Ore polynomials and application to coding theory

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Arithmetic, Geometry,
Cryptography and Coding Theory

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Ore polynomials

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Definition

The Ore ring $K[x; \varphi, \partial]$ is described as follows

- ☞ elements: usual univariate polynomials over K
- ☞ addition: standard addition
- ☞ multiplication: driven by $x\,a = \varphi(a)\,x + \partial(a), \quad \forall a \in K$

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&= \big(\varphi^2(a)\,\mathbf{x} + \partial\varphi(a)\big) \cdot \mathbf{x}
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$$(x^2 + ax + b) \cdot (x^2 + cx + d)$$

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&(\mathrm{x}^2 + a\mathrm{x} + b) \cdot (\mathrm{x}^2 + c\mathrm{x} + d) \\
&\quad = \mathrm{x}^4 + \mathrm{x}^2 c\mathrm{x} + \mathrm{x}^2 d + a\mathrm{x}^3 + a\mathrm{x}c\mathrm{x} + a\mathrm{x}d + b\mathrm{x}^2 + bc\mathrm{x} + d
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$\Longrightarrow$ Ore polynomials can admit many factorizations

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Given $A, B \in K[x; \varphi, \partial]$ with $B \neq 0$
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$K[x; \varphi, \partial]$ admits right gcd and left lcm

Application to coding theory

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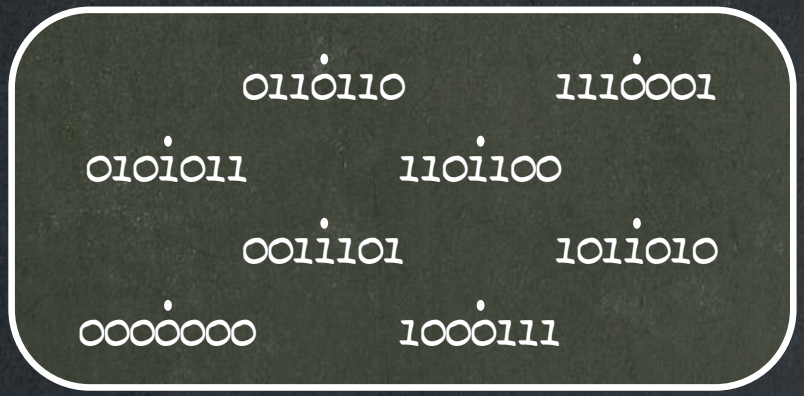
Example



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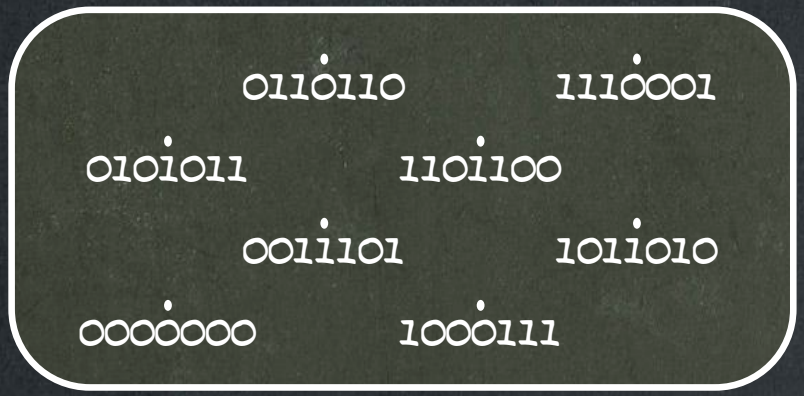

Theorem
(Singleton bound)

For any linear code:
$d + k \leq n + 1$

Reed Solomon codes and Gabidulin codes

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Let k, n be two positive integers with $k \leq n$

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The associated Reed Solomon code $RS(k; a_1, \ldots, a_n)$ is the image of

$$\begin{array}{ccc} K[x]_{<k} & \longrightarrow & K^n \\ f(x) & \mapsto & (f(a_1), \ldots, f(a_n)) \end{array}$$

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Let k, n be two positive integers with $k \leq n$

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Let k, n be two positive integers with $k \leq n$

Let $a_1, \ldots, a_n$ be $\mathbb{F}_p$-linearly indep. elements of K

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Let k, n be two positive integers with $k \leq n$

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Let $a_1, \ldots, a_n$ be $\mathbb{F}_p$-linearly indep. elements of $K = \mathbb{F}_q$

The associated Gabidulin code $RS(k; a_1, \ldots, a_n)$ is the image of

$$\begin{array}{ccc} K[x]_{<k} & \longrightarrow & K^n \\ f(x) & \mapsto & (f(a_1), \ldots, f(a_n)) \end{array}$$

Theorem

length = n

dimension = k

minimal distance = $n - k + 1$

Reed Solomon codes and Gabidulin codes

Let k, n be two positive integers with $k \leq n$

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The associated Gabidulin code $(k; a_1, \ldots, a_n)$ is the image of

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$$\left\{ c_0 x + c_1 x^p + \cdots + c_{k-1} x^{p^{k-1}} \right\} \quad (c_i \in K)$$

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Theorem

$\{$ [redacted] $\}$ $(c_i \in K)$

length = n

dimension = k

minimal distance = $n - k + 1$

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$$\boxed{x a = a^p x}$$

Theorem

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$$\longrightarrow K^n$$

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Let k, n be two positive integers with $k \leq n$

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The associated Gabidulin code $\mathrm{Gab}(k; a_1, \ldots, a_n)$

is the image of

$$K[x;\varphi]_{<k} \longrightarrow K^n$$

$$f(x) \mapsto$$

$$\left\{ c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} \right\} \qquad (c_i \in K)$$

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$$\boxed{x a = a^p x}$$

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dimension = k

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Theorem

length = n

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for the rank distance: $w_{rk}(U) =$

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$\left\{ c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} \right\}$

$x a = a^p x$ $\quad (c_i \in K)$

Theorem

length = n

dimension = k

minimal distance = $n - k + 1$

for the rank distance: $w_{rk}(u) = \mathrm{rank}\ u$

Reed Solomon codes and Gabidulin codes

Let k, n be two positive integers with $k \leq n$

Let $g_1, \ldots, g_n$ be $\mathbb{F}_p$-linearly indep. elements of $K = \mathbb{F}_q$

The associated Gabidulin code $\mathrm{Gab}(k; g_1, \ldots, g_n)$ is the image of

$$\begin{array}{ccc} K[x;\varphi]_{<k} & \longrightarrow & \mathrm{Hom}_{\mathbb{F}_p}(V, K) \\ f(x) & \mapsto & f(\varphi)_{|V} \end{array}$$

Theorem

length = n

dimension = k

minimal distance = $n - k + 1$

for the rank distance: $w_{rk}(u) = \mathrm{rank}\ u$

$$\left\{ c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} \right\}$$

$$\boxed{x a = a^p x} \qquad (c_i \in K)$$

Reed Solomon codes and Gabidulin codes

Let k, n be two positive integers with $k \leq n$

Let V be $\mathbb{F}_p$-linearly indep. elements of $K = \mathbb{F}_q$

The associated Gabidulin code $\mathrm{Gab}(k; a_1, \ldots, a_n)$ is the image of

$$\begin{array}{ccc} K[x;\varphi]_{<k} & \longrightarrow & \mathrm{Hom}_{\mathbb{F}_p}(V, K) \\ f(x) & \mapsto & f(\varphi)_{|V} \end{array}$$

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Reed Solomon codes and Gabidulin codes

Let k, n be two positive integers with $k \leq n$

Let V be [redacted] of $K = \mathbb{F}_q$

The associated Gabidulin code $\mathrm{Gab}(k; a_1, \ldots, a_n)$ is the image of

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$$\boxed{x\,a = a^p\,x}$$

Theorem

length = n

dimension = k

minimal distance = $n - k + 1$

for the rank distance: $w_{rk}(u) = \mathrm{rank}\ u$

Reed Solomon codes and Gabidulin codes

Let k, n be two positive integers with $k \leq n$

Let V be sub-$\mathbb{F}_p$-vector space of $K = \mathbb{F}_q$

The associated Gabidulin code $\mathrm{Gab}(k; a_1, \ldots, a_n)$ is the image of

$$\begin{array}{ccc} K[x;\varphi]_{<k} & \longrightarrow & \mathrm{Hom}_{\mathbb{F}_p}(V, K) \\ f(x) & \mapsto & f(\varphi)_{|V} \end{array}$$

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$$\boxed{x a = a^p x}$$

Theorem

length = n

dimension = k

minimal distance = $n - k + 1$

for the rank distance: $w_{rk}(u) = \mathrm{rank}\ u$

Reed Solomon codes and Gabidulin codes

Let k, n be two positive integers with $k \leq n$

Let V be sub-$\mathbb{F}_p$-vector space of $K = \mathbb{F}_q$

The associated Gabidulin code $\mathrm{Gab}(k; \quad)$

is the image of

$$\begin{array}{ccc} K[x;\varphi]_{<k} & \longrightarrow & \mathrm{Hom}_{\mathbb{F}_p}(V, K) \\ f(x) & \mapsto & f(\varphi)_{|V} \end{array}$$

$$\left\{ c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} \right\} \qquad (c_i \in K)$$

$$\boxed{x\,a = a^p\,x}$$

Theorem

length = n

dimension = k

minimal distance = $n - k + 1$

for the rank distance: $w_{rk}(u) = \mathrm{rank}\ u$

Reed Solomon codes and Gabidulin codes

Let k, n be two positive integers with $k \leq n$

Let V be sub-$\mathbb{F}_p$-vector space of $K = \mathbb{F}_q$

The associated Gabidulin code $\mathrm{Gab}(k; V)$ is the image of

$$\begin{array}{ccc} K[x;\varphi]_{<k} & \longrightarrow & \mathrm{Hom}_{\mathbb{F}_p}(V, K) \\ f(x) & \mapsto & f(\varphi)_{|V} \end{array}$$

$$\left\{ c_0 + c_1 x + \cdots + c_{k-1} x^{k-1} \right\} \qquad (c_i \in K)$$

$$\boxed{x\,a = a^p\,x}$$

Theorem

length = n

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Why do we care about the rank distance?

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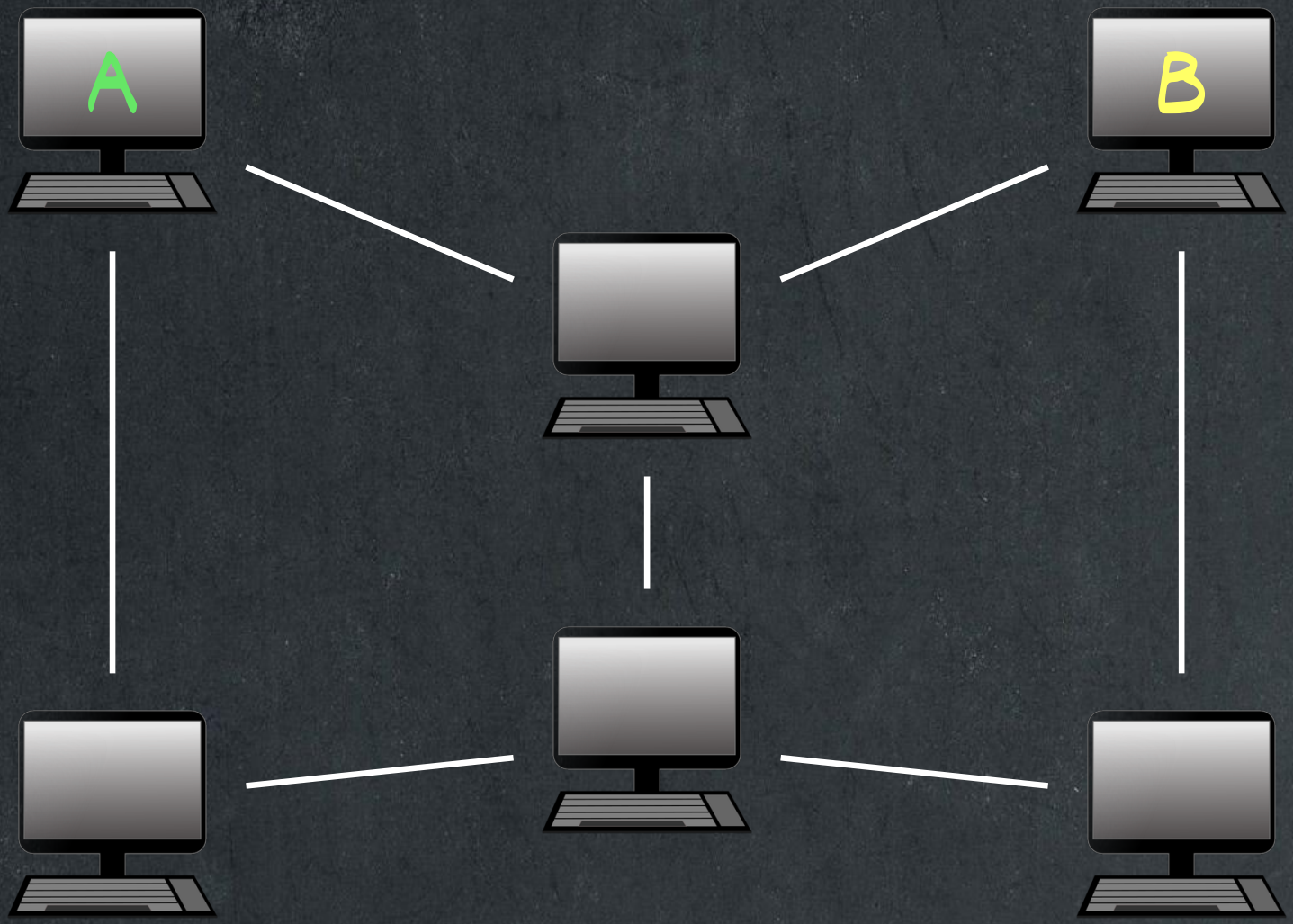

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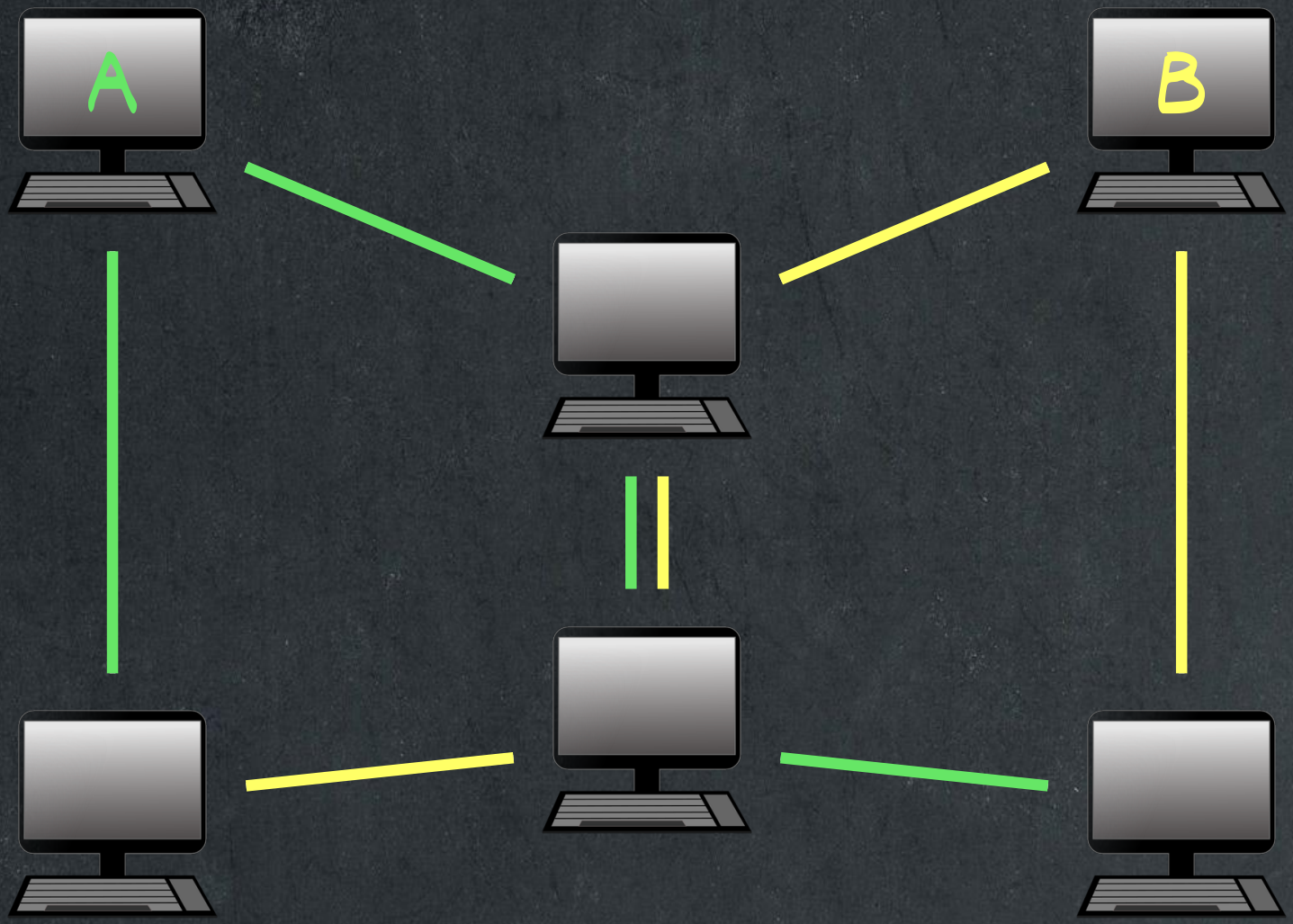

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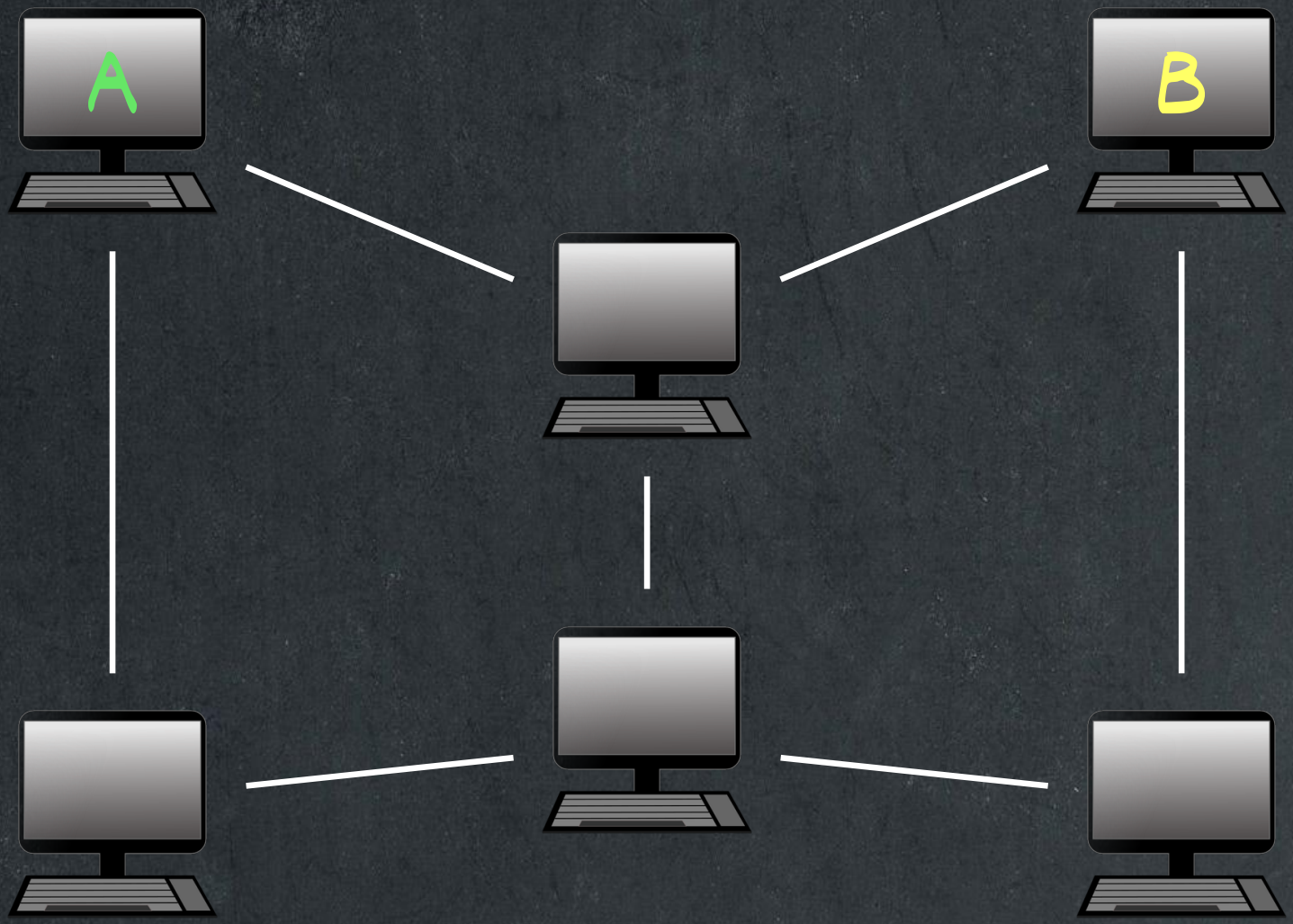

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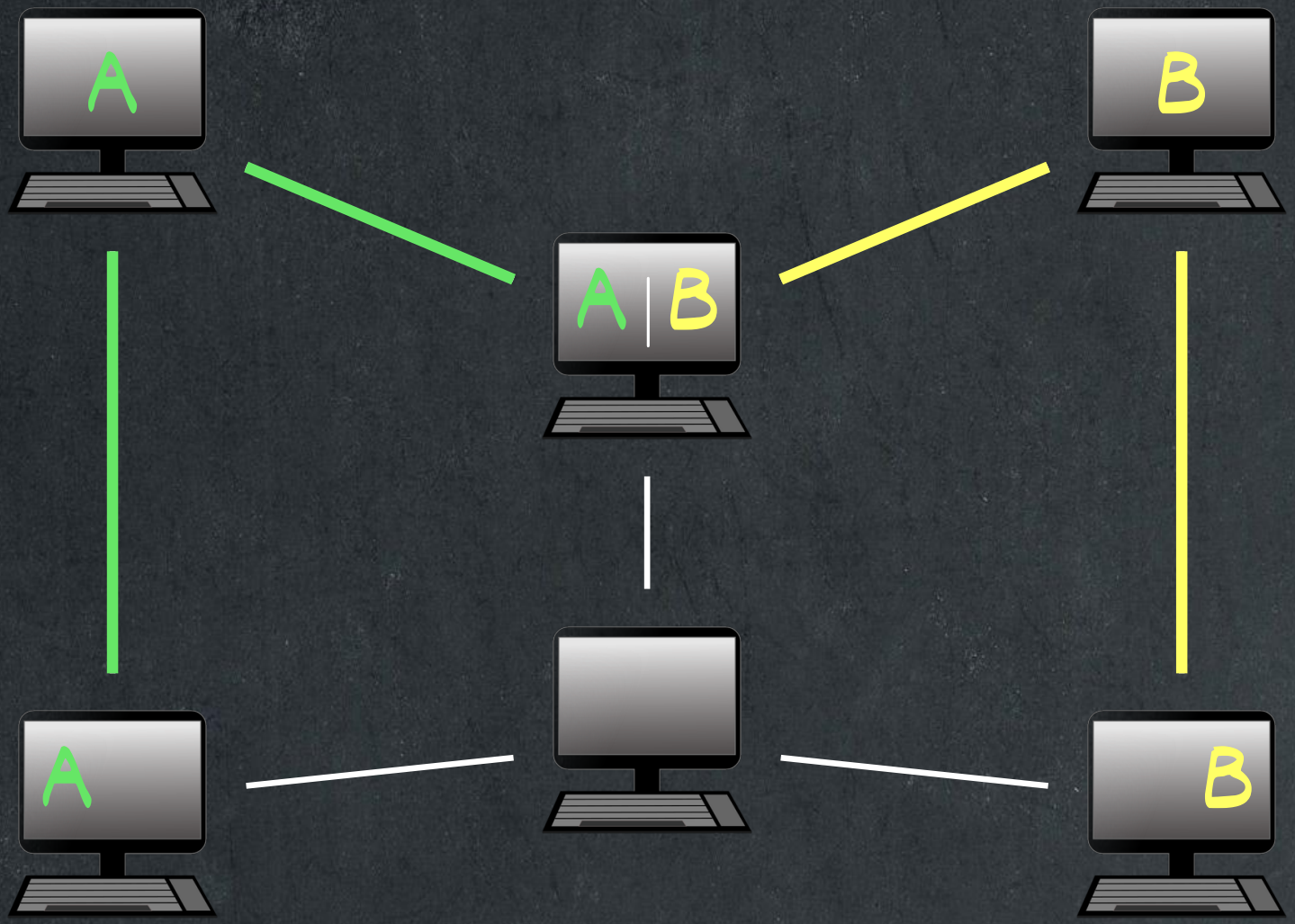

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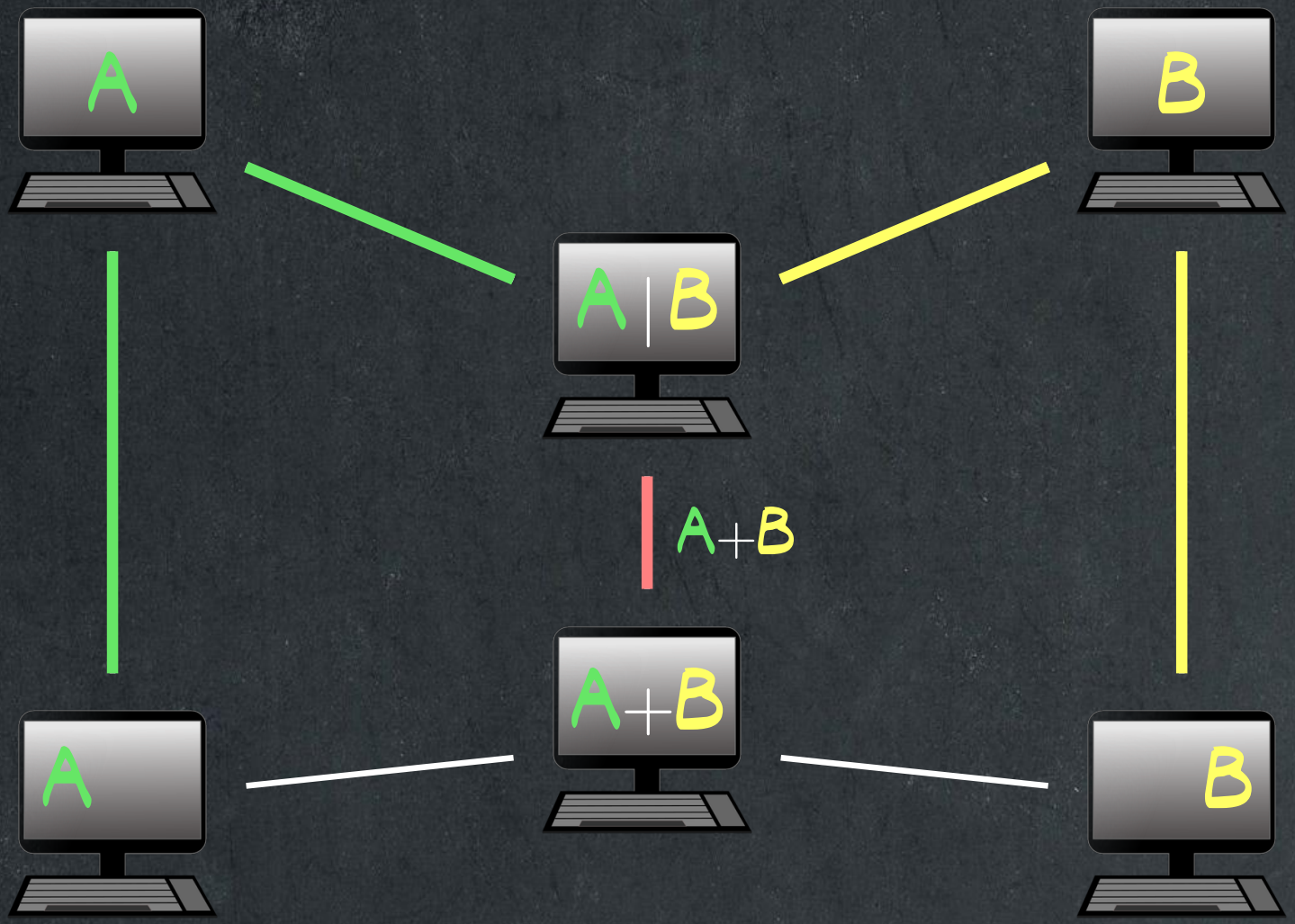

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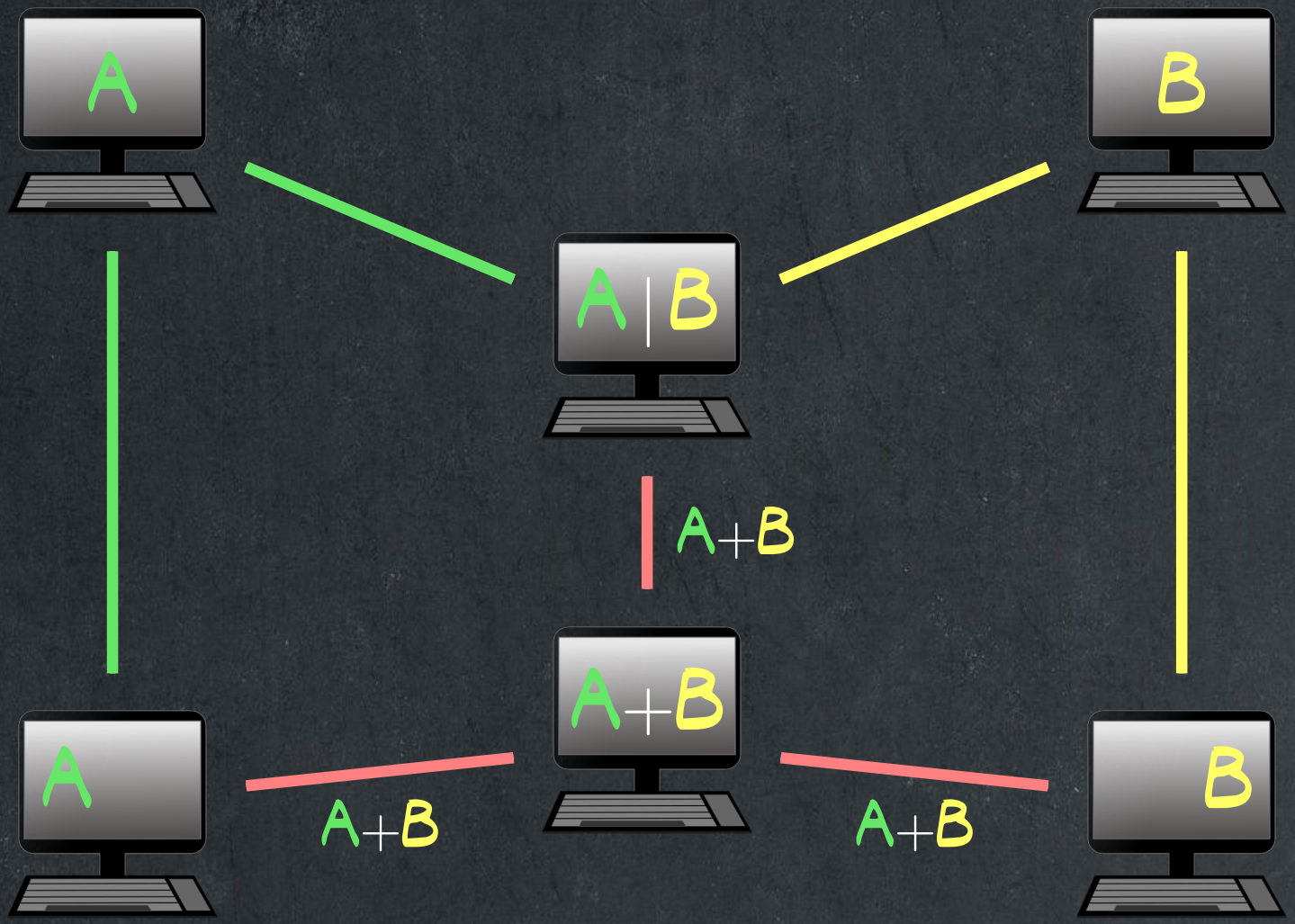

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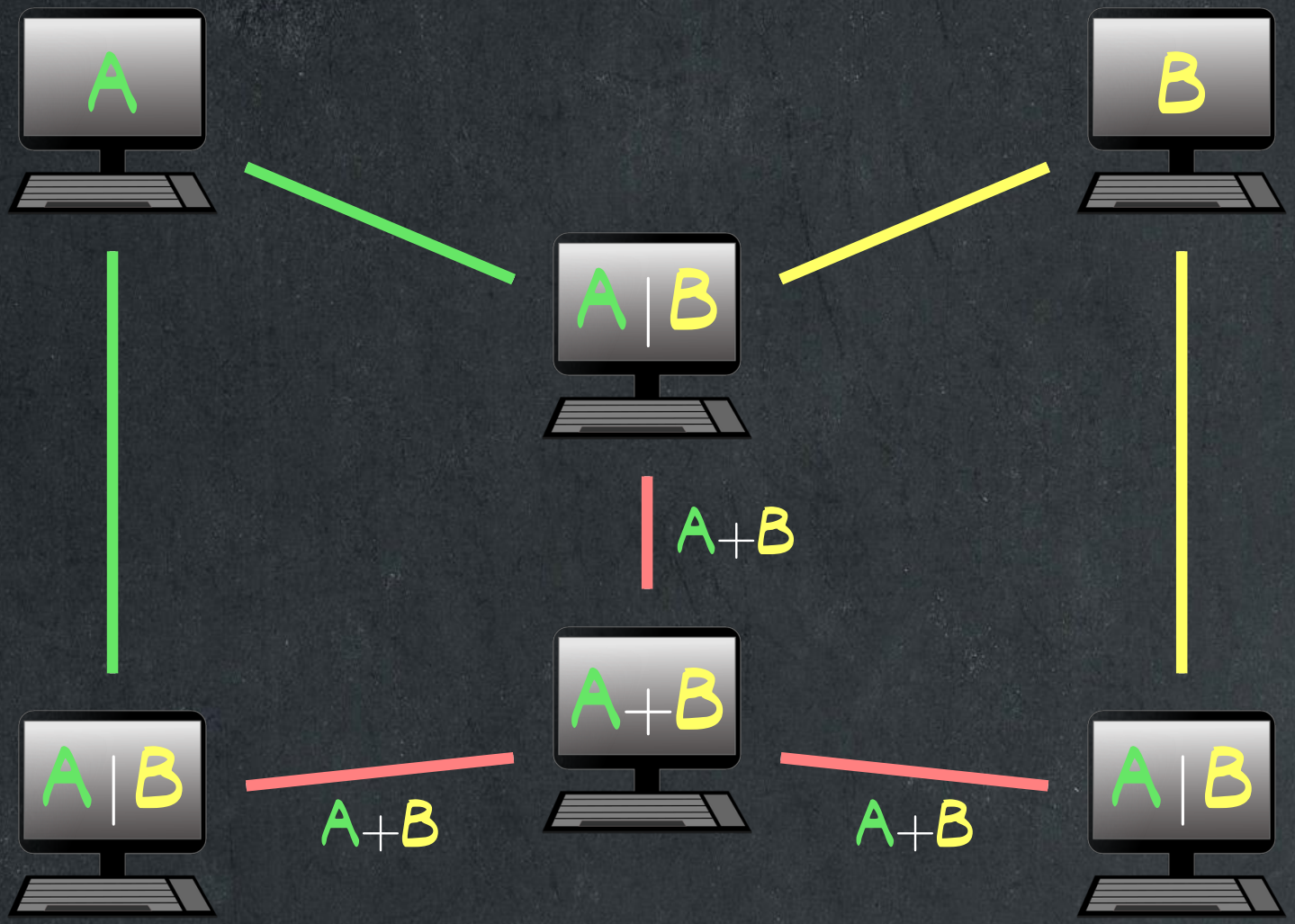

More evaluation morphisms

More evaluation morphisms

From now on, let K be a field equipped with $\varphi : K \to K$

More evaluation morphisms

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The semi-linear mappings $\alpha : K \to K$ are the $(a\varphi)$'s for $a \in K$

Linearized RS codes (after Martinez-Peñas)

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$$\begin{aligned} K[x;\varphi]_{<k} &\longrightarrow \mathrm{Hom}_F(V_1, K) \times \cdots \times \mathrm{Hom}_F(V_m, K) \\ f(x) &\mapsto \left(f(a_1\varphi)_{|V_1}, \ldots, f(a_m\varphi)_{|V_m}\right) \end{aligned}$$

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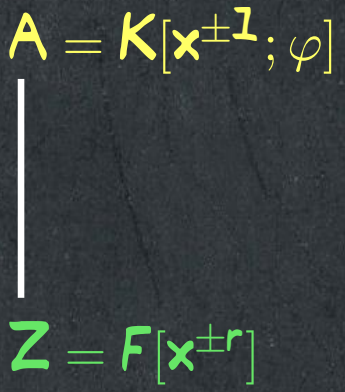

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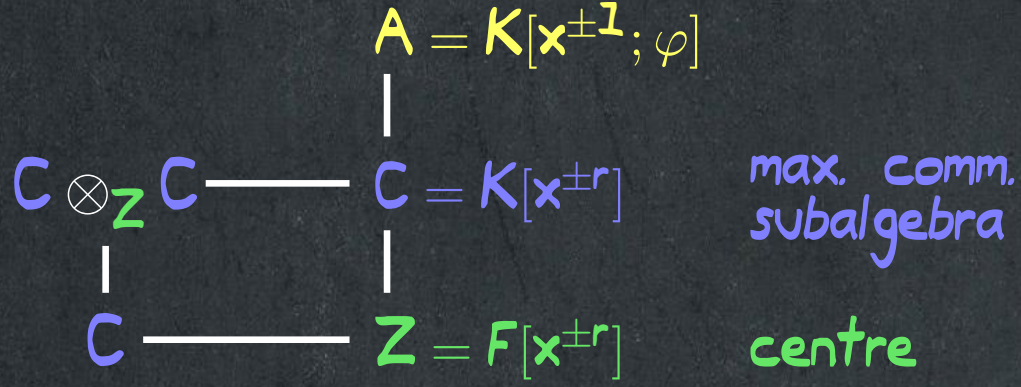

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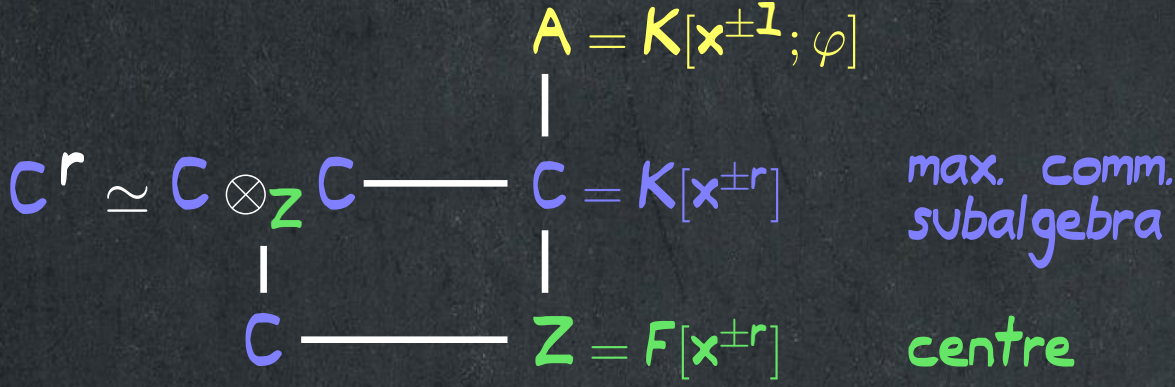

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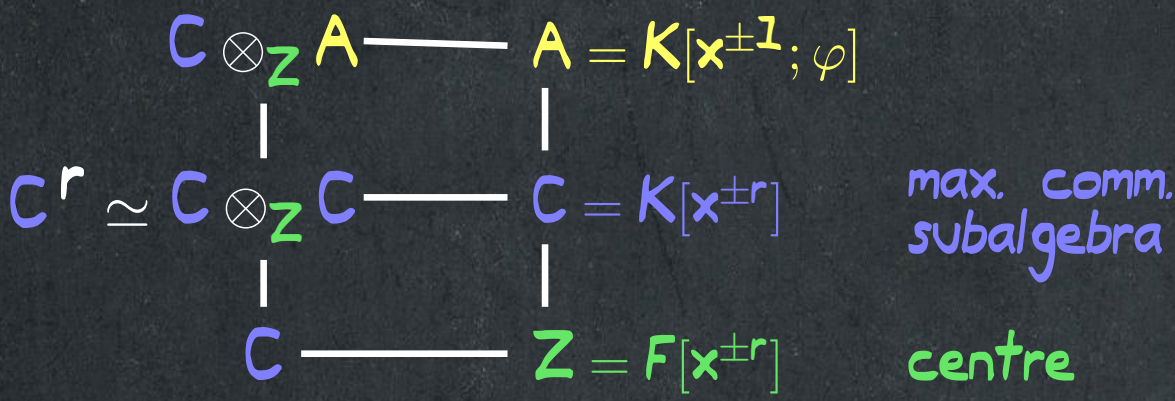

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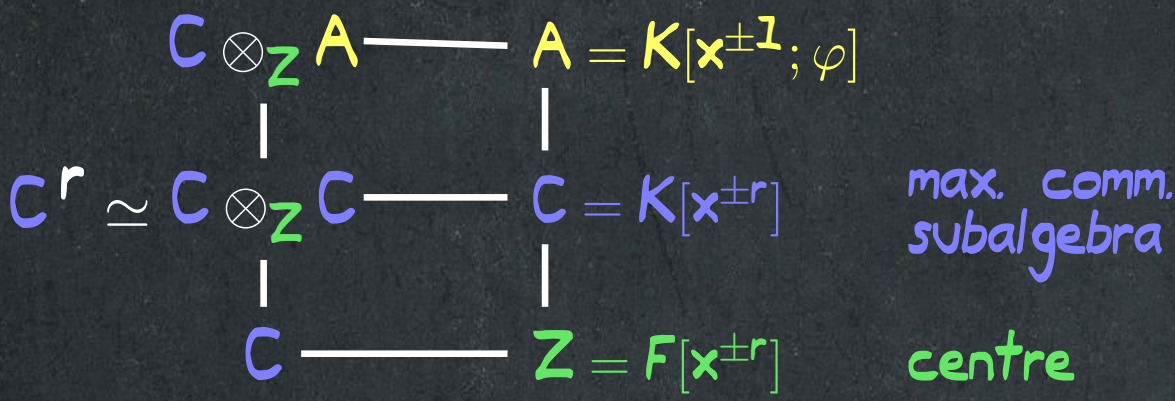

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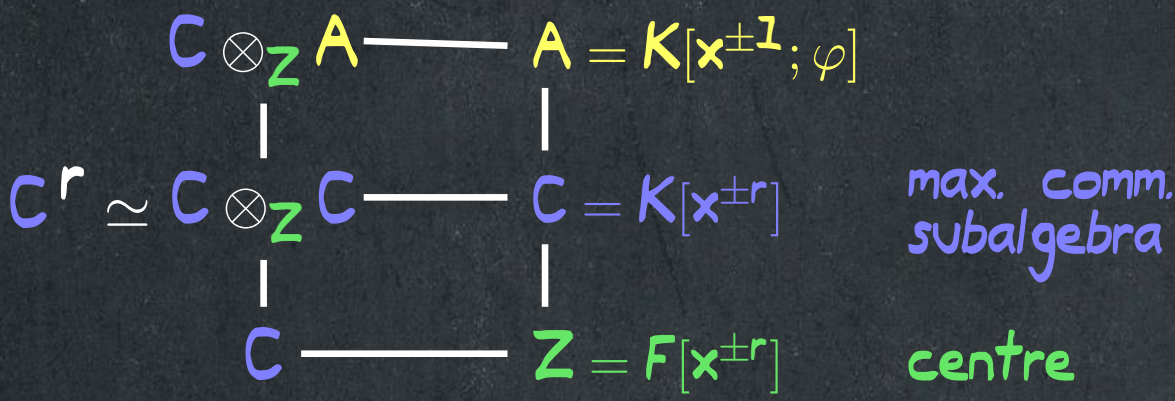

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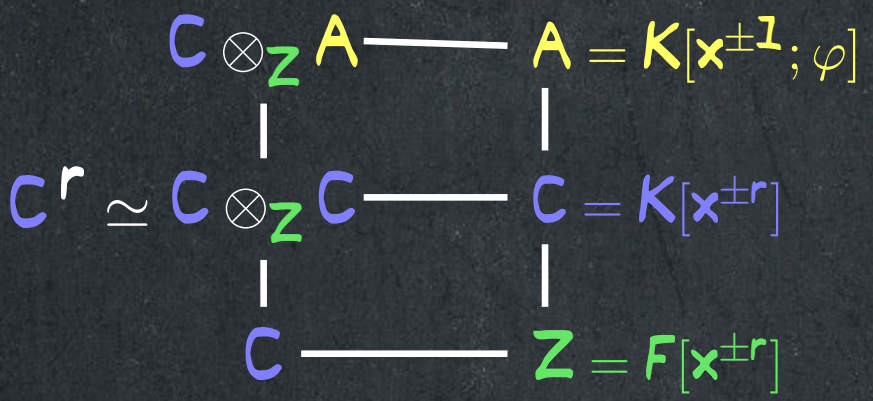

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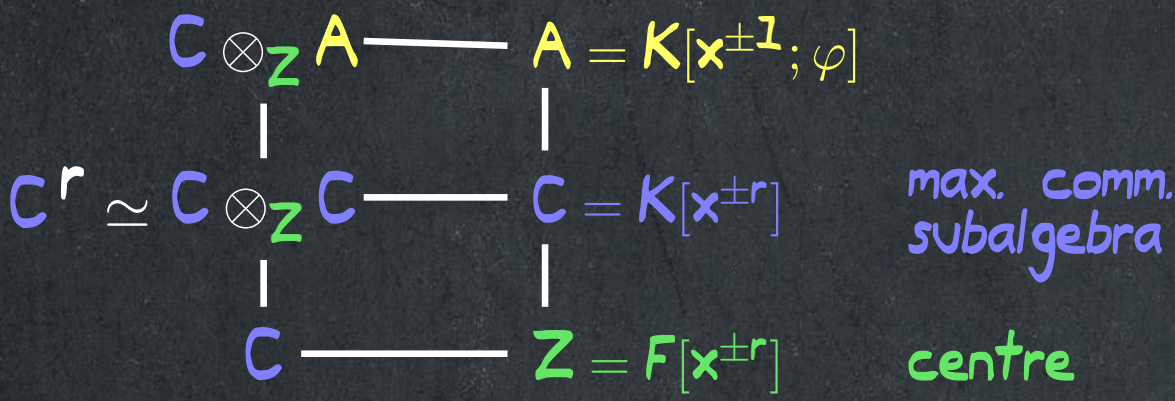

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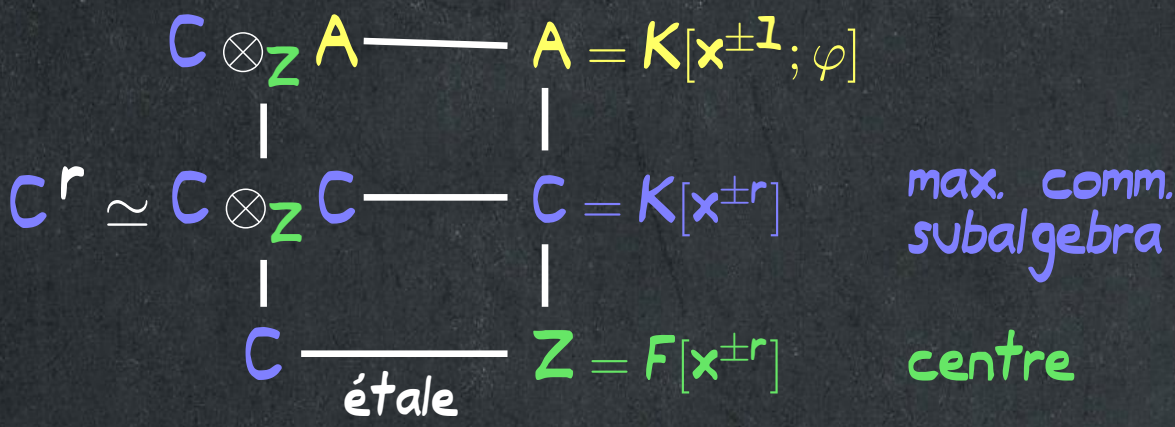

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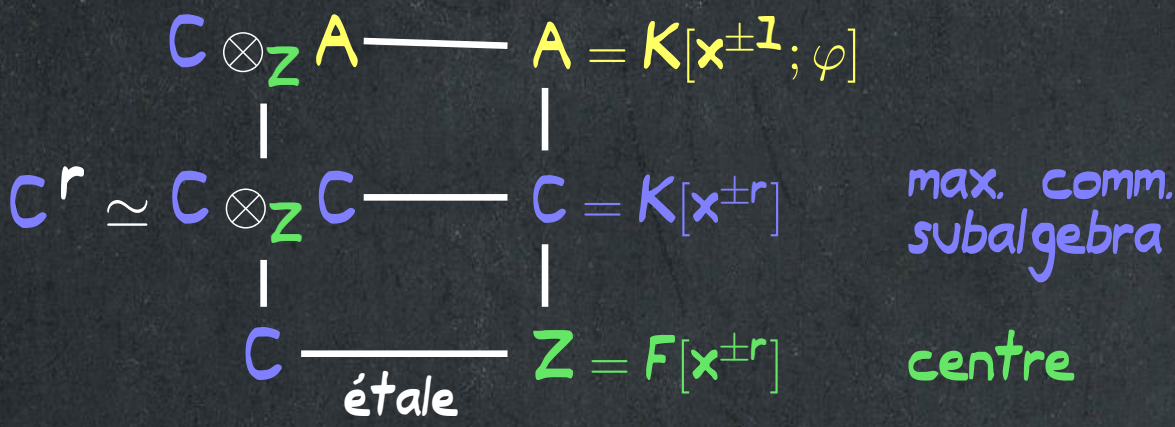

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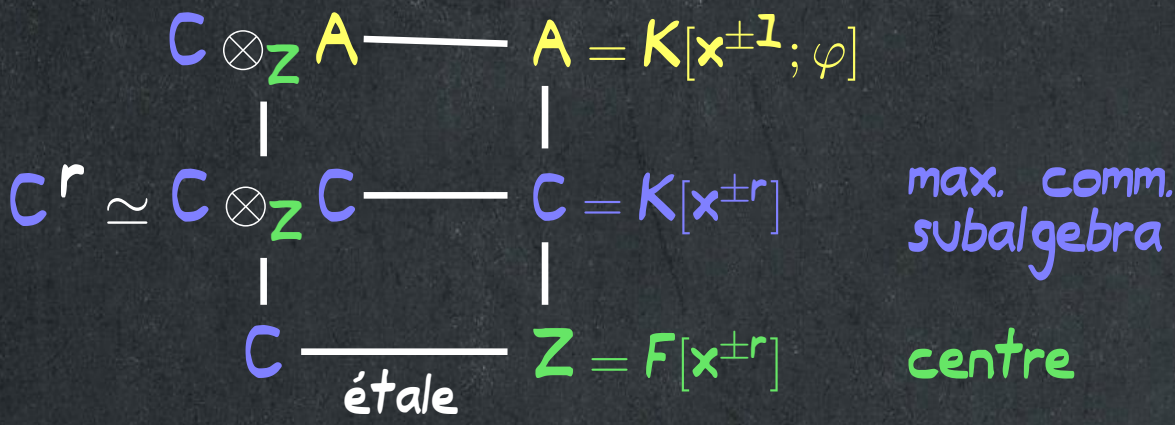

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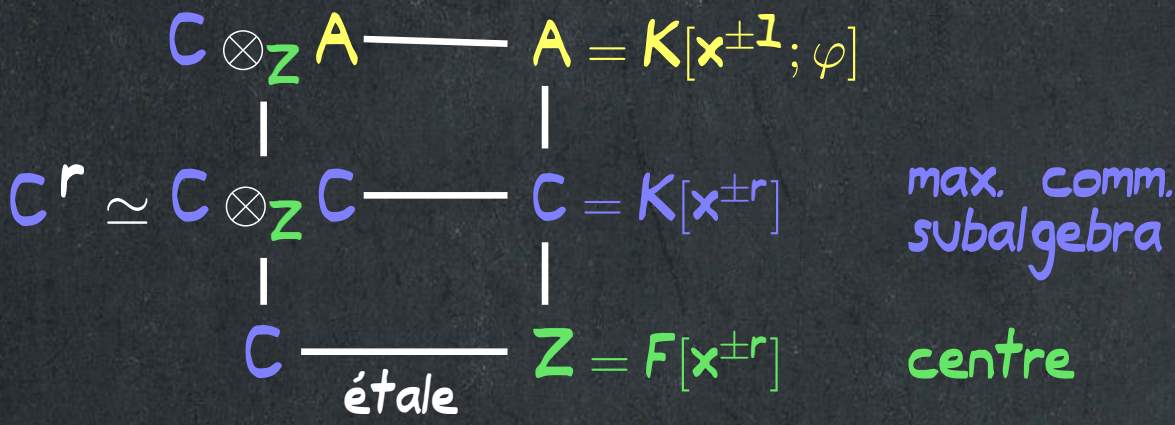

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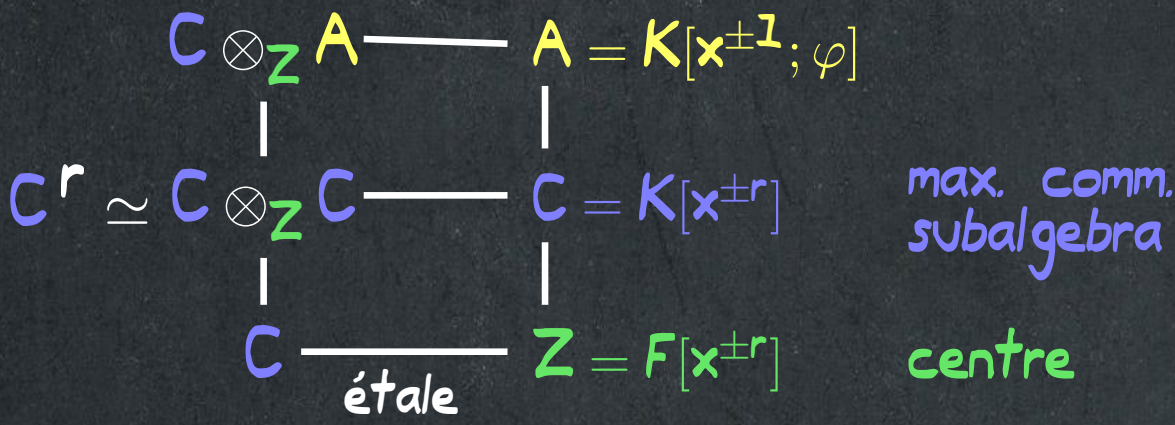

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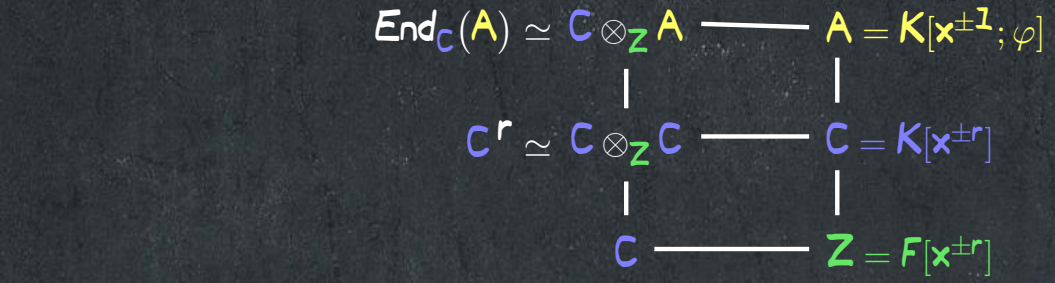

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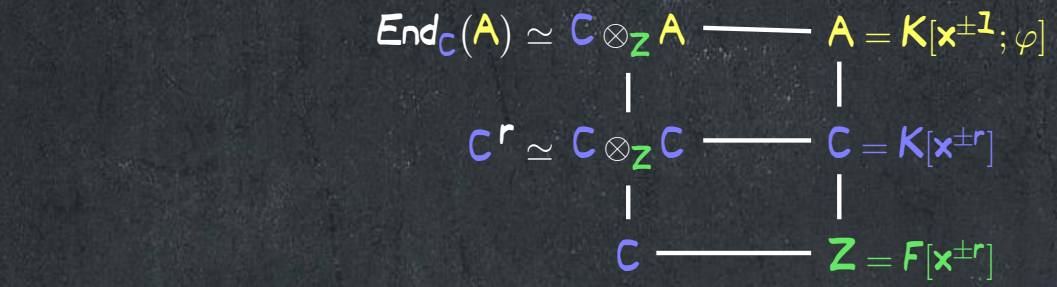

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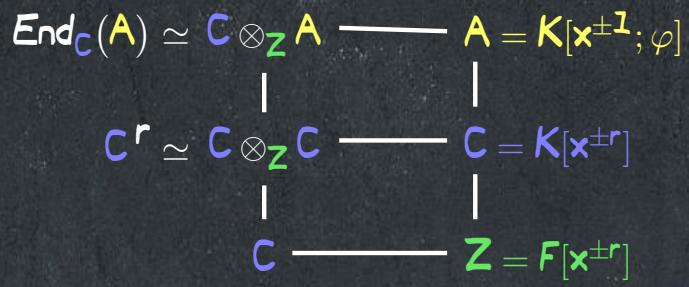

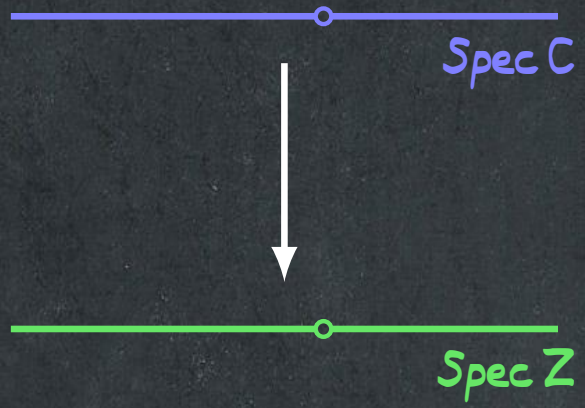

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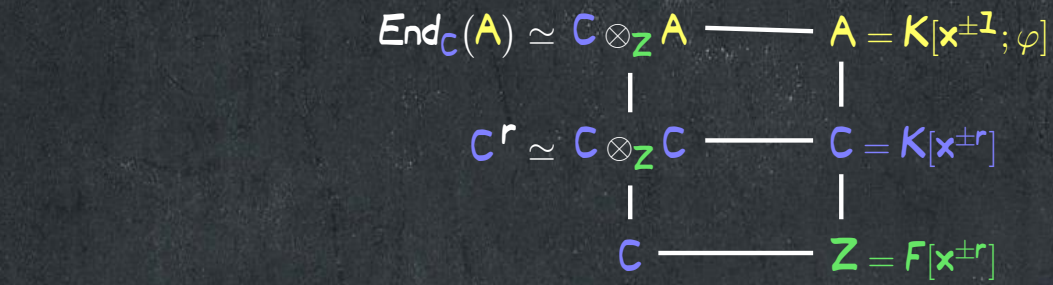

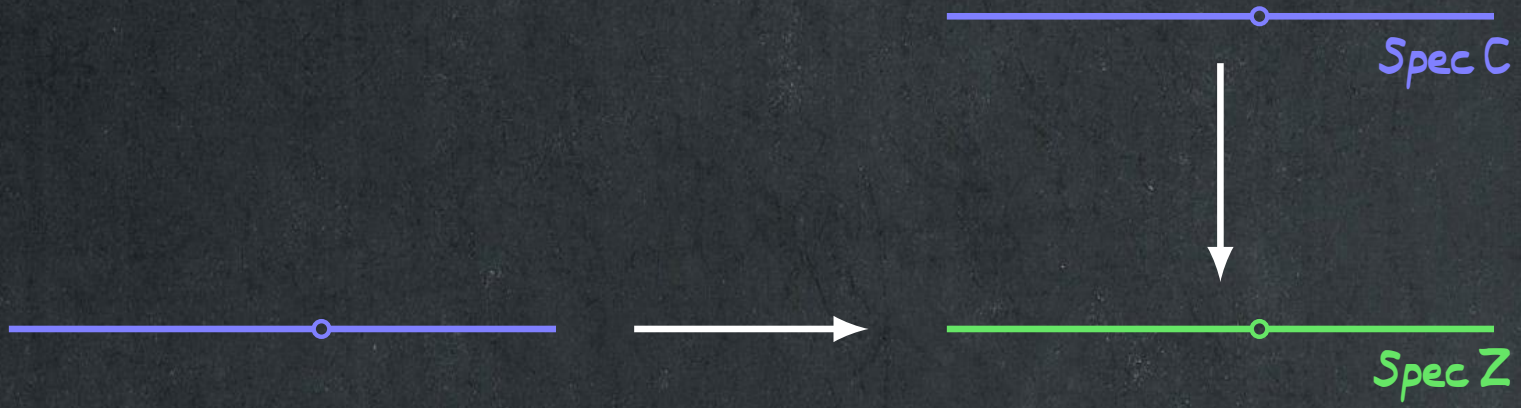

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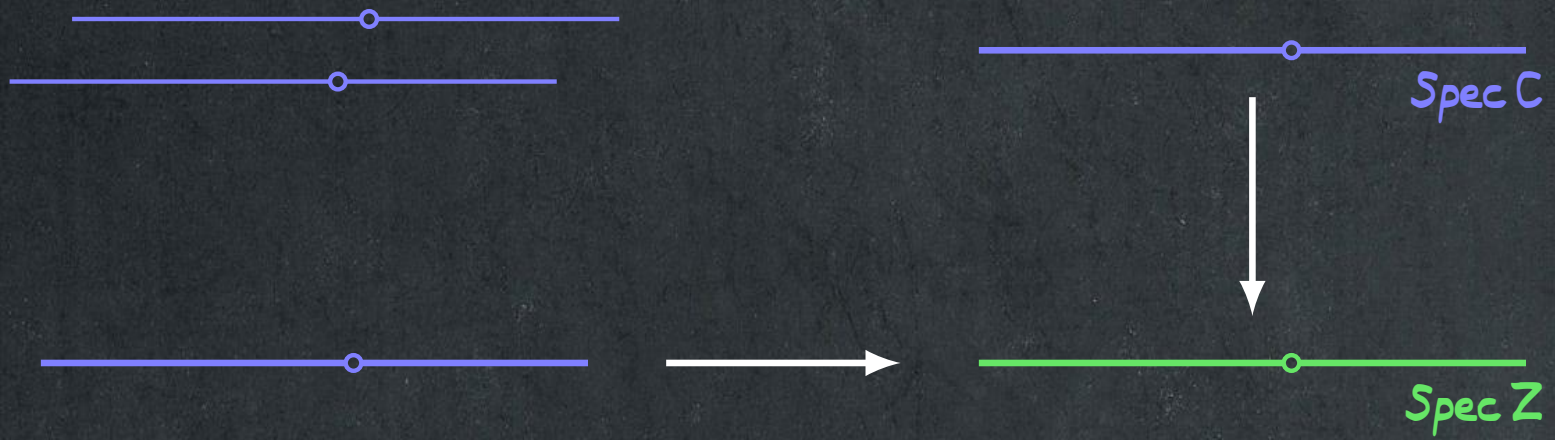

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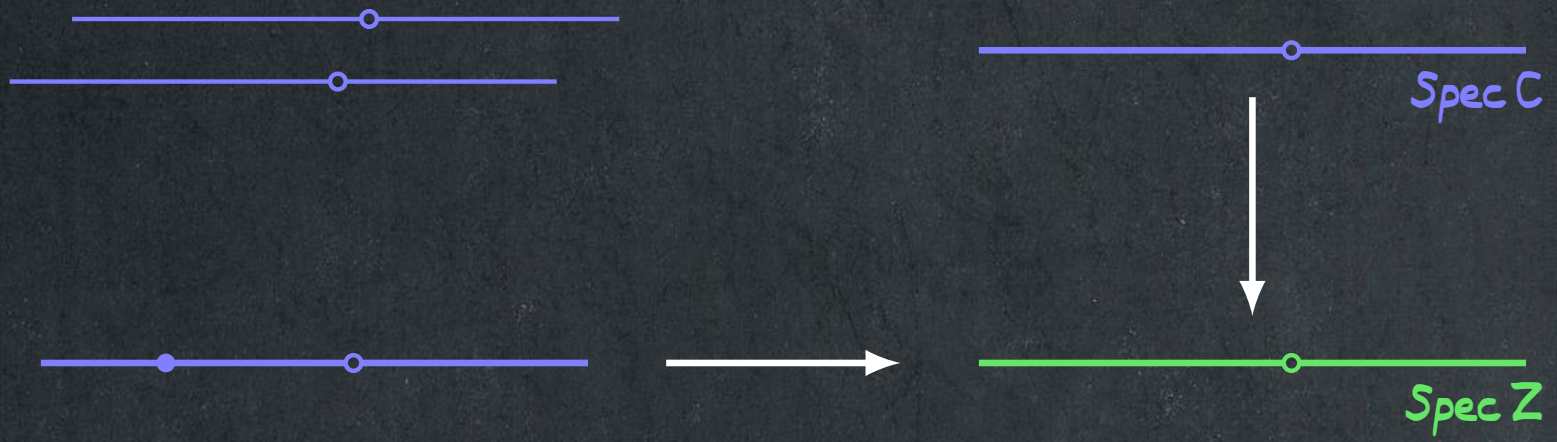

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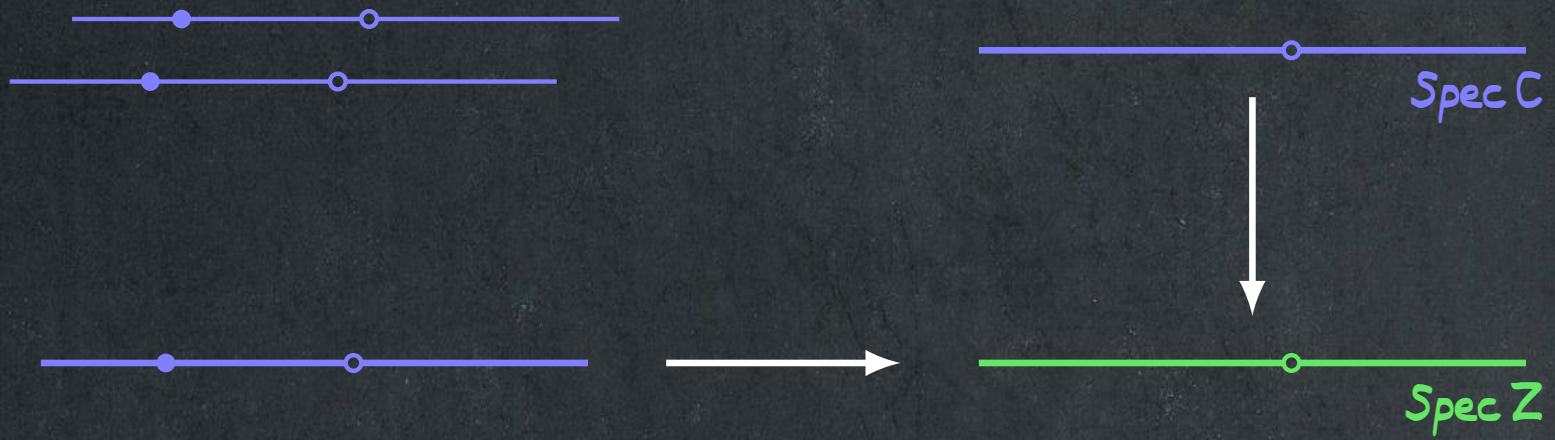

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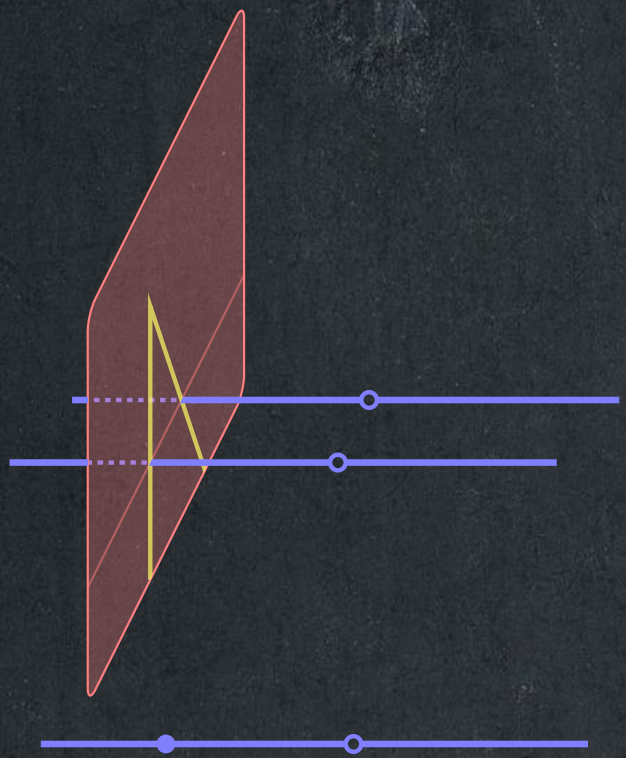

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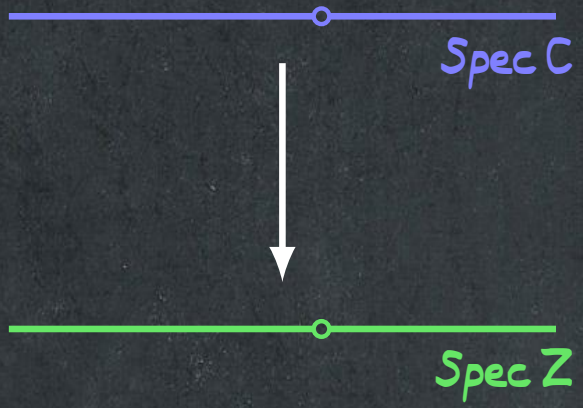

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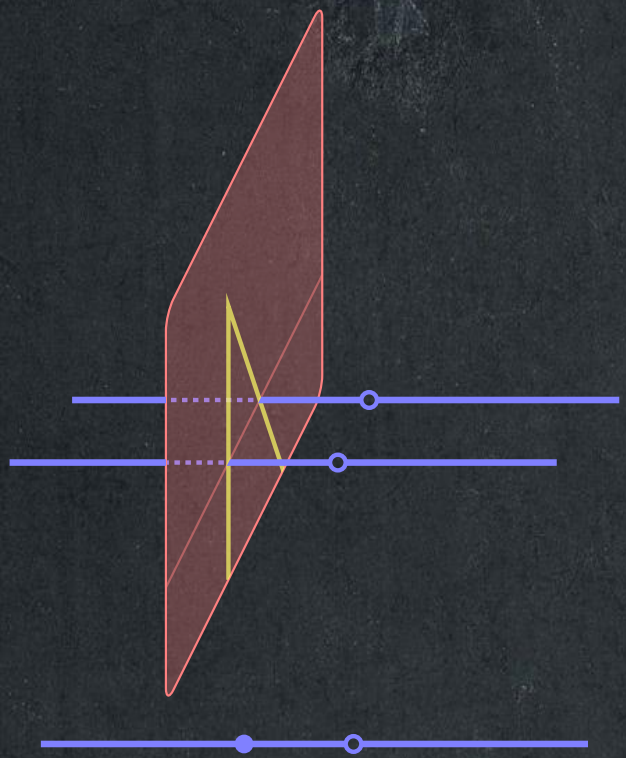

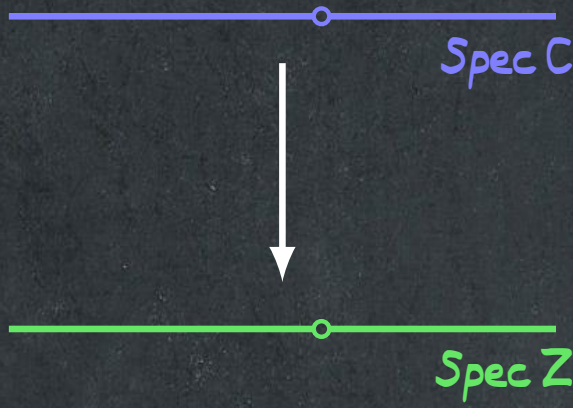

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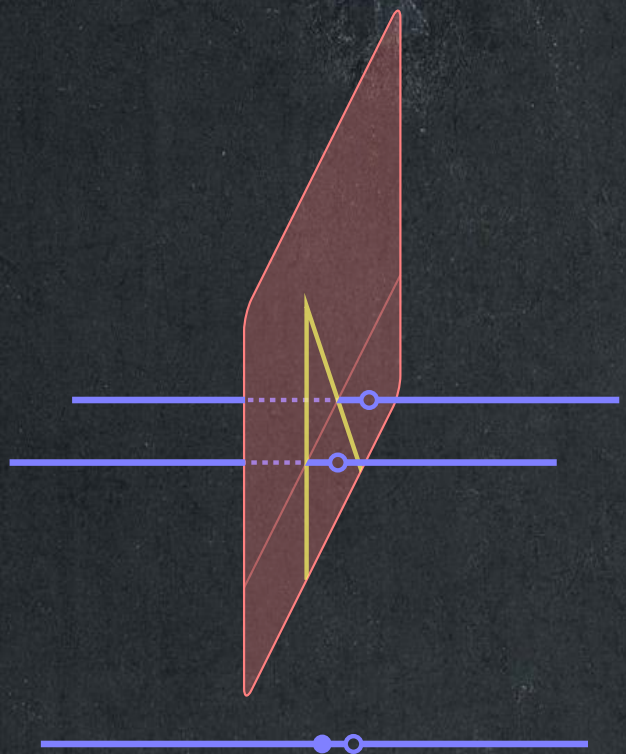

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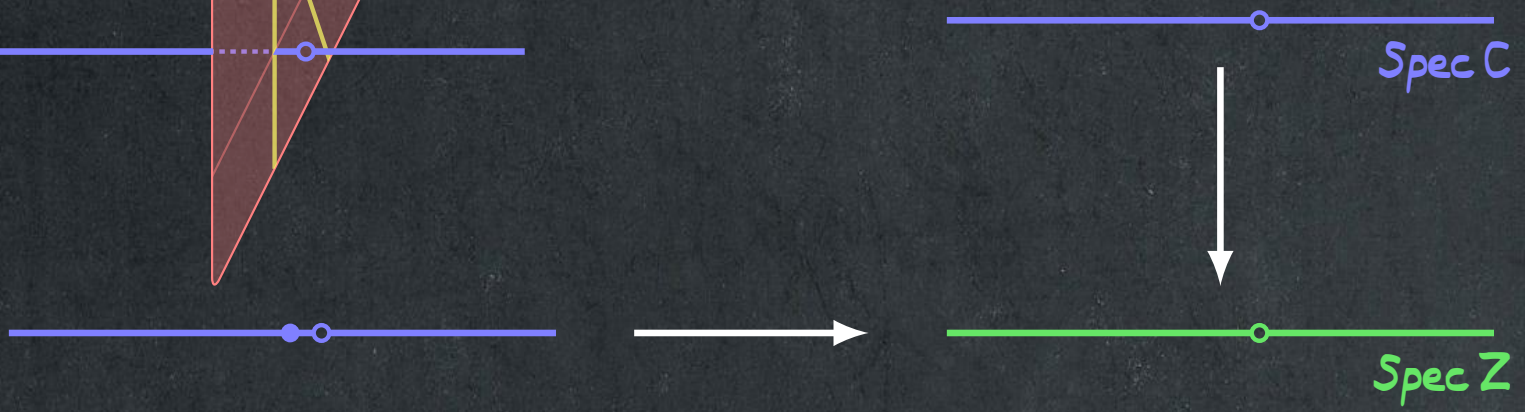

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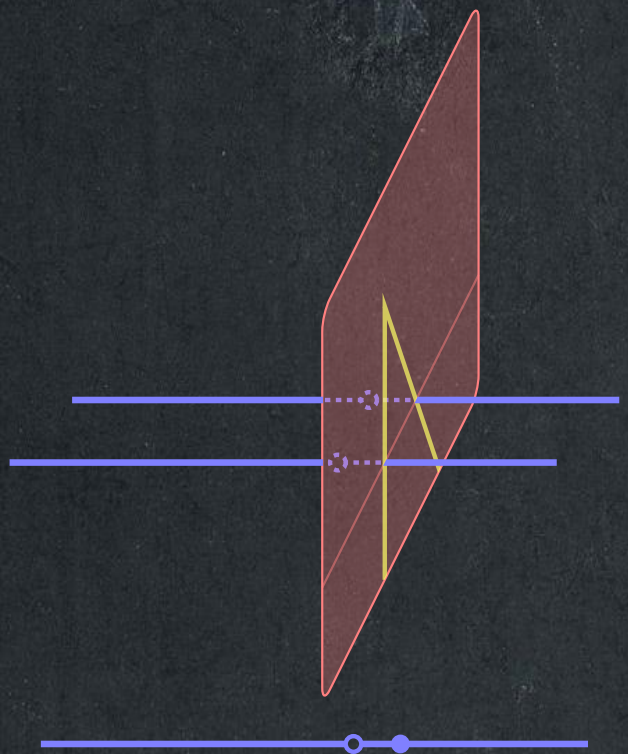

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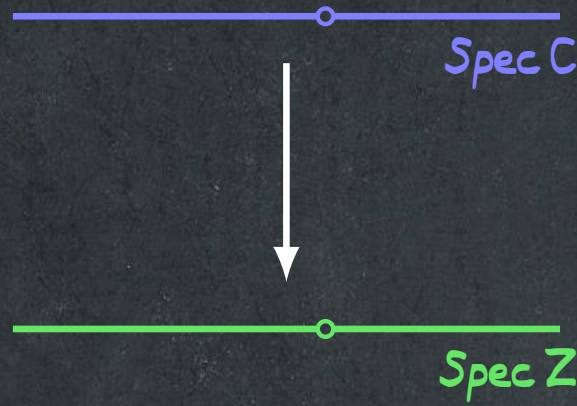

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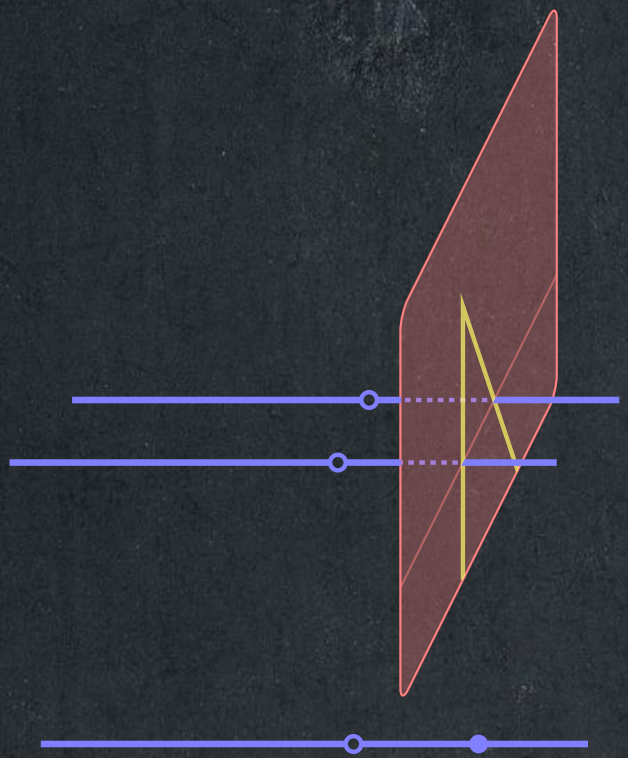

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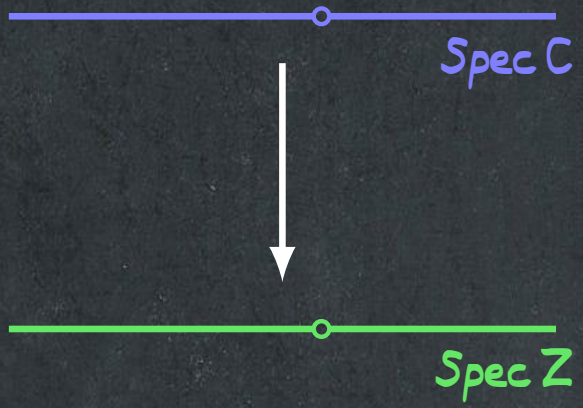

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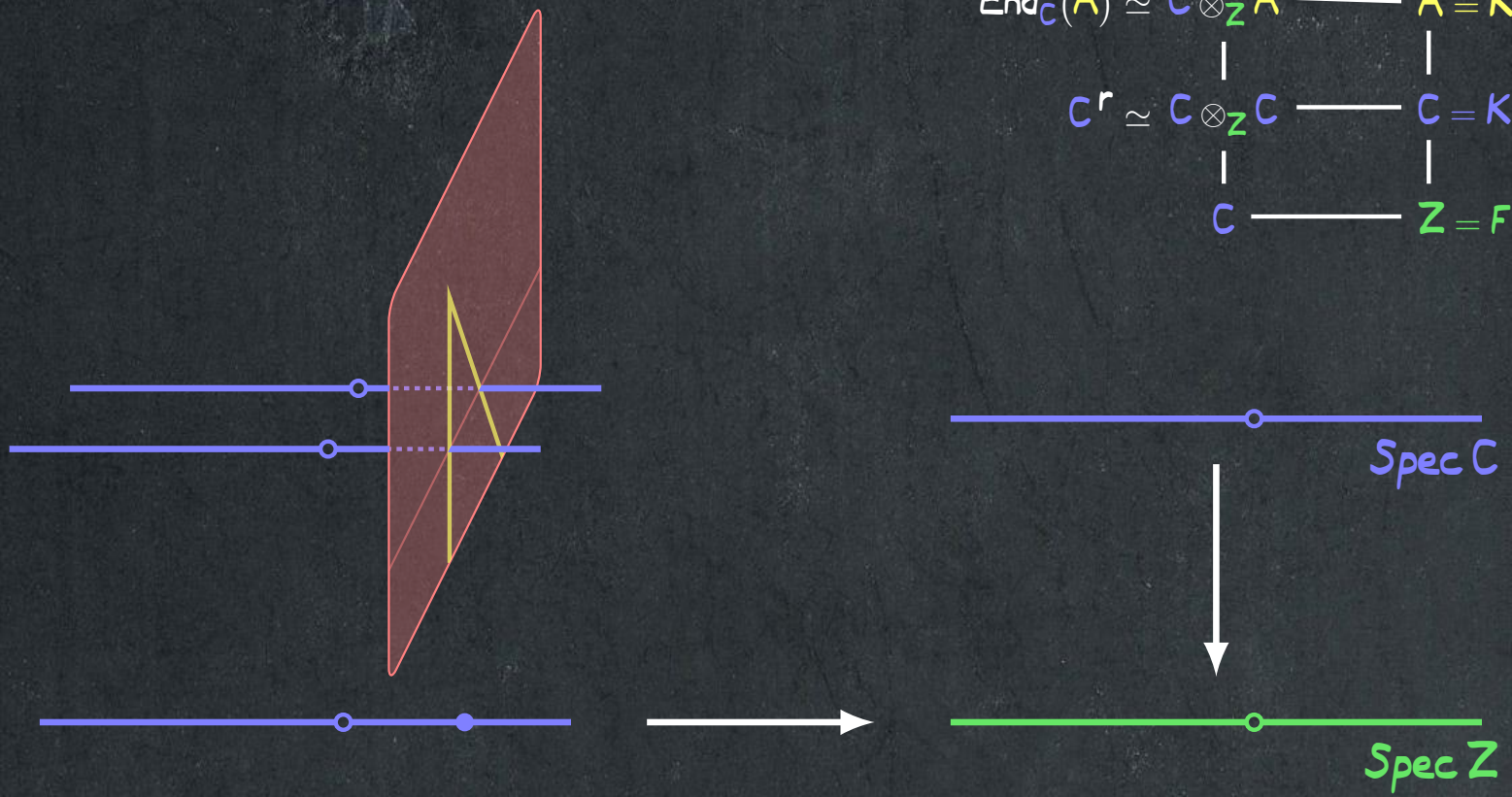

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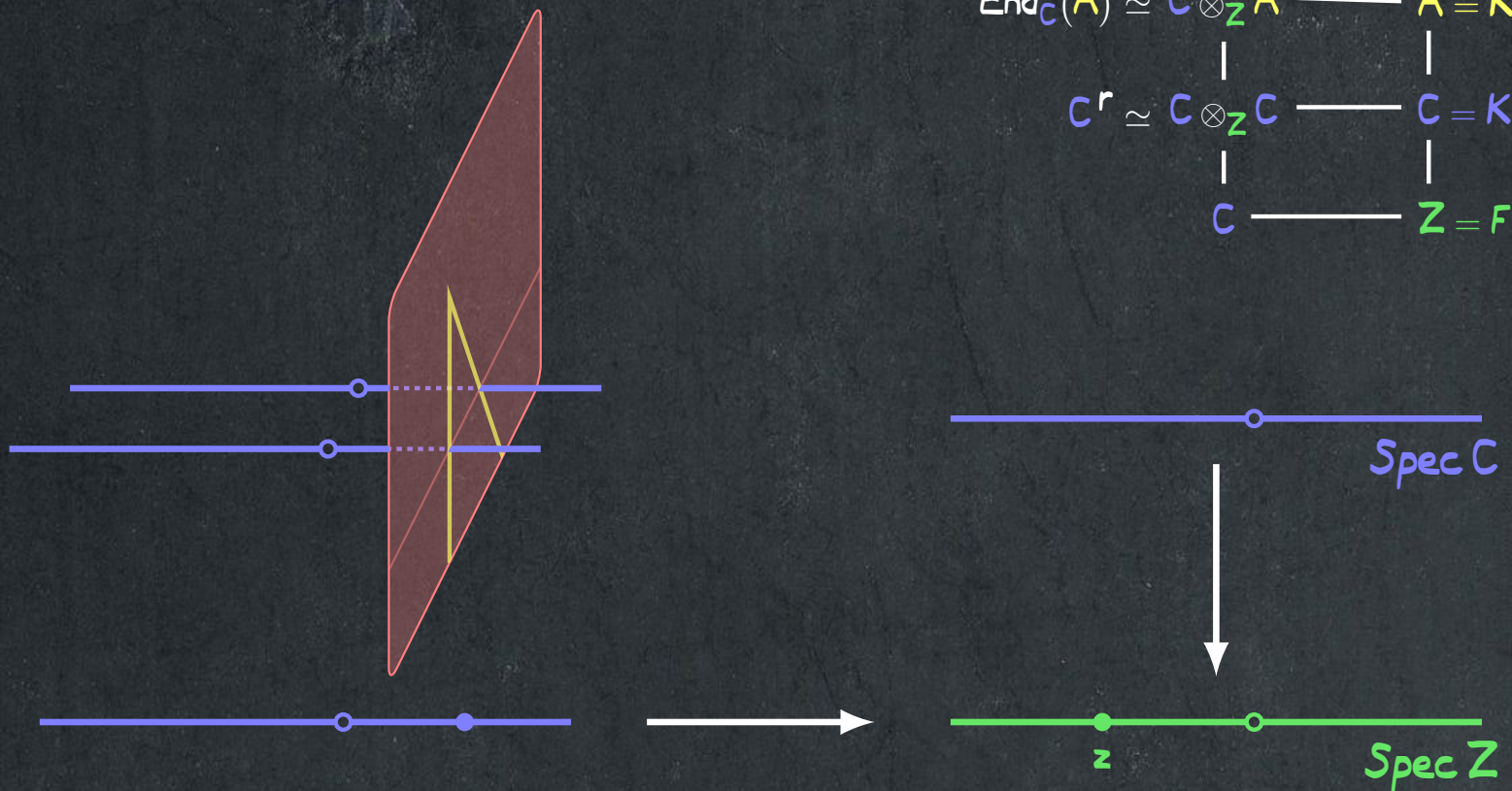

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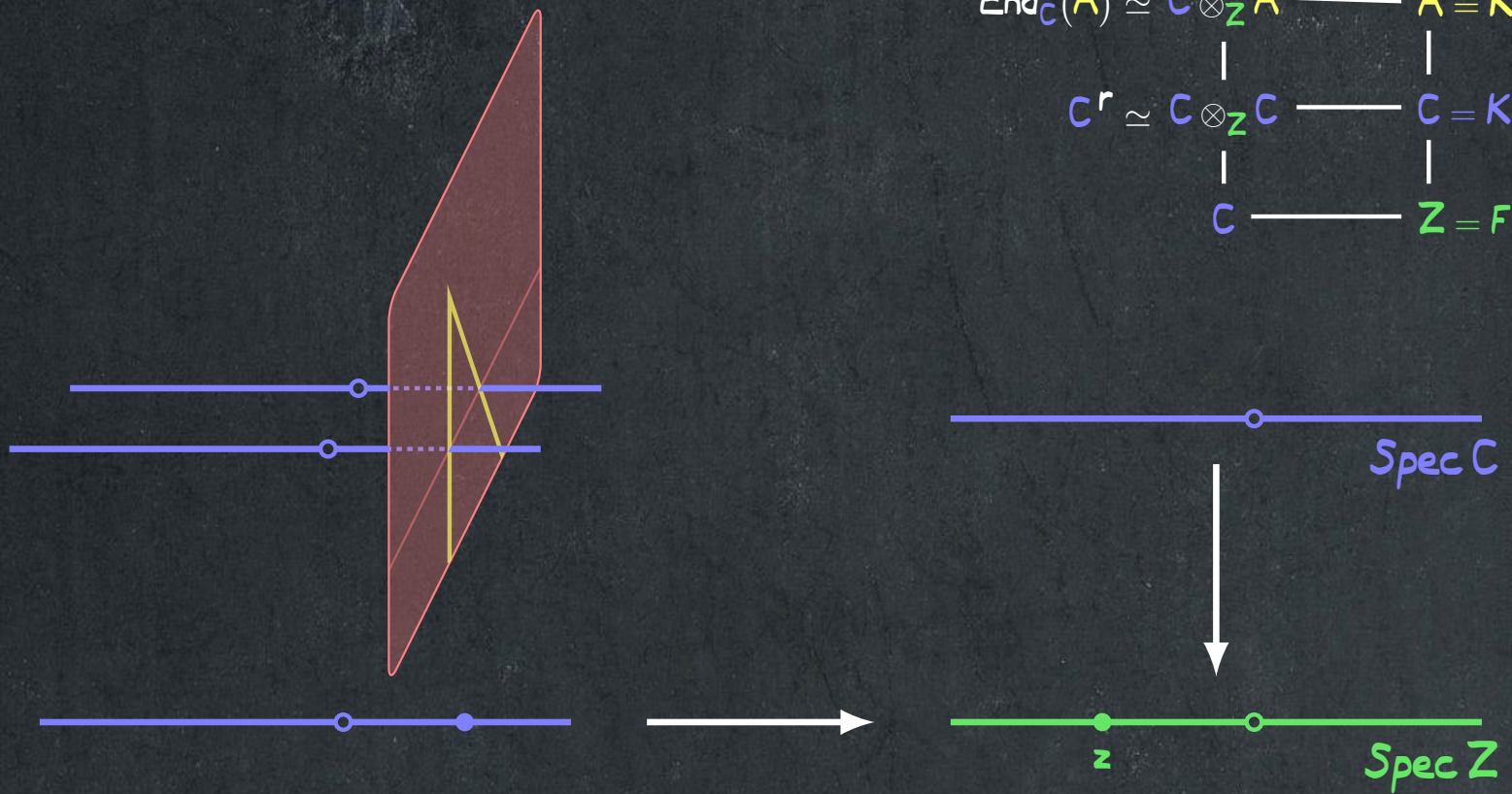

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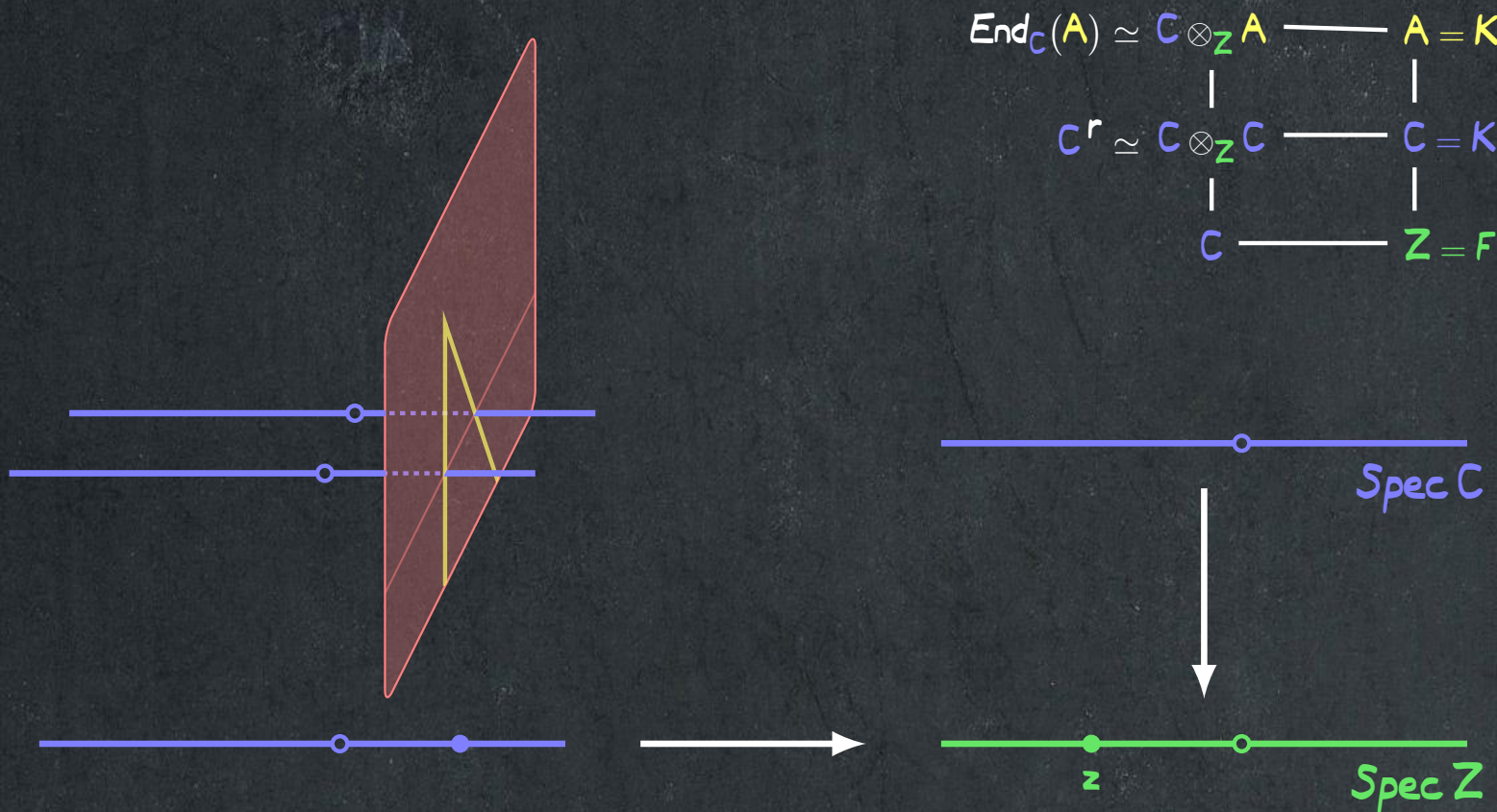

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$A_z = A/(x^r - z)$ simple central algebra over F

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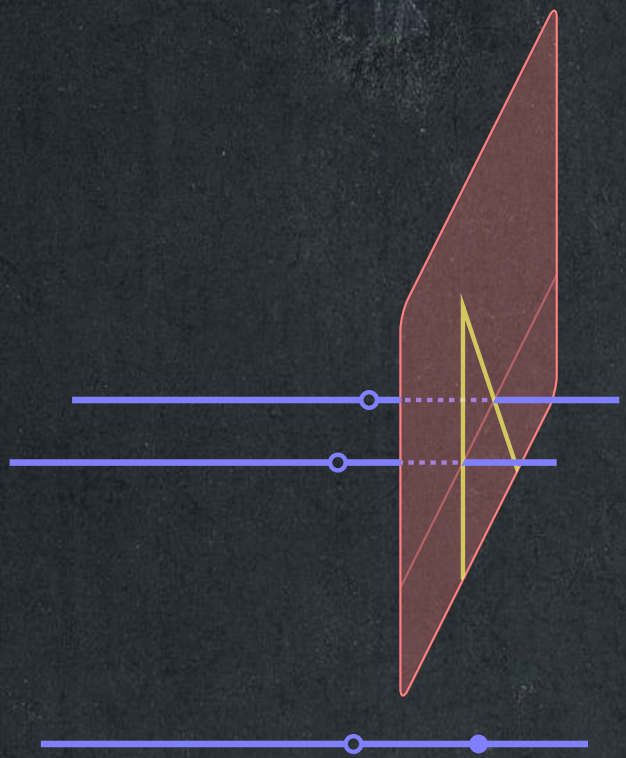

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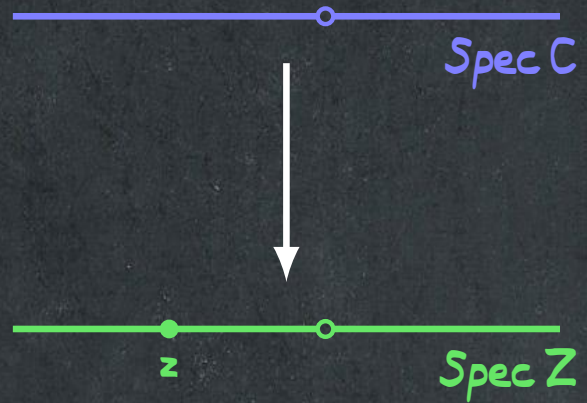

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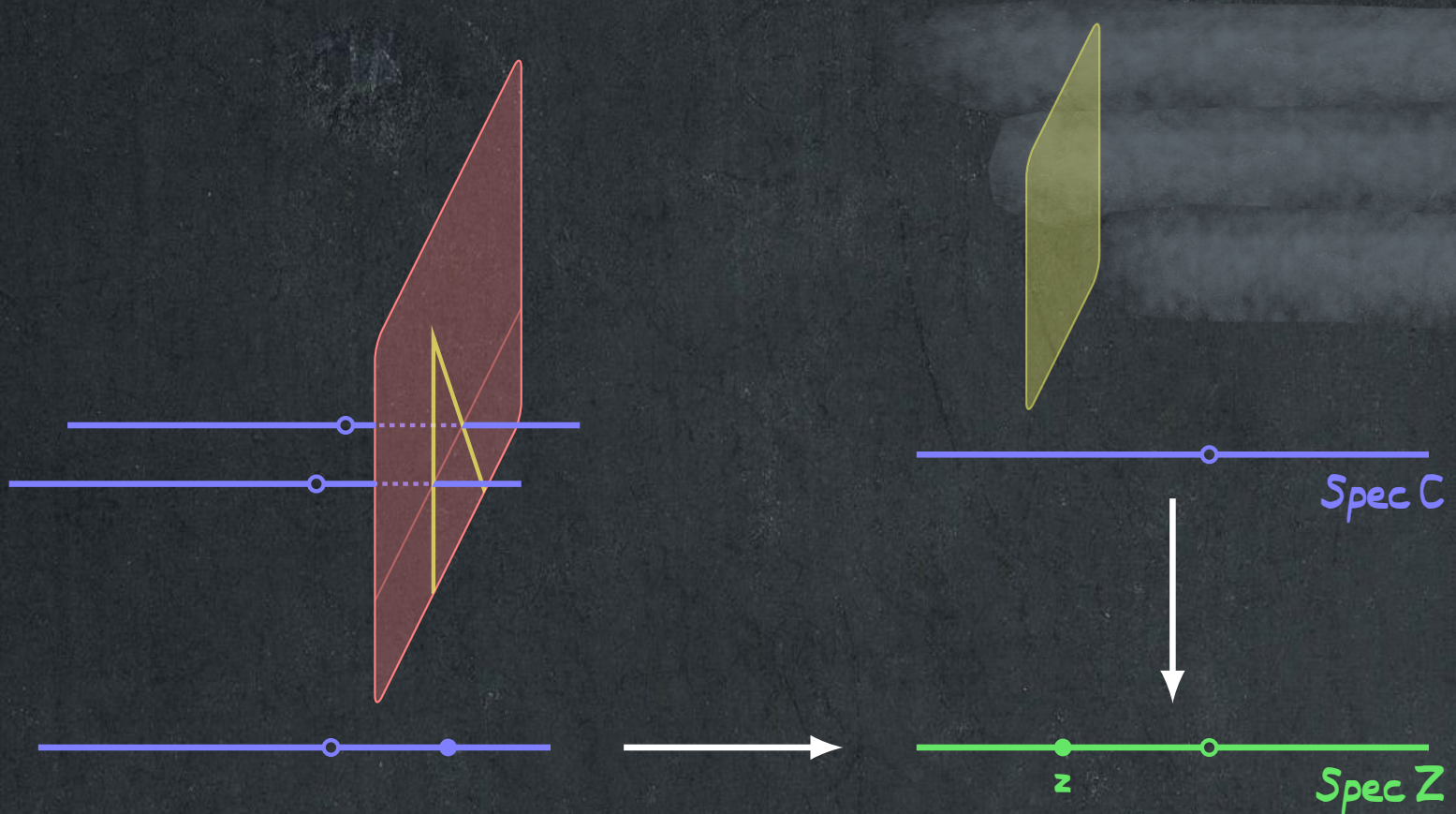

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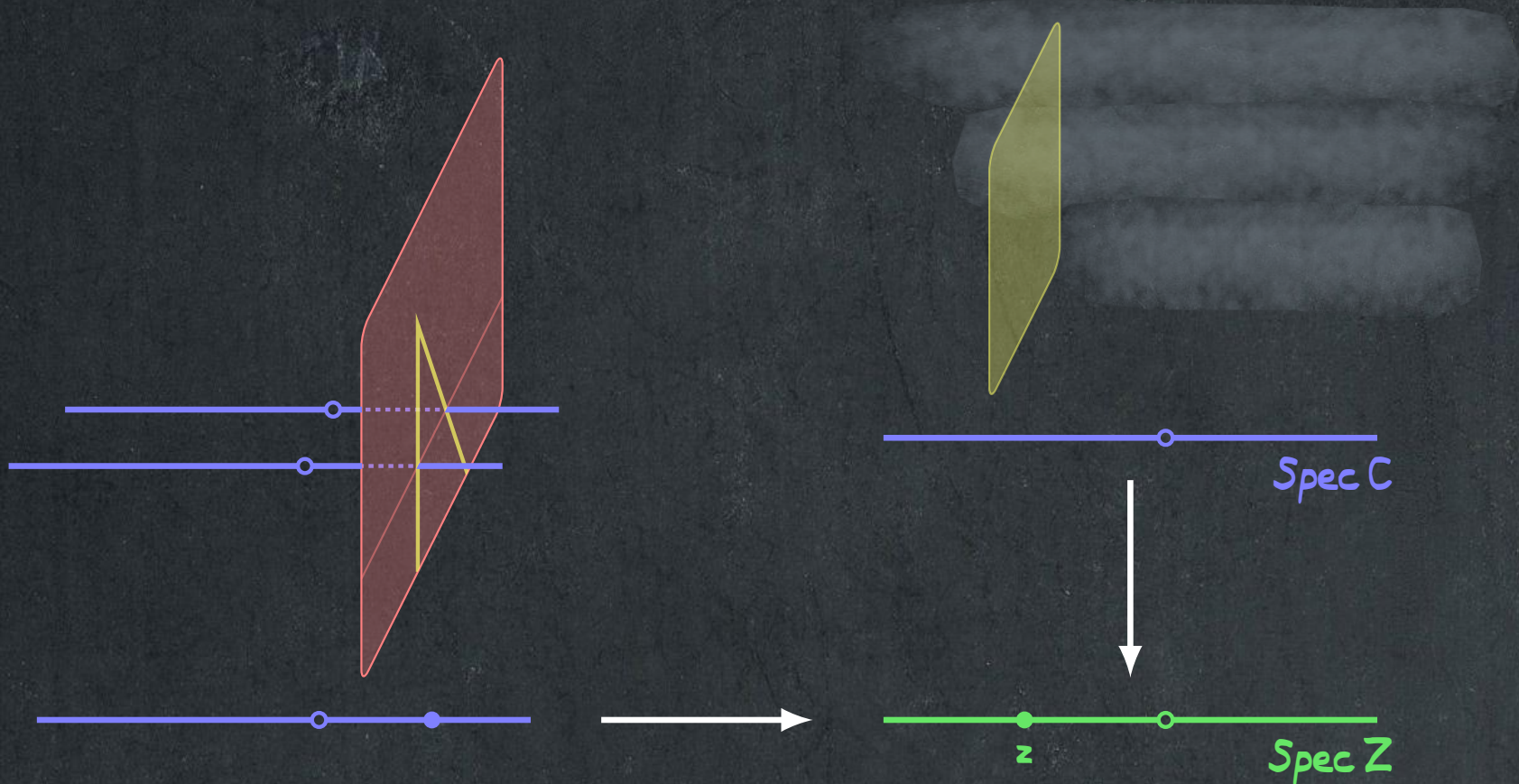

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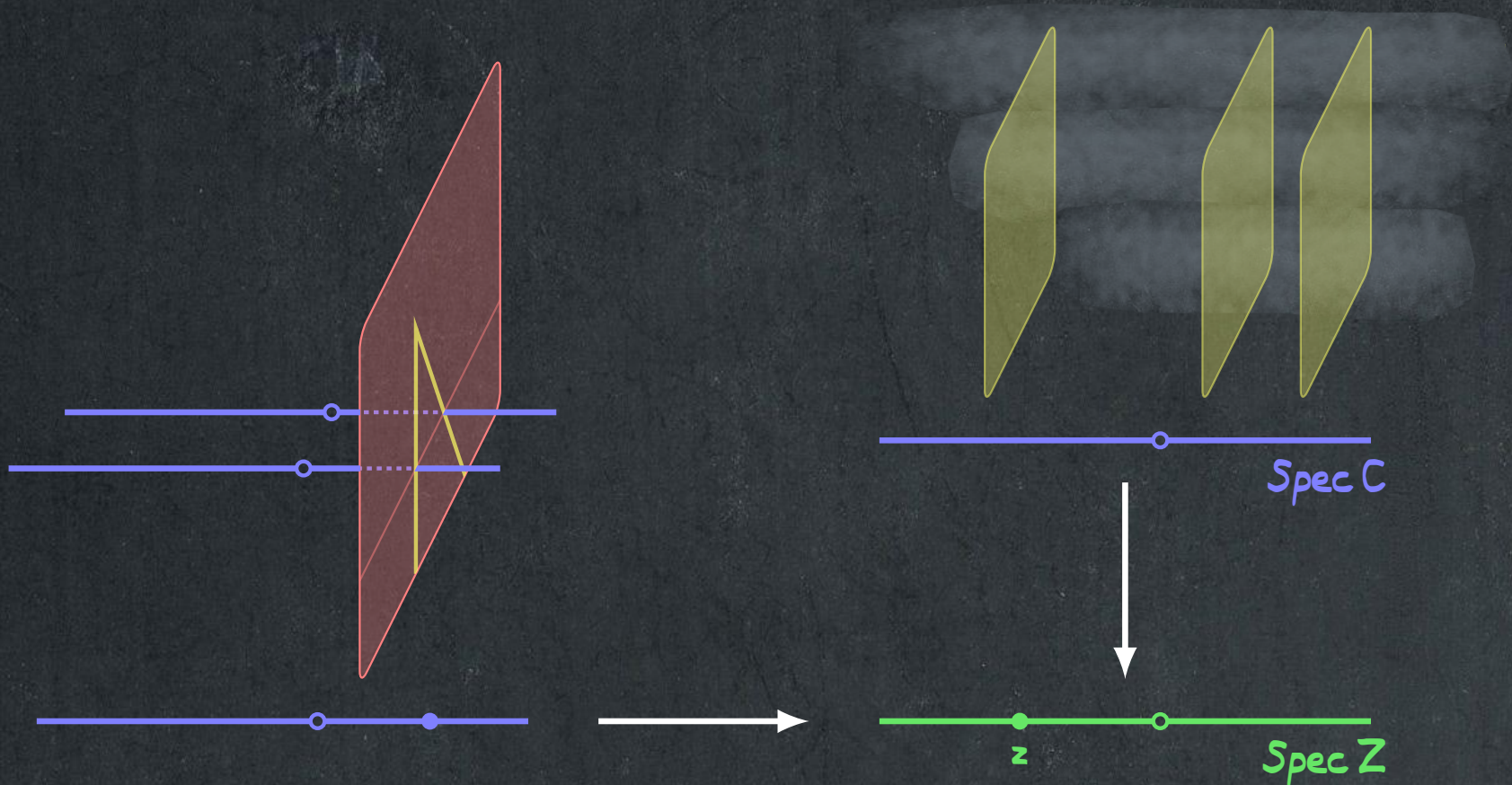

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Towards Linearized Geometric codes

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work in progress
with Amaury Durand

Duality

Duals of Reed Solomon codes

Duals of Reed Solomon codes

Theorem

The dual of $RS(k; a_1, \ldots, a_n)$

which was defined as the image of

$$\begin{aligned} K[x]_{<k} &\to K^n \\ f(x) &\mapsto (f(a_1), \ldots, f(a_n)) \end{aligned}$$

is the image of

$$\begin{aligned} K[x]_{<n-k} &\longrightarrow K^n \\ g(x) &\mapsto (\mathrm{res}_{a_1}(g/h), \ldots, \mathrm{res}_{a_n}(g/h)) \\ &\quad \text{with } h(x) = (x-a_1)\cdots(x-a_n) \end{aligned}$$

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Taylor expansion of Ore polynomials

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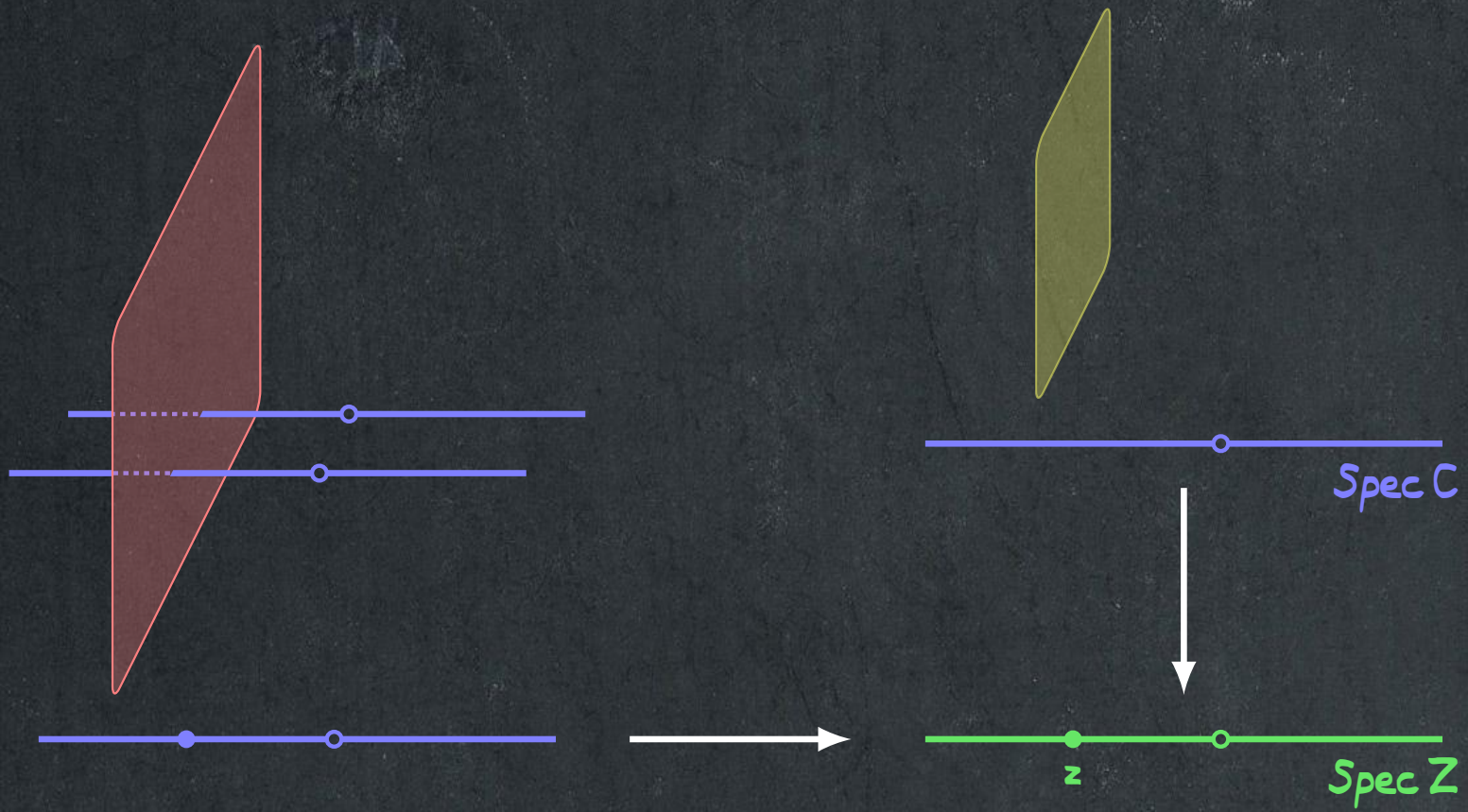

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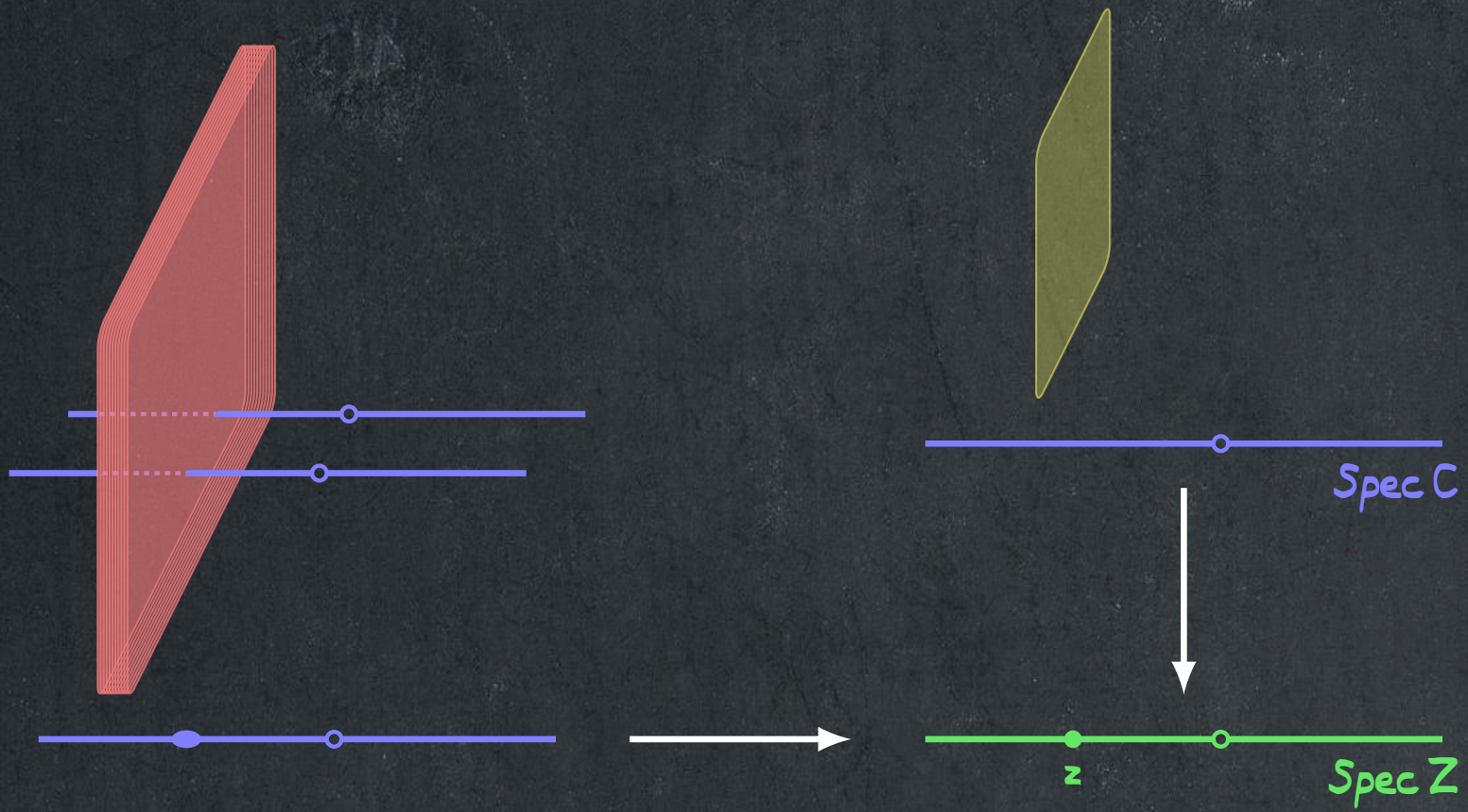

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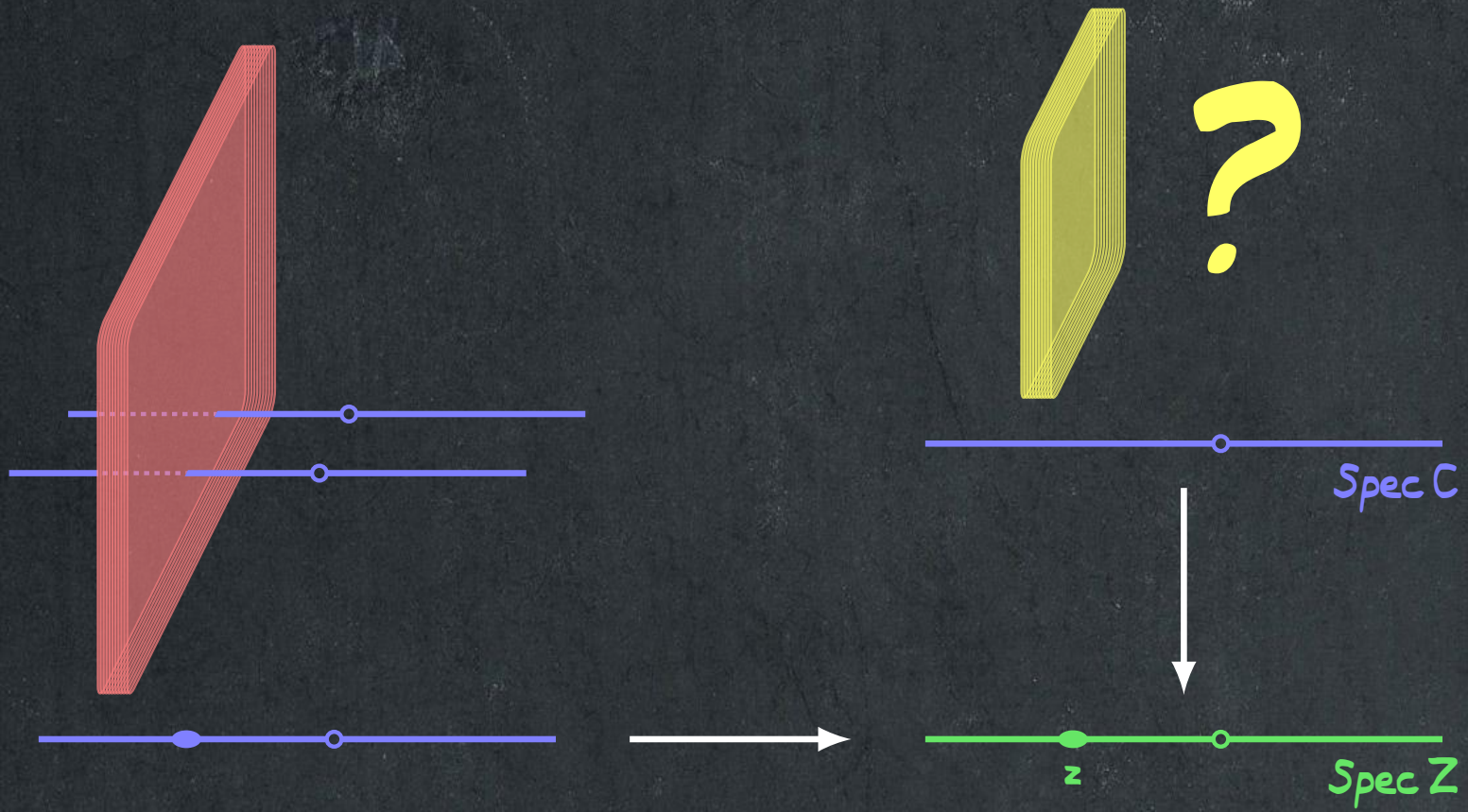

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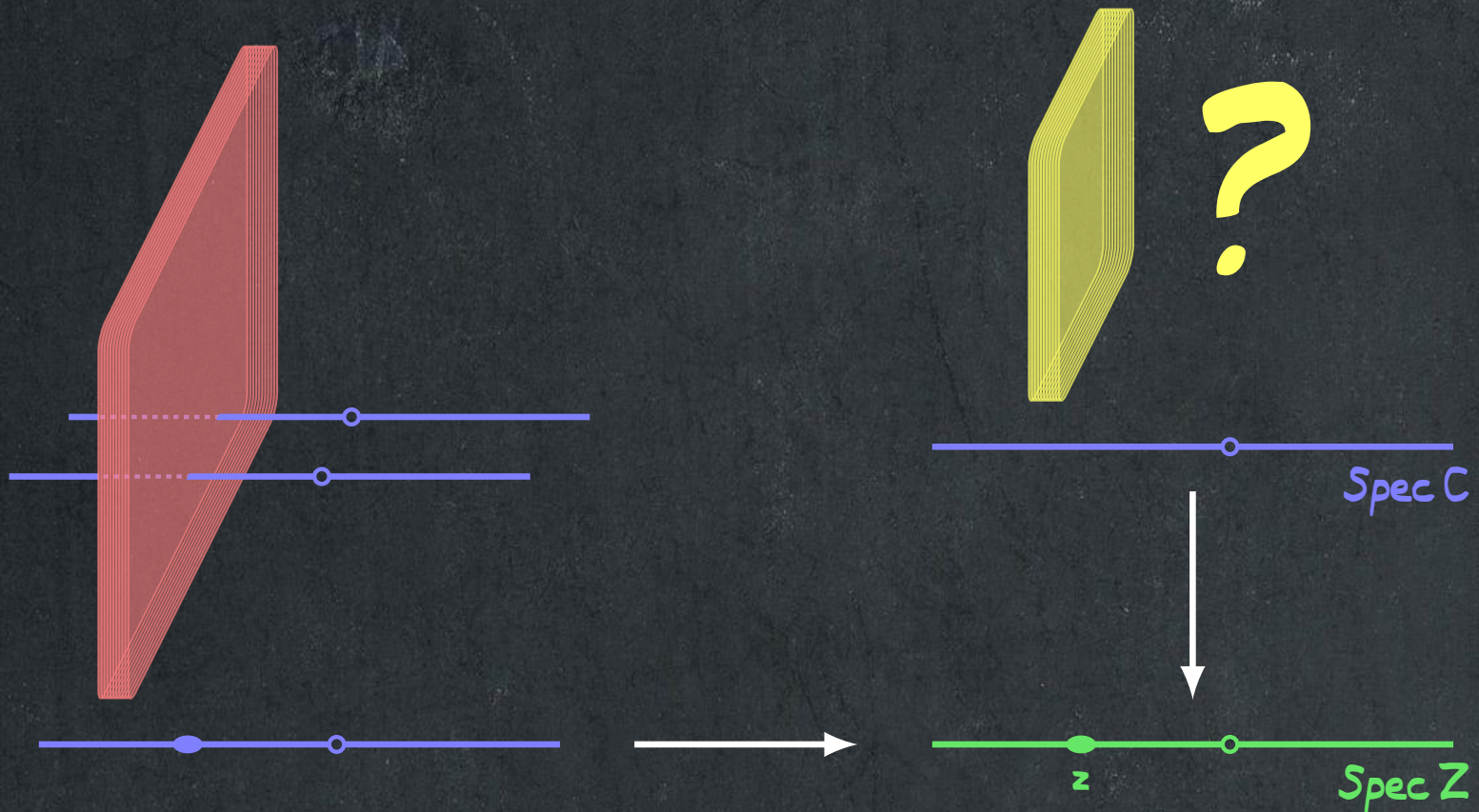

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Let N be an irreducible polynomial in Z

Then there is an isomorphism: $\varprojlim_m A/N^mA \simeq (A/NA)[[h]]$

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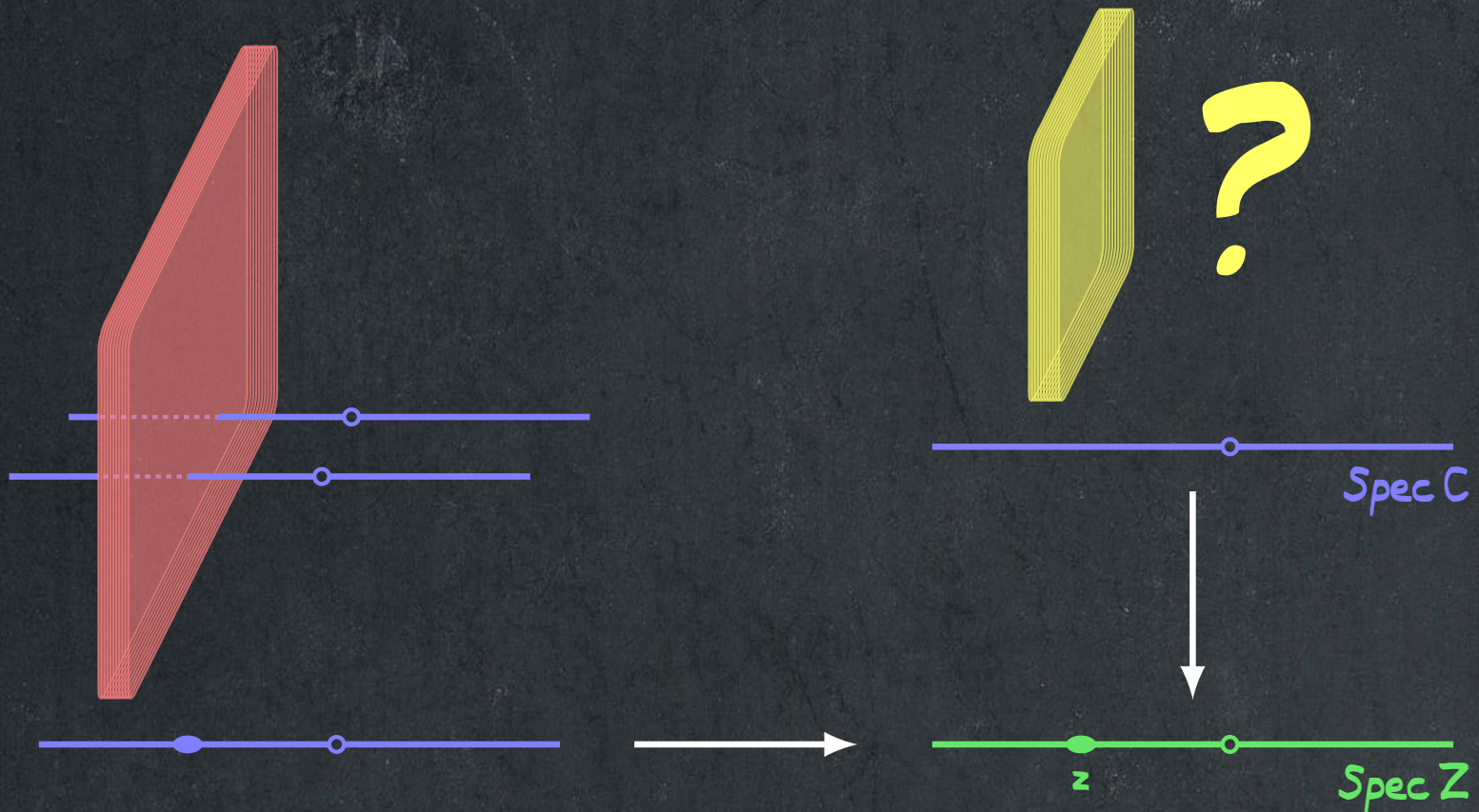

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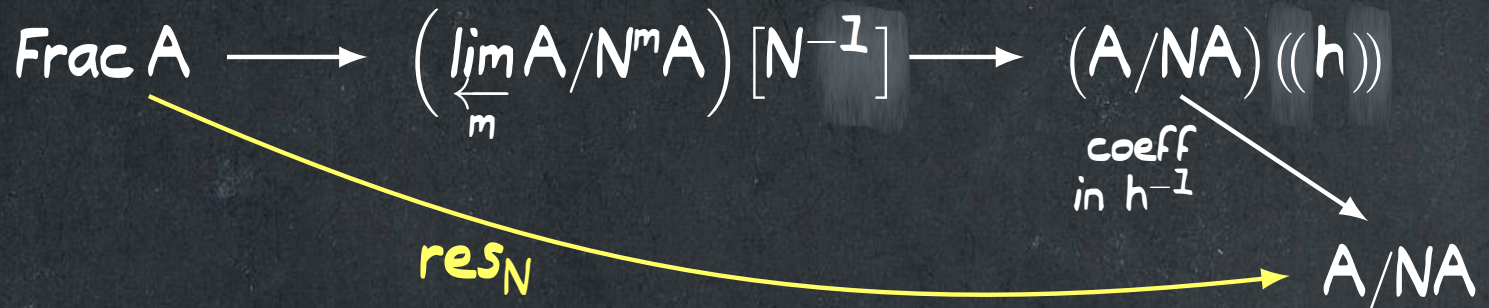

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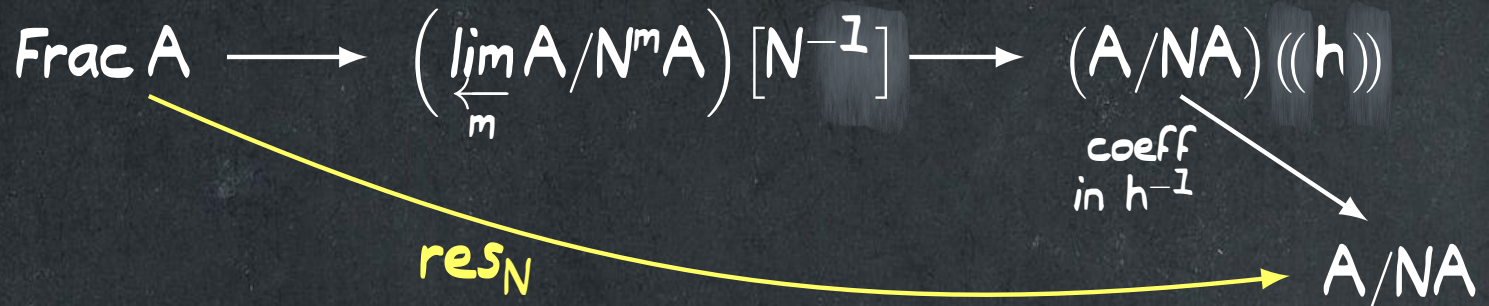

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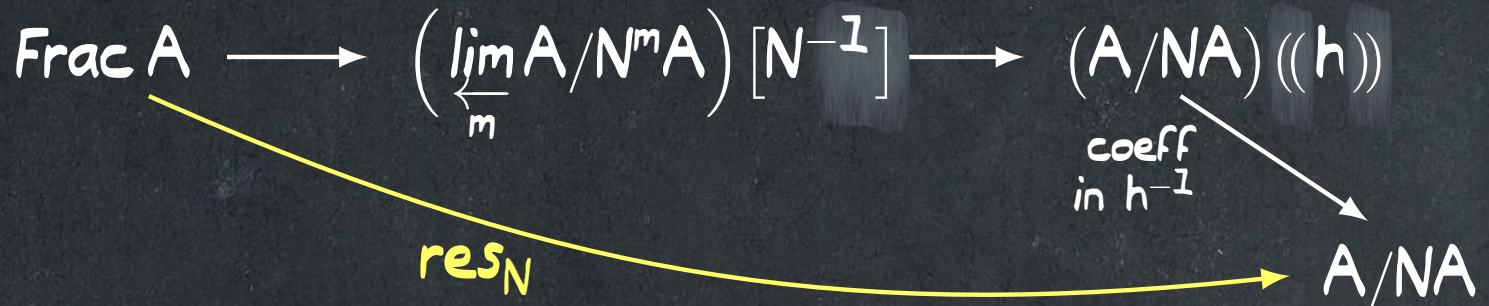

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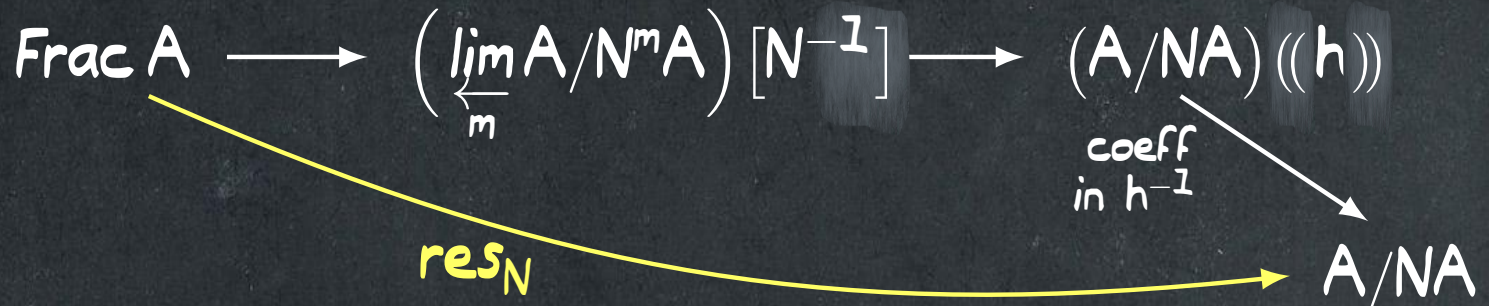

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Fact

All these pairings are symmetric and non-degenerate

Pulling back the pairing on Ore polynomials

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$\langle ev_{a^{-1}\varphi}(g), ev_{a\varphi}(f) \rangle_{\mathrm{End}_K(F)} = T_{rd}(g^* f)$ eval. at $x^r = N_{K/F}(a)$

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Duals of Gabidulin codes

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The dual of $\mathrm{LRS}(k; a_1, \ldots, a_m)$

which was defined as the image of

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is the image of

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$$\text{with } h(x) = x^{mr-r-k} \cdot (x^r - N_{K/F}(a_1)^{-1}) \cdots (x^r - N_{K/F}(a_n)^{-1})$$

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Corollary

Duals of LRS codes meet the Singleton bound

Thanks

for your attention