alpha^{i,\varepsilon})$ such that

$$\left| J^i(\mathbf{x_0},\bar\alpha^\varepsilon) - e^i \right| \le \varepsilon, \qquad J^i(\mathbf{x_0},\bar\alpha^\varepsilon) \le J^i(\mathbf{x_0},\alpha^i,(\bar\alpha^{j,\varepsilon})_{j\neq i}) + \varepsilon$$

for any $\alpha^i \in \mathcal{A}^i$. Note that $(\bar\alpha^{i,\varepsilon})_{i=1,\dots,N}$ is an $\varepsilon-$Nash equilibrium.

The unique MFG equilibrium

The ergodic MFG system : find (λ, u, m) such that

$$\begin{cases} -\Delta u(x) + H(Du(x), x) = F(\mu) + \lambda & \text{in } \mathbb{T}^d, \\ -\Delta \mu(x) - \operatorname{div}(\mu(x) H_p(Du(x), x)) = 0 & \text{in } \mathbb{T}^d, \\ \mu \geq 0, \ \int_{\mathbb{T}^d} \mu = 1. \end{cases}$$

Proposition

There is a unique MFG equilibrium, given by $(\lambda, u, m) = (\lambda_0 - F(\mu_0), u_0, \mu_0)$, where u_0 solves the ergodic problem

$$-\Delta u_0(x) + H(Du_0(x), x) = \lambda_0, \qquad \text{in } \mathbb{T}^d$$

and μ_0 is unique invariant measure

$$-\Delta \mu_0(x) - \operatorname{div}(\mu_0(x) H_p(Du_0(x), x)) = 0 \qquad \text{in } \mathbb{T}^d, \qquad \mu_0 \geq 0, \ \int_{\mathbb{T}^d} \mu_0 = 1.$$

Remark : It is known that $(\mu_0, -H_p(Du_0, x))$ is a minimum of

$$\inf_{(\mu,\alpha)} \int_{\mathbb{T}^d} L(\alpha(x), x)\mu(dx) = -\lambda_0$$

under the constraint $-\Delta \mu + \operatorname{div}(\mu\alpha) = 0$, $\mu \geq 0$, $\int_{\mathbb{T}^d} \mu = 1$.

Main results (1)

Convergence Theorem (C.-Rainer '19)

If $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is C^1, then there exists a sequence of symmetric Nash equilibrium payoffs $\mathbf{e}^N = (e^N, \dots, e^N)$ in the $N-$player game such that

$$\lim_{N\to+\infty} e^N = \tilde{e} := \inf_{\alpha,\mu} \int_{\mathbb{T}^d} L(\alpha(x), x)\mu(dx) + F(\mu),$$

where the infimum is taken over the pairs (α, μ) such that

$$-\Delta\mu + \mathrm{div}(\mu\alpha) = 0 \text{ in } \mathbb{T}^d, \qquad \mu \geq 0, \int_{\mathbb{T}^d} \mu = 1.$$

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Main results (2)

Theorem on efficiency (C.-Rainer '18)

If $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is C^1 and non constant, then

$$\tilde{e} := \inf_{\alpha,\mu} \int_{\mathbb{T}^d} L(\alpha(x), x)\mu(dx) + F(\mu) \ < \ \lambda_0 - F(\mu_0),$$

where $\lambda_0 - F(\mu_0)$ is the MFG equilibrium payoff.

Corollary

If $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is C^1 and non constant, there exists symmetric Nash equilibrium payoffs $\mathbf{e}^N = (e^N, \dots, e^N)$ which does not converge to the the MFG equilibrium payoff :

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Sketch of proof of Theorem 1

- W.l.o.g., we assume that F is non-constant, so that $\tilde{e} < -\lambda_0 + F(\mu_0)$.
- As F is C^1 and L coercive, there exists at least one minimizer $(\tilde{\mu}, \tilde{\alpha})$ of the problem

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- By duality arguments, one can prove that there exists $\tilde{u}$ and $\tilde{\lambda}$ with $\tilde{\alpha}(x) = -H_p(x, D\tilde{u}(x))$ and such that $(\tilde{\lambda}, \tilde{u}, \tilde{m})$ solves the MFG system

$$\begin{cases} -\Delta\tilde{u}(x) + H(D\tilde{u}(x), x) = \dfrac{\delta F}{\delta m}(x, \tilde{\mu}) + \tilde{\lambda} & \text{in } \mathbb{T}^d, \\ -\Delta\tilde{\mu}(x) - \mathrm{div}(\tilde{\mu}(x) H_p(D\tilde{u}(x), x)) = 0 & \text{in } \mathbb{T}^d, \\ \tilde{\mu} \geq 0, \displaystyle\int_{\mathbb{T}^d} \tilde{\mu} = 1. \end{cases}$$

- Let $e^N := \int_{\mathbb{T}^d} L(\tilde{\alpha}(x), x)\tilde{\mu}(dx) + \int_{(\mathbb{T}^d)^{N-1}} F(m_{\mathbf{x}}^{N,1})\tilde{\mu}(dx_2) \dots \tilde{\mu}(dx_N)$.

  Then $e^N \to \tilde{e} := \inf_{\alpha,\mu} \int_{\mathbb{T}^d} L(\alpha(x), x)\mu(dx) + F(\mu) < -\lambda_0 + F(\mu_0)$.

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Sketch of proof (continued)

- Fix $\varepsilon > 0$ and let $T, \delta > 0$ to be chosen below.
- We define the strategies $\beta^{N,T,\delta,i}$ as follows : Given $(X^1, \dots, X^N) \in (\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}^d))^N$, let

$$\theta^{N,T,\delta}(X^1, \dots, X^N) = \inf \left\{ t \geq T, \sup_{j \in \{1,\dots,N\}} \mathbf{d}_1\Big(\frac{1}{t}\int_0^t \delta_{X^j_s} ds, \tilde{\mu}\Big) \geq \delta \right\}.$$

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$$\beta^{N,T,\delta,i}(X^1, \dots, X^N)_t = \begin{cases} \tilde{\alpha}(X^i_t) & \text{if } t \leq \theta^{N,T,\delta}(X^1, \dots, X^N) \\ \alpha_0(X^i_t) & \text{otherwise,} \end{cases}$$

where $\alpha_0(x) = -H_p(Du_0(x), x)$.
(recall that (λ_0, u_0, m_0) is the solution of the MFG system).

- If no player deviates, then there exists $T = T(N, \delta, \varepsilon)$ such that

$$\mathbf{P}\left[\theta^{N,T,\delta}(\tilde{X}^1, \dots, \tilde{X}^N) < +\infty\right] << 1$$

and the payoff of each player is close to e^N.

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Sketch of proof (end)

- If Player *i* deviates and plays α^i, then
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  - or the deviation is detected, i.e., $\theta < +\infty$. Then the other players switch to α_0. Thus (for *N* large)

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^1 \delta_{X^i_s} ds = \mu_0 \qquad \text{so that } \limsup_{t\to+\infty} \frac{1}{t}\int_0^t F(m^{N,i}_{\mathbf{X}_t})\, dt \sim F(\mu_0).$$

Hence Player's i's payoff is close to

$$\sim \limsup_T \frac{1}{T}\int_{\mathbb{T}^d} L(X_s, \alpha^i_s)ds + F(\mu_0) \geq \int_{\mathbb{T}^d} L(\alpha_0(x), x)\mu_0(x)dx + F(\mu_0)$$
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- Taking expectation, we get $J^i(\mathbf{x_0}, \alpha^i, (\bar{\alpha}^{\varepsilon,j})_{j\neq i}) \geq J^i(\mathbf{x_0}, \bar{\alpha}^{\varepsilon}) - \varepsilon$.

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Conclusion and open problems

So far,

- We have characterized the efficiency of MFG
  (in most interesting cases, the MFG system is not efficient)
- We have (roughly) quantified the lack of efficiency.
- Application to a mean field limit.

Open problems.

- The upper bound relies on a structure condition : Is this necessary ?
- Obtain quantitatives properties independent of the regularity of the system.
- Efficiency for MFG in which the interaction is also through the distribution of the controls.

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