Mean Field Games on Unbounded Networks and the Graphon MFG Equations

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CROWDS models and control

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Program

- Major-Minor Agent Systems and MFG Equilibria
- LQG PO Major-Minor Agent MFG Theory
- Populations of Agents Distributed on Networks: Motivation + Introduction to Graphon Theory
- Graphon Control Systems
- Graphon Mean Field Games
- LQG-GMFG Example

Basic Formulation of Nonlinear Major-Minor MFG Systems

Problem Formulation:

- Notation: Subscript 0 for the major agent $\mathcal{A}_0$ and an integer valued subscript for minor agents $\{\mathcal{A}_i : 1 \le i \le N\}$.
- The states of $\mathcal{A}_0$ and $\mathcal{A}_i$ are $\mathbb{R}^n$ valued and denoted $z_0^N(t)$ and $z_i^N(t)$.

State Dynamics of the Major and N Minor Agents:

$$
\begin{aligned}
dz_0^N(t) &= \frac{1}{N}\sum_{j=1}^{N} f_0(t, z_0^N(t), u_0^N(t), z_j^N(t))dt \\
&\quad + \frac{1}{N}\sum_{j=1}^{N} \sigma_0(t, z_0^N(t), z_j^N(t))dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \le t \le T, \\
dz_i^N(t) &= \frac{1}{N}\sum_{j=1}^{N} f(t, z_i^N(t), z_0^N(t), u_i^N(t), z_j^N(t))dt \\
&\quad + \frac{1}{N}\sum_{j=1}^{N} \sigma(t, z_i^N(t), z_j^N(t))dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \le i \le N.
\end{aligned}
$$

MFG Nonlinear Major-Minor Agent Formulation

Performance Functions for Major and Minor Agents:

$$J_0^N(u_0^N; u_{-0}^N) := E\int_0^T \Big(\frac{1}{N}\sum_{j=1}^N L_0[t, z_0^N(t), u_0^N(t), z_j^N(t)]\Big)dt,$$

$$J_i^N(u_i^N; u_{-i}^N) := E\int_0^T \Big(\frac{1}{N}\sum_{j=1}^N L[t, z_i^N(t), z_0^N(t), u_i^N(t), z_j^N(t)]\Big)dt.$$

- The major agent has non-negligible influence on the mean field (mass) behaviour of the minor agents. (A consequence will be that the system mean field is no longer a deterministic function of time.)

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}^N, \mathbb{P})$: a complete filtered probability space
- $\mathcal{F}_t^N := \sigma\{z_j(0), w_j(s) : 0 \leq j \leq N, 0 \leq s \leq t\}$ Mtlly. Ind. ICs, Ind. BMs.
- $\mathcal{F}_t^{w_0} := \sigma\{z_0(0), w_0(s) : 0 \leq s \leq t\}$.

Basic Formulation of Nonlinear MFG Systems

Controlled McKean-Vlasov Equations:

- Infinite population limit dynamics:

$$
\begin{aligned}
dx_t &= f[x_t, u_t, \mu_t]dt + \sigma dw_t \\
f[x, u, \mu_t] &\triangleq \int_{\mathbb{R}} f(x, u, y)\mu_t(dy)
\end{aligned}
$$

Given ICs, a solution to the MKV SDE is a pair $(x_t, \mu_t(dx); 0 \le t < T)$

- Infinite population limit cost:

$$
\inf_{u \in \mathcal{U}} J(u, \mu) \triangleq \inf_{u \in \mathcal{U}} \mathbb{E} \int_0^T L[x_t, u_t, \mu_t]dt
$$

where $\mu_t(\cdot) =$ measure of the population state distribution

Information Patterns and Nash Equilibria

Information Patterns:

Local to Agent i: $\mathcal{F}_i \triangleq \sigma(x_i(\tau); \tau \leq t), \;\; 1 \leq i \leq N$
$\mathcal{U}_{loc,i}$: $\mathcal{F}_i$ adapted control + system parameters

Global with respect to the Population:

$\mathcal{F}^N \triangleq \sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)$
$\mathcal{U}$: $\mathcal{F}^N$ adapted control + system parameters

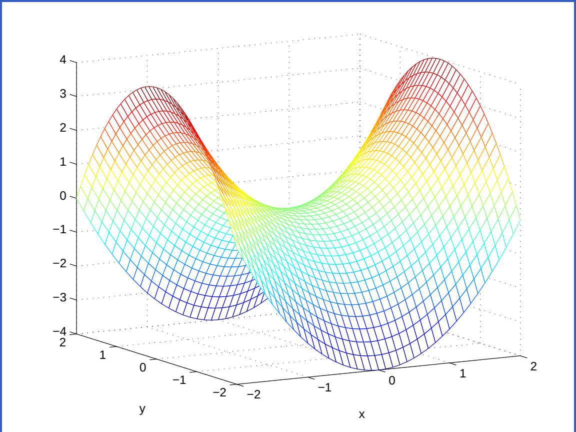
Definition: Nash Equilibrium: Unilateral Move Yields No Gain

The set of controls $\mathcal{U}^a = \{u_i^a;\, u_i^a$ adapted to $\mathcal{U}_{loc,i},\, 1 \leq i \leq N\}$ generates a *Nash Equilibrium* w.r.t. the performance functions $\{J_i; 1 \leq i \leq N\}$ if, for each i,

$$J_i(u_i^a, u_{-i}^a) = \inf_{u_i \in \mathcal{U}} J_i(u_i, u_{-i}^a)$$

Saddle Point Nash Equilibrium

- Agent y is a maximizer
- Agent x is a minimizer



ϵ-Nash Equilibrium

ϵ-Nash Equilibria:

Given $\varepsilon > 0$, the set of controls $\mathcal{U}^0 = \{u_i^0; 1 \le i \le N\}$ generates an *ε-Nash Equilibrium* w.r.t. the performance functions $\{J_i; 1 \le i \le N\}$ if, for each i,

$$J_i(u_i^0, u_{-i}^0) - \varepsilon \le \inf_{u_i \in \mathcal{U}} J_i(u_i, u_{-i}^0) \le J_i(u_i^0, u_{-i}^0)$$

Fundamental Mean Field Game MV HJB-FPK Theory

- Mean Field Game Pair (HMC, 2006, LL, 2006-07):
  Assuming that for any given strategy (i.e. control law) the infinite population limits exist for the population dynamics and performance functions, then:
  (i) the generic agent best response (BR) is generated by an MKV-HJB equation and
  (ii) the corresponding generic agent state distribution is generated by an MV-FPK equation (equivalently MKV SDE):

[MF-HJB] $$-\frac{\partial V}{\partial t} = \inf_{u \in U} \left\{ f[x,u,\mu_t] \frac{\partial V}{\partial x} + L[x,u,\mu_t] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}$$

$$V(T,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}$$

[MF-FPK] $$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial \{f[x,u,\mu]p(t,x)\}}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p(t,x)}{\partial x^2}$$

([MF-MKV SDE] $dx_t = f[x_t, \varphi(t,x|\mu_t), \mu_t]dt + \sigma dw_t$)

[MF-BR] $u_t = \varphi(t,x|\mu_t), \quad (t,x) \in [0,T] \times \mathbb{R}$

Basic Mean Field Game MV HJB-FPK Theory

Theorem (Huang, Malhamé, PEC, CIS'06)

Subject to technical conditions (i.e. uniform cty.+ boundedness on all functions + their derivatives + Lipschitz cty. wrt. controls):
(i) the MKV MFG Equations have a unique solution with the best response control generating a unique Nash equilibrium given by

$$u_i^0 = \varphi(t, x|\mu_t), \quad 1 \leq i \leq N.$$

Furthermore,
(ii) $\forall \epsilon > 0 \;\; \exists N(\epsilon)$ s.t. $\forall N \geq N(\epsilon)$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}} J_i^N(u_i, u_{-i}^0) \leq J_i^N(u_i^0, u_{-i}^0),$$

where $u_i \in \mathcal{U}$ is adapted to $\mathcal{F}^N := \{\sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)\}$.

Outline of Proof of Basic Result

Outline of Proof:

Restrict Lipschitz constants so that a Banach contraction argument gives existence and uniqueness via an iterated closed loop from mean field measure to control (from HJB) to measure (from FPK).

Major-Minor NL MFG theory: Mojtaba Nourian, PEC, SICON, 2013.

The Three Key Ideas of Mean Field Game Theory

Three Key Ideas:

- Nash Equilibrium Non-Cooperative Game Theoretic Equilibrium given by the solution to a **stochastic control problem** (wrt the distribution of the mass of agents)
- Dynamic Regeneration of Equilibrium: Generic Agent Mean Field Equilbrium is **regenerated** when all agents use the MFG BR strategies)
- Drastic Simplification of Dynamic Games: Infinite Population Control Strategies Yield **simple** Approximate Nash Equilbria for Large Finite Populations

Next on the Program

- Major-Minor Agent State Estimation and MFG Equilibria
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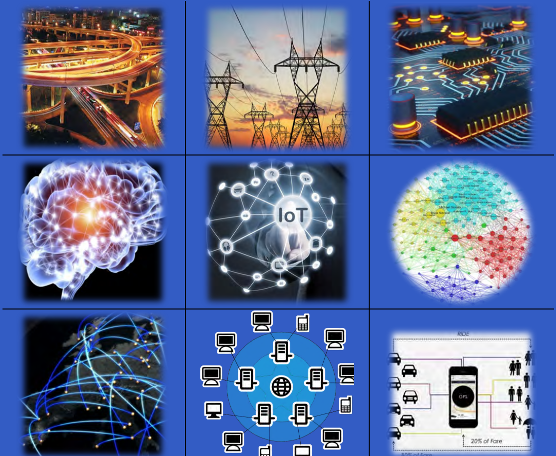
Separated and Linked Populations

Seek an MFG theory of flocking and swarming.

Motivation for Application of Graphon Theory in Systems and Control

Networks are ubiguitous, and are often growing in size and complexity: Online Social Networks, Brain Networks, Grid Networks, Transportation Networks, IoT, etc.



Motivation for a Graphon Theory of Systems and Control

A Common Feature of Networks of Dynamical Systems: Local nodes possess intrinsic states which evolve due to interactions with other nodes.

- Power grids (loads, generators and energy storage units)
- Epidemic networks
- Brain networks
- Social networks (opinions) and Fish Schooling
- Networks of computational devices
- Crowds?

Range of System Networks Behaviours: freely evolving, or locally controlled, and (or) globally controlled.

Motivation for Application of Graphon Theory in Systems and Control

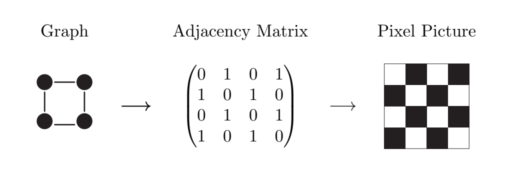
Shall consider a class of complex networks characterized by:

- Large number of nodes (in principle millions/billions of nodes)
- Complex connections which are asymptotically dense at each node (but sparse case is important)
- Intrinsically capable of growth in size

The recently developed mathematical theory of graphons provides a methodology for analyzing arbitrarily complex networks. (Sparse theory is developing.)

Introduction to Graphons

Graphs, Adjacency Matrices and Pixel Pictures

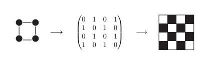


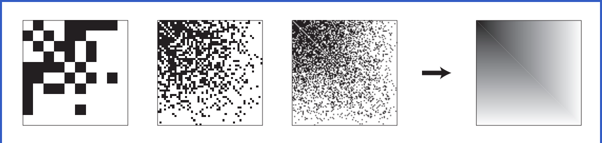
Graph, Adjacency Matrix, Pixel Picture

The whole pixel picture is presented in a unit square $[0,1] \times [0,1]$, so the square elements have sides of length $\frac{1}{N}$, where N is the number of nodes.

Introduction to Graphons

Graph Sequence Converging to Graphon





Graph Sequence Converging to its Limit

Graphons: bounded symmetric Lebesgue measurable functions

$$\mathbf{W} : [0,1]^2 \to [0,1]$$

interpreted as weighted graphs on the vertex set $[0,1]$.

$$\mathbf{G_0^{sp}} := \{\mathbf{W} : [0,1]^2 \to [0,1]\} \qquad \mathbf{G_1^{sp}} := \{\mathbf{W} : [0,1]^2 \to [-1,1]\}$$

$$\mathbf{G_R^{sp}} := \{\mathbf{W} : [0,1]^2 \to R\}$$

L. Lov´asz, Large Networks and Graph Limits. American Mathematical Soc., 2012, vol. 60.

Cut norm

$$\|\mathbf{W}\|_\Box := \sup_{M,T\subset[0,1]} \left| \int_{M\times T} \mathbf{W}(x,y)dxdy \right| \tag{1}$$

Cut metric

$$d_\Box(\mathbf{W},\mathbf{V}) := \inf_\phi \|\mathbf{W}^\phi - \mathbf{V}\|_\Box \tag{2}$$

L^2 metric

$$d_{L^2}(\mathbf{W},\mathbf{V}) := \inf_\phi \|\mathbf{W}^\phi - \mathbf{V}\|_2 \tag{3}$$

where $\mathbf{W}^\phi(x,y) = \mathbf{W}(\phi(x),\phi(y))$.

Since $\|W\|_\Box \leq \|W\|_{L^2}$ for any graphon W, convergence in d_{L^2} implies convergence in $d_\Box$.

Theorem

The graphon spaces $(\mathbf{G}_{\mathbf{0}}^{\mathbf{sp}}, d_{\square})$*, and hence the closed subsets of any* $(\mathbf{G}_{\mathbf{R}}^{\mathbf{sp}}, \mathbf{d}_{\square})$*, are compact.*

Graphon $\mathbf{W} \in \mathbf{G_1^{sp}}$ as an operator:

$$\mathbf{W} : L^2[0,1] \rightarrow L^2[0,1]$$

Operation on $\mathbf{v} \in L^2[0,1]$:

$$[\mathbf{W}\mathbf{v}](x) = \int_0^1 \mathbf{W}(x,\alpha)\mathbf{v}(\alpha)d\alpha \tag{4}$$

Operator product :

$$[\mathbf{U}\mathbf{W}](x,y) = \int_0^1 \mathbf{U}(x,z)\mathbf{W}(z,y)dz \tag{5}$$

where $\mathbf{U}, \mathbf{W} \in \mathbf{G_1^{sp}}$

$\mathbf{A} \in \mathbf{G}_{\mathbf{1}}^{\mathbf{sp}}$ is the infinitesimal generator of the uniformly continuous semigroup

$$\mathbf{S}_{\mathbf{A}}(t) := e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!} \tag{6}$$

The initial value problem of the graphon differential equation

$$\dot{\mathbf{y}}_{\mathbf{t}} = \mathbf{A}\mathbf{y}_{\mathbf{t}}, \quad \mathbf{y}_{\mathbf{0}} \in L^2[0,1], \quad 0 \leq t \leq T \tag{7}$$

has a solution given by

$$\mathbf{y}_{\mathbf{t}} = e^{\mathbf{A}t}\mathbf{y}_{\mathbf{0}}, \quad \mathbf{y}_{\mathbf{t}} \in L^2[0,1], \quad 0 \leq t \leq T. \tag{8}$$

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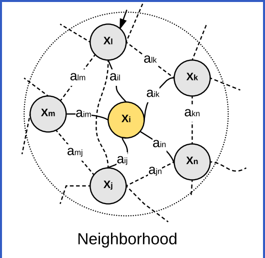
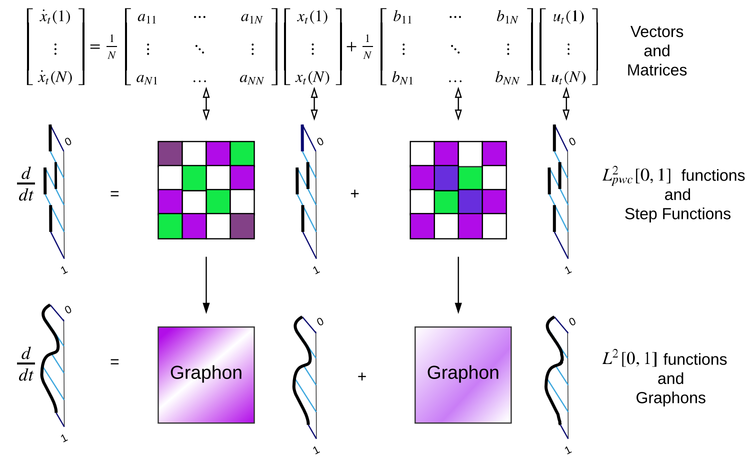
The dynamics of the i^{th} agent in the network



$$\dot{x}_t^i = \frac{1}{N}\sum_{j=1}^{N} a_{ij} x_t^j + \frac{1}{N}\sum_{j=1}^{N} b_{ij} u_t^j$$

$x_t^i \in R^1$: state
$u_t^i \in R^1$: control

Consider the scalar case for simplicity.



Compactness of graphon space ensures subsequence limits exists.

Infinite dimensional linear systems

$$LS^{\infty}: \quad \begin{array}{l} \dot{\mathbf{x}}_{\mathbf{t}} = \mathbf{A}\mathbf{x}_{\mathbf{t}} + \mathbf{B}\mathbf{u}_{\mathbf{t}}, \quad 0 \leq t \leq T \\ \mathbf{x}_{\mathbf{0}} \in L^2[0,1], \quad \mathbf{A} \in \mathbf{G}_{\mathbf{1}}^{\mathbf{sp}}, \mathbf{B} \in \mathcal{G}_{AI} \end{array}$$

$\mathbf{x}_{\mathbf{t}} \in L^2[0,1]$: system state; $\mathbf{u}_{\mathbf{t}} \in L^2[0,1]$: control input

$$(H1) \quad \begin{cases} \text{(i)} & \mathbf{A} \text{ generates a uniformly continuous} \\ & \text{semigroup } e^{t\mathbf{A}} \text{ on } L^2[0,1], \\ \text{(ii)} & \mathbf{B} \in \mathcal{L}(L^2[0,1]; L^2[0,1]), \end{cases}$$

Subject to $H1$ there **exists a unique solution** $\mathbf{x} \in C([0,T]; L^2[0,1])$ to LS^{∞} for any $\mathbf{x}_{\mathbf{0}} \in L^2[0,1]$ and any $\mathbf{u} \in L^2([0,T]; L^2[0,1])$.

Definition An infinite dimensional linear system $(\mathbf{A};\mathbf{B})$ is *exactly controllable* if on any time interval [0, t] $(0 < t < \infty)$ any initial state in the state space X can be steered to any target state in X.

Note: In the present case, a state $x \in X$ is an equivalence class of $L^2[0,1]$ functions.

Controllability Gramian $\mathbf{W_t} : L^2[0,1] \to L^2[0,1]$

$$\mathbf{W_t} := \int_0^t e^{\mathbf{A}(t-s)} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t-s)} ds, \quad t > 0.$$

A necessary and sufficient condition for exact controllability on $[0,T]$ is the uniform positive definiteness of $\mathbf{W}_T$:

$$(\mathbf{W}_T h, h) \geq c_T \|h\|^2$$

for all $h \in L^2[0,1]$, where $c_T > 0$ and $\| \cdot \|$ is the $L^2[0,1]$ norm (Bensoussan et al., 2007, Curtain et al,1995)

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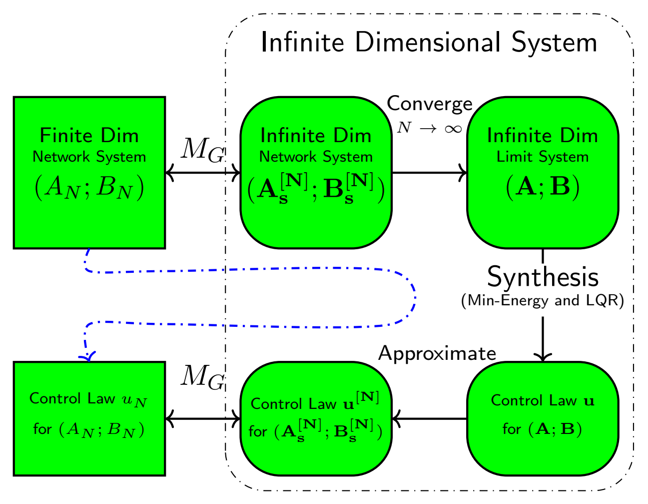
Methodology for Controlling Systems on Complex Networks



Control Design Procedure for Network Systems via Graphon Limits

Methodology for Controlling Systems on Complex Networks

Theorem (S.Gao, PEC 2017b)

Consider a sequence of graphon systems $\{(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})\}$ *converging to a graphon system* $(\mathbf{A}; \mathbf{B})$ *in the* L^2 *operator norm as* $N \to \infty$*:* $\mathbf{A_s^{[N]}} \to \mathbf{A}$ *and* $\mathbf{B_s^{[N]}} \to \mathbf{B}$*. Then*
1. There exists a control $\mathbf{v}^{[\mathbf{N}]} \in L^2[0,1]$ *for* $(\mathbf{A_s^{[N]}}; \mathbf{B_s^{[N]}})$ *approximating the control* $\mathbf{v} \in L^2[0,1]$ *for* $(\mathbf{A}; \mathbf{B}) \in L^2[0,1]$*such that*

$$\begin{aligned} \|\mathbf{x_T}(\mathbf{v}) - \mathbf{x_T^N}(\mathbf{v}^{[\mathbf{N}]})\|_2 \leq & \|\mathbf{A_\Delta^N}\|_2 \|\mathbf{B}\|_2 \int_0^T e^{T-\tau}(T-\tau) \cdot \|\mathbf{v}_\tau\|_2 d\tau \\ & + \|\mathbf{B_\Delta^N}\|_2 \int_0^T e^{(T-\tau)\|\mathbf{A_s^{[N]}}\|_2} \cdot \|\mathbf{v}_\tau\|_2 d\tau, \end{aligned} \tag{9}$$

where $\mathbf{A_\Delta^N} = \mathbf{A} - \mathbf{A_s^{[N]}}$ *and* $\mathbf{B_\Delta^N} = \mathbf{B} - \mathbf{B_s^{[N]}}$*.*
2. Furthermore, $\lim_{N \to \infty} \|\mathbf{x_T}(\mathbf{v}) - \mathbf{x_T^N}(\mathbf{v}^{[\mathbf{N}]})\|_2 = 0$*.*

Lemma (S.Gao, PEC 2017b)

Suppose $\mathbf{A}_s^{[N]} \to \mathbf{A}$ *in the* $L^2[0,1]^2$ *operator norm as* $N \to \infty$. *Then for any* $v \in L^2[0,1]$ *there exists a control* $\mathbf{u}^{[N]} \in L^2[0,1]$ *for* $(\mathbf{A}_s^{[N]}; I)$ *approximating the control* $\mathbf{u}$ *for* $(\mathbf{A}; I)$ *such that*

$$\begin{aligned}\|\mathbf{x_T}(\mathbf{u}) - \mathbf{x_T^N}(\mathbf{u^{[N]}})\|_2 \leq & \|\mathbf{A_\Delta^N}\|_2 \int_0^T e^{T-\tau}(T-\tau)\|\mathbf{u}_\tau\|_2 d\tau \\ & + \|\int_0^T [\mathbf{u}_\tau - \mathbf{u}_\tau^N] d\tau\|_2,\end{aligned} \tag{10}$$

where $\mathbf{A_\Delta^N} = \mathbf{A} - \mathbf{A_s^{[N]}}$.

$$\mathbf{u_t^{[N]}}(\alpha) = N \int_{P_i} \mathbf{u_t}(\beta) d\beta, \quad \forall \alpha \in P_i, \tag{11}$$

with the uniform partition P^N.

Minimum Energy Graphon Control

The Minimum Energy state to state control problem for a graphon system $(\mathbf{A}; \mathbf{B})$:

$$
\begin{aligned}
& inf_u J(u) \\
& s.t. \quad \text{Inital state } x_0 \rightarrow \text{Target state } x_T,
\end{aligned}
$$

where the control energy is given by

$$J(u) := \int_0^T \|u_\tau||_2^2 d\tau = \int_0^T \int_0^1 u_\tau(\alpha)^2 d\alpha d\tau$$

Recall: Definition: The controllability Gramian
$\mathbf{W_t} : L^2[0,1] \to L^2[0,1]$

$$\mathbf{W_t} := \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t-s)} ds, \quad t > 0.$$

Recall: Fact: $(\mathbf{A};\mathbf{B})$ exactly controllable $\Leftrightarrow$ $\mathbf{W}$ uniformly positive definite.

If $(\mathbf{A};\mathbf{B})$ exactly controllable the Optimal control law:

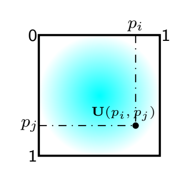
$$\mathbf{u}^*_\tau = \mathbf{B}^T e^{\mathbf{A}^T(t-\tau)}\mathbf{W_t}^{-1}(\mathbf{x_t} - e^{\mathbf{A}t}\mathbf{x_0}), \quad \tau \in [0,t] \qquad (12)$$

A method for generating a class of generic dynamic network examples with finite graphs converging to a given graphon U:

To obtain a network system $(A_N; I_N)$:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} A_{Nij}x_j + u_i, \quad x_i, u_i \in R, i \in \{1, ..., N\} \qquad (13)$$

where A_{Nij} is randomly generated from the graphon limit $\mathbf{U}$ (bounded and almost everywhere continuous).
Sample independently and uniformly N points $\{p_i\}_{i=1}^{N}$ from $[0, 1]$

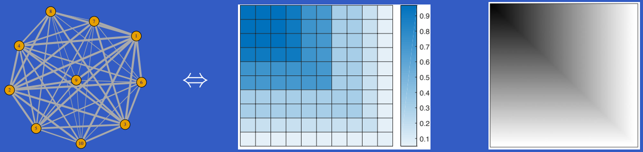


$$A_{N_{ij}} = \mathbf{U}(p_i, p_j)$$

Minimum Energy Graphon Control

Example I

Uniform Attachment Graphon: $\mathbf{U}(x, y) = 1 - \max(x, y)$, $x, y \in [0, 1]$.



Weighted Graph Generated from $\mathbf{U}$, its Stepfunction and Graphon Limit

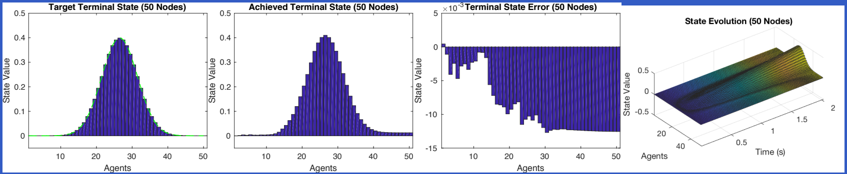
Minimum Energy Graphon Control

Example I

Uniform Attachment Graphon: $\mathbf{U}(x, y) = 1 - \max(x, y)$, $x, y \in [0, 1]$.

$$\dot{x}_t = \frac{1}{N} A_N x_t + u_t, \quad x_t \in R^N, u_t \in R^N$$

Simulation: Control generated *analytically* from graphon limit and sampled for input to 50 node network system.



Minimum Energy Target State Control on Network with 50 Nodes

Graphon Linear Quadratic Regulation

For a graphon system $(\mathbf{A};\mathbf{B})$ find the infimum of the performance function

$$\text{OCP:} \quad J(\mathbf{u}) = \int_0^T \left[\|\mathbf{C}\mathbf{x}_\tau\|^2 + \|\mathbf{u}_\tau\|^2\right] d\tau + \langle \mathbf{P}_0 \mathbf{x}_T, \mathbf{x}_T \rangle$$

over all controls $\mathbf{u} \in L^2(0,T;L^2(0,1))$ where $\mathbf{C}$ and $\mathbf{P}_0$ satisfy:

$(H2)$
- (iii) $\mathbf{P}_0 \in \mathcal{L}(L^2[0,1])$ is hermitian and non-negative,
- (iv) $\mathbf{C} \in \mathcal{L}(L^2[0,1];L^2[0,1])$

Graphon Linear Quadratic Regulation

Let $\mathbf{P}$ solve the following Riccati equation:

$$\dot{\mathbf{P}} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T\mathbf{C}, \quad \mathbf{P}(0) = \mathbf{P}_0. \tag{14}$$

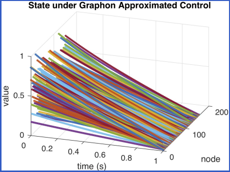
Applying (Bensoussan et al, 2007) and specializing the Hilbert space there to be $L^2[0,1]$ space, we have:

Theorem

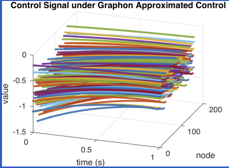
Assume that $(H2)$ is verified. Then the Riccati Equation (14) has a unique (mild) solution $\mathbf{P} \in C_s([0,T); \Sigma^+(L^2[0,1]))$ and the closed loop system under LQR optimal control over $[0,T]$ is given by

$$\begin{aligned} &\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t - \mathbf{B}\mathbf{B}^*\mathbf{P}(T-t)\mathbf{x}_t, \\ &t \in [0,T], \mathbf{x}_0 \in L^2[0,1]. \end{aligned} \tag{15}$$

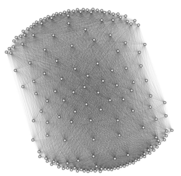
Sinusoidal Graphon: $\mathbf{U}(x, y) = cos(\pi(x - y)),\ x, y \in [0, 1].$
Control generated *analytically* from graphon limit; sampled for input at 160 nodes.



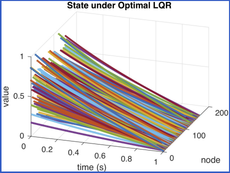
State Evolution under Graphon Control



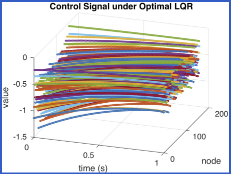
Control Input of Graphon Control



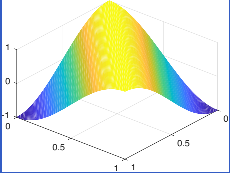
Network of 160 Nodes



State Evolution under Optimal LQR



Control Input of Optimal LQR

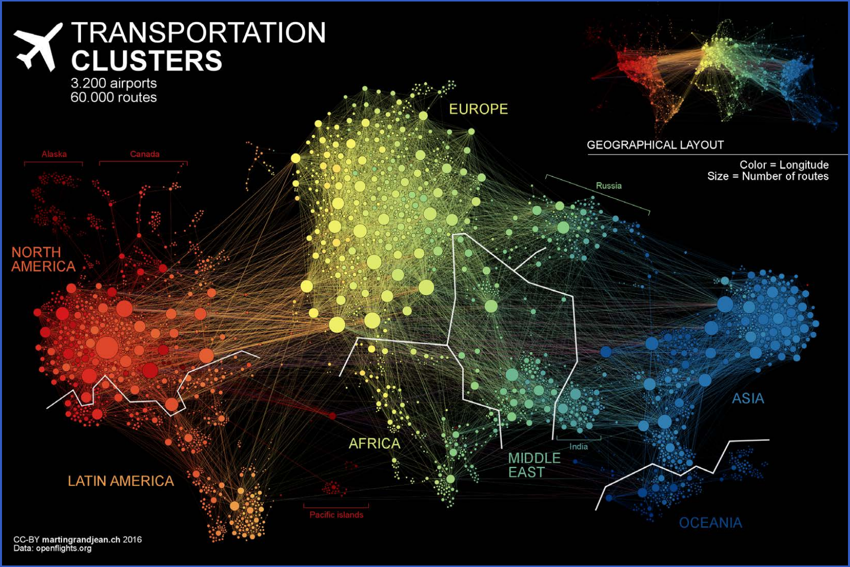


Graphon Limit

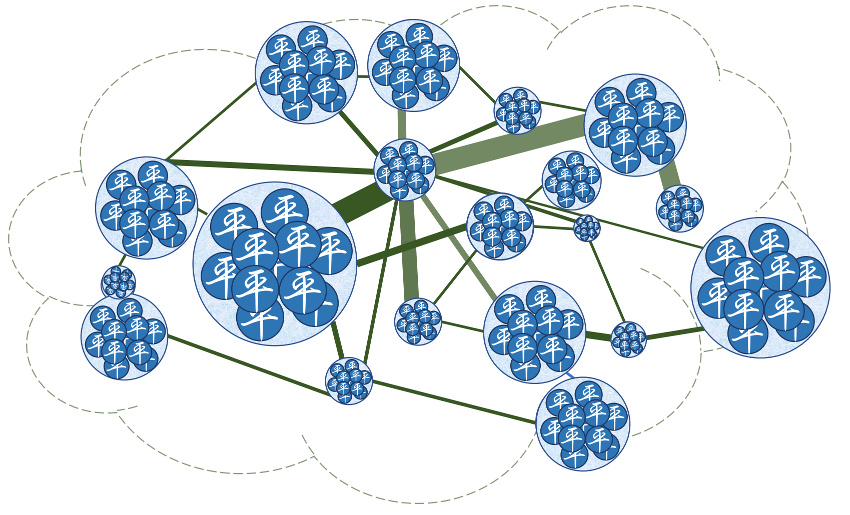
Next on the Program

- Major-Minor MFG Theory
- Populations of Agents Distributed on Networks: Motivation + Introduction to Graphons
- Graphon Control Systems
- Graphon Mean Field Games
- LQG - MFG Example

Graphon Mean Field Games



Graphon Mean Field Games - Motivation



The Graphon Mean Field Game Equations (i)

$$\text{[HJB]}(\alpha) \quad -\frac{\partial V_\alpha(t,x)}{\partial t} = \inf_{u\in U}\left\{\widetilde{f}[x,u,\mu_G;g_\alpha]\frac{\partial V_\alpha(t,x)}{\partial x} + \widetilde{l}[x,u,\mu_G;g_\alpha]\right\} + \frac{\sigma^2}{2}\frac{\partial^2 V_\alpha(t,x)}{\partial x^2},$$

$$V_\alpha(T,x) = 0, \qquad (t,x)\in[0,T]\times\mathbb{R}^n, \quad \alpha\in[0,1],$$

$$\text{[FPK]}(\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} = -\frac{\partial\{\widetilde{f}[x,u^0(x_\alpha,\mu_G;g_\alpha)p_\alpha(t,x)\}}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p_\alpha(t,x)}{\partial x^2},$$

$$\text{[BR]}(\alpha) \quad u^0(x_\alpha,\mu_G;g_\alpha) = \arg\inf_u H(x_\alpha,u,\mu_G;g_\alpha), =: \varphi(t,x_t|\mu_G;g_\alpha)$$

Graphon Mean Field Games : GMFG

The Graphon Mean Field Game Equations (ii)

The *graphon local mean field* μ_α, the corresponding set of all the *local mean fields* $\mu_G = \{\mu_\beta; 0 \leq \beta \leq 1\}$, and the graphon function $g_\alpha = \{g(\alpha, \beta); 0 \leq \beta \leq 1\}$ are inter-related by the FPK and the defining integral relation

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] := \int_{[0,1]} \int_R f(x_\alpha, u_\alpha, x_\beta) g(\alpha, \beta) \mu_\beta(dx_\beta) d\beta$$

which gives the complete graphon mean field dynamics via the sum

$$\tilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] := f_0(x_\alpha, u_\alpha) + f[x_\alpha, u_\alpha, \mu_G; g_\alpha].$$

The graphon mean field cost functions $\widetilde{l}[x, u, \mu_G; g_\alpha]$ are defined similarly.

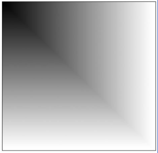
Graphon Mean Field Games : GMFG

We retrieve the simple standard MFG framework when the agents' dynamics and costs are uniform, and, further, the network is totally connected with uniform link weights giving $\{g(\alpha, \beta) = 1; 0 \leq \alpha, \beta \leq 1\}$. Since then the FPK equations and integral equations have a solution where all the local graphon mean fields are equal, i.e. $\mu_{t,\alpha} =: \mu_t$, for all α.

Image of a non-uniform graphon with function

$$g(\alpha, \beta) = 1 - \max(\alpha, \beta),$$

$$\alpha, \beta \in [0, 1]$$



Graphon Mean Field Games : GMFG

Theorem 1: Existence and Uniqueness of Solutions to the GMFG Equation Systems (PEC, Huang, 2017)
Subject to technical conditions, there exists a unique solution to the graphon dynamical GMFG equations, which (i) gives the feedback control best response (BR) strategy $\varphi(t, x_t|\mu_G; g_\alpha)$ *depending only upon the agent's state and the graphon local mean fields* (i.e. $(x_t, \mu_G; g_\alpha)$), *and* (ii) *generates a Nash equilibrium.*

Graphon Mean Field Games : GMFG 2

Theorem 2: ϵ-Nash Equilibria for GMFG System (PEC, Huang, 2018)
Let the conditions of Theorem 1 hold together with the continuity of the graphon function $\mathbf{G} = \{g(\alpha, \beta), 0 \leq \alpha, \beta \leq 1\}$. Then the joint strategy $\{u_i^o(t) = \varphi(t, x_t | \mu_G; g_\alpha)$ yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$.
Namely, $\forall \epsilon > 0 \;\; \exists N(\epsilon)$ s.t. $\forall N \geq N(\epsilon)$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}} J_i^N(u_i, u_{-i}^0) \leq J_i^N(u_i^0, u_{-i}^0),$$

where $u_i \in \mathcal{U}$ is adapted to $\mathcal{F}^N := \{\sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)\}$.

Finally on the Program

- Major-Minor MFG Theory
- Populations of Agents Distributed on Networks: Motivation + Introduction to Graphons
- Graphon Control Systems
- Graphon Mean Field Games
- LQG - MFG Example

LQG-GMFG Example - Finite Population (1)

Linear Quadratic Gaussian - GMFG Systems: Example

Individual Agent's Dynamics:

$$dx_i = (Ax_i + Bu_i)dt + \Sigma dw_i, \quad 1 \leq i \leq N.$$

- x_i: state of the ith agent
- u_i: control
- w_i: disturbance (standard Wiener process)
- $\mathcal{V}_k$: set of vertices: index set $\{1, ..., N_k\}$
- C_ℓ: set of agents in the ℓth cluster

For $x_i \in C_q$ and symmetric adjacency matrix $M = [m_{q\ell}]$:

$$z_i = \frac{1}{|\mathcal{V}_k|} \sum_{\ell \in \mathcal{V}_k} m_{q\ell} \frac{1}{|C_\ell|} \sum_{j \in C_\ell} x_j$$

LQG-GMFG Example - Finite Population (2)

Individual Agent's Cost:

$$J_i(u_i, \nu_i) \triangleq \mathbb{E} \int_0^T \Big[[(x_i - \nu_i)^\intercal Q (x_i - \nu_i) + u_i^\intercal R u_i] dt$$
$$+ (x_i(T) - \nu_i(T))^\intercal Q_T (x_i(T) - \nu_i(T)) \Big], \quad 1 \leq i \leq N,$$

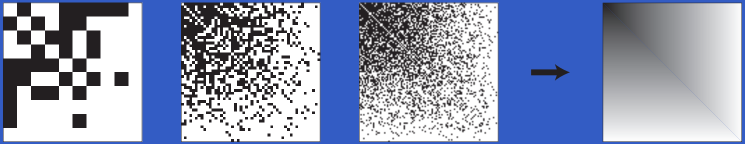
where $Q, Q_T \geq 0, R > 0$, and $\nu_i \triangleq \gamma(z_i + \eta)$ is the process tracked by agent i.

Main features:

- Agents may be linearly coupled via (i) their dynamics (omitted in this example) and (ii) running costs over a finite bidirectional graph of clusters
- Tracked process ν_i:
  - i stochastic
  - ii depends on other agents' control laws
  - iii depends on the location in the graph of x_i's cluster

LQG-GMFG Example - Infinite Population (1)

The sequence and limit of the underlying graph sequence chosen for this example: The Uniform Attachement Graph (LL2012)



Mean field coupling at any agent in cluster C_α in the limit:

$$z_\alpha = \int_{[0,1]} [M(\alpha,\beta) \int_{R^n} x_\beta \mu_\beta (dx_\beta)] d\beta, \qquad \alpha, \beta \in [0,1]$$

LQG-GMFG Example - Infinite Population (2)

Individual Agent's Dynamics:

$$dx_\alpha = (Ax_\alpha + Bu_\alpha)dt + \Sigma dw_\alpha, \quad \alpha \in [0,1].$$

Individual Agent's Cost:

$$J_\alpha(u_\alpha, \nu_\alpha) \triangleq \mathbb{E} \int_0^T \Big[[(x_\alpha - \nu_\alpha)^\intercal Q (x_\alpha - \nu_\alpha) + u_\alpha^\intercal R u_\alpha] dt + (x_\alpha(T) - \nu_\alpha(T))^\intercal Q_T (x_\alpha(T) - \nu_\alpha(T)) \Big]$$

where $Q, Q_T \geq 0, R > 0$ and $\nu_\alpha \triangleq \gamma(z_\alpha + \eta)$.

Graphon local mean field at agent α for the Uniform Attachment Graph:

$$z_\alpha = \int_{[0,1]} \big[(1 - \max(\alpha, \beta)) \int_{R^n} x_\beta \mu_\beta(dx_\beta) \big] d\beta, \qquad \alpha, \beta \in [0,1].$$

LQG-GMFG Example - Infinite Population (3)

Infinite Population Nash Equilibrium generated via optimal tracking (BR) control applied for each agent in each cluster C_α:

$$u_\alpha(t) = -R^{-1}B^\intercal[\Pi_t x_\alpha(t) + s_\alpha(t)] \tag{16}$$

$$-\dot{\Pi}_t = A^\intercal \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^\intercal \Pi_t + Q, \quad \Pi_T = Q_T \tag{17}$$

$$-\dot{s}_\alpha(t) = \left(A - BR^{-1}B^\intercal \Pi_t\right)^\intercal s_\alpha(t) - Q\nu_\alpha(t), \quad s_\alpha(T) = Q_T \nu_\alpha(T) \tag{18}$$

Graphon local mean field and tracked process (cost coupling)

$$z_\alpha = \int_{[0,1]} M(\alpha,\beta)\bar{x}_\beta d\beta, \quad \nu_\alpha \triangleq \gamma(z_\alpha + \eta), \quad \alpha \in [0,1] \tag{19}$$

Mean of State Process x_β

$$\bar{x}_\beta \triangleq \lim_{|C_\beta|\to\infty} \frac{1}{|C_\beta|} \sum_{j \in C_\beta} x_j = \int_{R^n} x_\beta \mu_\beta(dx_\beta) \tag{20}$$

$$\dot{\bar{x}}_\alpha = (A - BR^{-1}B^\intercal \Pi_t)\bar{x}_\alpha - BR^{-1}B^\intercal s_\alpha, \quad \alpha \in [0,1]. \tag{21}$$