Quenched CLT for stationary random fields under projective criteria

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Plan of talk

Setting

- Let $(\Omega, \mathcal{K}, P)$ be a probability space.
- Let $T,\ S : \Omega \to \Omega$ be two commuting, invertible, measure preserving transformations.
- Let $\mathcal{F}_{0,0}$ be a sub-sigma algebra of $\mathcal{K}$. For $(i,j) \in \mathbb{Z}^2$, define :

$$\mathcal{F}_{i,j} = S^{-j} T^{-i} \mathcal{F}_{0,0}, \tag{1}$$

  Assume that $\mathcal{F}_{0,0} \subset \mathcal{F}_{0,1}$ and $\mathcal{F}_{0,0} \subset \mathcal{F}_{1,0}$.
- Define random field

$$X_{i,j}(\omega) = f(T^i S^j(\omega)) \tag{2}$$

  where $f : \Omega \to \mathbb{R}$ is $\mathcal{F}_{0,0}$measurable.
- $P^\omega(\cdot) = P(\cdot|\mathcal{F}_{0,0})(\omega), \forall \omega \in \Omega$, also denote by E^ω the expectation corresponding to P^ω.
- We are interested in the quenched asymptotics of partial sums

$$S_{k,j} = \sum_{u,v=1}^{k,j} X_{u,v}$$

Orthomartingale approximation for random fields

Definition (Quenched CLT)

We say that $(X_{i,j})_{i,j\in\mathbb{Z}}$ satisfies the quenched CLT if for almost all $\omega \in \Omega$,

$$\frac{1}{\sqrt{nm}} S_{n,m} \Rightarrow \mathcal{N}(\mu, \sigma^2) \text{ under } P^{\omega} \text{ when } n \wedge m \to \infty.$$

Definition (Orthomartingale approximation)

We say that a random field $(X_{n,m})_{n,m\in Z}$ admits a martingale approximation if there is a field of martingale differences $(D_{n,m})_{n,m\in Z}$ such that

$$\lim_{n\wedge m\to\infty} \frac{1}{nm} E^{\omega}(S_{n,m} - M_{n,m})^2 = 0 \text{ for almost all } \omega \in \Omega, \tag{3}$$

where $M_{k,j} = \sum_{u,v=1}^{k,j} D_{u,v}$.

- Peligrad and Volný (2018) established a quenched CLT for orthomartingales: for a field of stationary orthomartingale differences $(X_{i,j})_{i,j\in\mathbb{Z}}$ with $EX_{0,0}^2 \log(1 + |X_{0,0}|) < \infty$, assume $(\mathcal{F}_{i,j})_{i,j\in\mathbb{Z}}$ is commuting and S (or T) is ergodic, then for almost all $\omega \in \Omega$,

$$\frac{1}{\sqrt{nm}} S_{n,m} \Rightarrow \mathcal{N}(0, \sigma^2) \text{ under } P^{\omega} \text{ when } n \wedge m \to \infty$$

Quenched CLT-summation over squares

- Projection operators: For an integrable random variable X, we introduce the projection operators defined by (Notation $E(X|\mathcal{F}_{a,b}) = E_{a,b}(X)$)

$$P_{\tilde{u},v}(X) := (E_{u,v} - E_{u-1,v})(X)$$
$$P_{u,\tilde{v}}(X) := (E_{u,v} - E_{u,v-1})(X).$$

By using property of commuting filtrations, we have

$$\mathcal{P}_{u,v}(\cdot) := P_{\tilde{u},v} \circ P_{u,\tilde{v}} = (E_{u,v} - E_{u,v-1} - E_{u-1,v} + E_{u-1,v-1})(\cdot)$$

Theorem (Quenched CLT-summation over square)

Assume that $(X_{n,m})_{n,m\in Z}$ is defined by $X_{i,j} = f(T^i S^j(\omega))$ and the filtrations are commuting. Also assume that T (or S) is ergodic and in addition

$$\sum_{u,v\geqslant 0} ||\mathcal{P}_{0,0}(X_{u,v})||_2 < \infty. \tag{4}$$

Then, for almost all $\omega \in \Omega$,

$$\frac{1}{n}(S_{n,n} - R_{n,n}) \Rightarrow N(0,\sigma^2) \text{ under } P^{\omega} \text{ when } n \to \infty.$$

$$\text{where } R_{n,n} = E_{n,0}(S_{n,n}) + E_{0,n}(S_{n,n}) - E_{0,0}(S_{n,n}).$$

Quenched CLT-summation over rectangles

- Young function: a convex, even function $\phi : \mathbb{R} \to [0, \infty)$ satisfying

$$\lim_{x \to 0} \frac{\phi(x)}{x} = 0 \text{ and } \lim_{x \to \infty} \frac{\phi(x)}{x} = \infty.$$

- For any measurable function $f : \Omega \to \mathbb{R}$, the Luxemburg norm of f is define by

$$||f||_\phi = \inf\{k \in (0, \infty) : E\phi(|f|/k) \leqslant 1\}. \quad (5)$$

Theorem (Summation over rectangles)

Assume now that $|| \cdot ||_2$ *in the first theorem is reinforced to the Luxemburg norm, that is*

$$\sum_{u,v \geqslant 0} ||\mathcal{P}_{0,0}(X_{u,v})||_\phi < \infty, \quad (6)$$

where $\phi(x) = x^2 \log(1 + |x|)$ *and* $|| \cdot ||_\phi$ *is the Luxemburg norm. Then, for almost all* $\omega \in \Omega$*,*

$$\frac{1}{(nm)^{1/2}}(S_{n,m} - R_{n,m}) \Rightarrow N(0, \sigma^2) \textit{ under } P^\omega \textit{ when } n \wedge m \to \infty, \quad (7)$$

$$\textit{where } R_{n,m} = E_{n,0}(S_{n,m}) + E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m}).$$

Quenched CLT for random fields

Corollary

Assume that the conditions of the second theorem hold. If

$$\lim_{n\wedge m\to\infty}\frac{E_{0,0}\left(E_{0,m}^{2}(S_{n,m})\right)}{nm}\to 0 \text{ and } \lim_{n\wedge m\to\infty}\frac{E_{0,0}\left(E_{n,0}^{2}(S_{n,m})\right)}{nm}\to 0 \text{ a.s. }, \quad (8)$$

then for almost all $\omega\in\Omega$

$$\frac{1}{(nm)^{1/2}}S_{n,m}\Rightarrow N(0,\sigma^{2}) \text{ under } P^{\omega} \text{ when } n\wedge m\to\infty. \quad (9)$$

If the conditions of the first Theorem hold and (8) holds with $m=n$ *, then for almost all* $\omega\in\Omega$*,*

$$\frac{1}{n}S_{n,n}\Rightarrow N(0,\sigma^{2}) \text{ under } P^{\omega} \text{ when } n\to\infty. \quad (10)$$

Sketch of Proof: $\lim_{n\wedge m\to\infty}\frac{1}{nm}E_{0,0}(R_{n,m}^{2})=0$ a.s. by the regularity conditions together with

$$E_{0,0}(R_{n,m}^{2})=E_{0,0}\left(E_{n,0}^{2}(S_{n,m})\right)+E_{0,0}\left(E_{0,m}^{2}(S_{n,m})\right)-E_{0,0}^{2}(S_{n,m})$$

A sufficient condition for quenched CLT

For the sake of applications, we provide a sufficient condition which will take care of both the generalized Hannan condition and the regularity assumptions.

Corollary

Assume that the filtrations are commuting. Also assume that T (or S) is ergodic and in addition for $\delta \geqslant 0$

$$\sum_{u,v\geqslant 1} \frac{||E_{1,1}(X_{u,v})||_{2+\delta}}{(uv)^{1/(2+\delta)}} < \infty. \quad (11)$$

(a) If $\delta = 0$, then a quenched CLT holds for $S_{n,n}/n$ holds.
(b) If $\delta > 0$, then the quenched convergence $S_{n,m}/\sqrt{nm}$ holds.

Sketch of proof:

- **Step 1.** Condition (11) implies

$$\lim_{n\wedge m\to\infty} \frac{1}{nm} E_{0,0}(R^2_{n,m}) = 0 \text{ a.s.}$$

- **Step 2.** Condition (11) implies

$$\sum_{u,v\geqslant 0} ||\mathcal{P}_{0,0}(X_{u,v})||_{2+\delta} < \infty. \quad (12)$$

Main tools for proving these results

The tools used in the proofs are

- martingale approximation
- multi-dimensional ergodic theorem
- Hölder inequlity
- Rosenthal inequalities for martingales
- Decompling inequlities for U-statistics of independent random variables.

Idea of the proofs

Proof of the main theorem:

- Construction of martingales
  $\lim_{n\wedge m\to\infty} \mathcal{P}_{1,1}(S_{n,m}) = \sum_{u,v\geqslant 1} \mathcal{P}_{1,1}(X_{u,v}) := D_{1,1}$ a.s. and in L^2.
  For each $i, j \in \mathbb{Z}$,

$$M_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{m} D_{i,j} \text{ where } D_{i,j} = \sum_{(u,v)\geqslant(i,j)} \mathcal{P}_{i,j}(X_{u,v})$$

- The martingale approximation:
  $\lim_{n\wedge m\to\infty} \frac{1}{nm} E_{0,0}\left(S_{n,m} - R_{n,m} - M_{n,m}\right)^2 = 0.$ a.s.

To validate this approximation, we first look at the decomposition of $S_{n,m}$:

$$S_{n,m} - R_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathcal{P}_{i,j}\Big(\sum_{u=i}^{n}\sum_{v=j}^{m} X_{u,v}\Big)$$

where

$$R_{n,m} \; = E_{n,0}(S_{n,m}) + E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m}).$$

Using the orthogonality of the martingale differences field $(\mathcal{P}_{i,j} - D_{i,j})_{i,j\in\mathbb{Z}}$,

$$\frac{1}{nm} E_{0,0}\left(S_{n,m} - R_{n,m} - M_{n,m}\right)^2 = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} E_{0,0}\left(\mathcal{P}_{i,j}\Big(\sum_{u=i}^{n}\sum_{v=j}^{m} X_{u,v}\Big) - D_{i,j}\right)^2.$$

Define the operators

$$Q_1(f) = E_{0,\infty}(\hat{T}f);\ \ Q_2(f) = E_{\infty,0}(\hat{S}f) \text{ where } \hat{T}f = f \circ T, \hat{S}f = f \circ S.$$

Then we can write

$$E_{0,0}\left(\mathcal{P}_{i,j}(X_{u,v})\right)^2 = Q_1^i Q_2^j (\mathcal{P}_{0,0}(X_{u-i,v-j}))^2.$$

$$E_{0,0}\left(\mathcal{P}_{i,j}(\sum_{u=i}^{n}\sum_{v=j}^{m} X_{u,v}) - D_{i,j}\right)^2 = E_{0,0}\left(\sum_{u=n+1}^{\infty}\sum_{v=j}^{m}\mathcal{P}_{i,j}(X_{u,v}) + \sum_{u=i}^{\infty}\sum_{v=m+1}^{\infty}\mathcal{P}_{i,j}(X_{u,v})\right)^2$$

So, we have

$$\frac{1}{nm}E_{0,0}\left(S_{n,m} - R_{n,m} - M_{n,m}\right)^2 \leqslant 2(I_{n,m} + II_{n,m})$$

where

$$I_{n,m} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} Q_1^i Q_2^j \left(\sum_{u=n+1-i}^{\infty}\sum_{v=0}^{\infty}|\mathcal{P}_{0,0}(X_{u,v})|\right)^2$$

and

$$II_{n,m} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} Q_1^i Q_2^j \left(\sum_{u=0}^{\infty}\sum_{v=m+1-j}^{\infty}|\mathcal{P}_{0,0}(X_{u,v})|\right)^2.$$

Let c be a fixed integer satisfying $c < n$. We decompose $I_{n,m}$ into two parts

$$\frac{1}{nm}\sum_{i=1}^{n-c}\sum_{j=1}^{m} Q_1^i Q_2^j \left(\sum_{u=n+1-i}^{\infty}\sum_{v=0}^{\infty} |\mathcal{P}_{0,0}(X_{u,v})|\right)^2 := A_{n,m}(c)$$

and

$$\frac{1}{nm}\sum_{i=n-c+1}^{n}\sum_{j=1}^{m} Q_1^i Q_2^j \left(\sum_{u=n+1-i}^{\infty}\sum_{v=0}^{\infty} |\mathcal{P}_{0,0}(X_{u,v})|\right)^2 := B_{n,m}(c)$$

By the ergodic theorem for Dunford-Schwartz operators (Krengel (1985), Theorem 1.1, ch.6), for each c fixed

$$\lim_{c\to\infty}\lim_{n\wedge m\to\infty} A_{n,m}(c) = 0 = \lim_{n\wedge m\to\infty} B_{n,m}(c) \text{ a.s.}$$

Thus

$$\lim_{n\wedge m\to\infty}\frac{1}{nm} E_{0,0}\left(S_{n,m} - R_{n,m} - M_{n,m}\right)^2 = 0.$$

A preparatory lemma to prove the corollary

Lemma

$\sum_{u,v\geqslant 1} \frac{||E_{1,1}(X_{u,v})||_{2+\delta}}{(uv)^{1/(2+\delta)}} < \infty$ *implies*

$$\sum_{u\geqslant 1} \frac{1}{u^{1/(2+\delta)}} \sum_{v\geqslant 0} \left\| P_{0,\tilde{0}}(X_{u,v}) \right\|_{2+\delta} < \infty, \tag{13}$$

and for any $u \geqslant 0$

$$\sum_{v=1}^{\infty} ||P_{0,\tilde{0}}(X_{u,v})||_{2+\delta} < \infty. \tag{14}$$

Proof of Lemma: By the Hölder's inequality and the Rosenthal inequality for martingales , we have ($C_\delta > 0$ denotes a generic constant depending on δ)

$$\sum_{v\geqslant 1} \|P_{-u,-\tilde{v}}(X_{0,0})\|_{2+\delta} \leqslant \sum_{n\geqslant 0} (2^n)^{\frac{1+\delta}{2+\delta}} \left(\sum_{v=2^n}^{2^{n+1}-1} \|P_{-u,-\tilde{v}}(X_{0,0})\|_{2+\delta}^{2+\delta} \right)^{\frac{1}{2+\delta}}$$

$$\leqslant C_\delta \sum_{n\geqslant 0} (2^n)^{\frac{1+\delta}{2+\delta}} \left\| \sum_{v=2^n}^{2^{n+1}-1} P_{-u,-\tilde{v}}(X_{0,0}) \right\|_{2+\delta} \leqslant 2C_\delta \sum_{n\geqslant 0} (2^n)^{\frac{1+\delta}{2+\delta}} \|E_{-u,-2^n}(X_{0,0})\|_{2+\delta} .$$

Examples

Let $|\mathbf{n}| = n_1 \cdots n_d$.

Example (Linear field)

Let $(\xi_{\mathbf{n}})_{\mathbf{n} \in Z^d}$ be a random field of independent, identically distributed random variables which are centered and $E\left(|\xi_{\mathbf{0}}|^{2+\delta}\right) < \infty$. For $\mathbf{k} \geqslant \mathbf{0}$ define

$$X_{\mathbf{k}} = \sum_{\mathbf{j} \geqslant \mathbf{0}} a_{\mathbf{j}} \xi_{\mathbf{k}-\mathbf{j}} \ .$$

Assume that

$$\sum_{\mathbf{k} \geqslant \mathbf{1}} \frac{1}{|\mathbf{k}|^{1/(2+\delta)}} \left(\sum_{\mathbf{j} \geqslant \mathbf{k}-\mathbf{1}} a_{\mathbf{j}}^2 \right)^{\frac{1}{2}} < \infty.$$

(a) If $\delta = 0$, then the quenched CLT holds for $S_{n,\cdots,n}/n^{d/2}$.
(b) If$\delta > 0$, the quenched CLT holds for $S_{\mathbf{n}}/\sqrt{|\mathbf{n}|}$.

Sketch of Proof: (Using independence of $\xi_{\mathbf{n}}$ and the Rosenthal inequality)

$$E_{\mathbf{1}}(X_{\mathbf{k}}) = \sum_{\mathbf{j} \geqslant \mathbf{k}-\mathbf{1}} a_{\mathbf{j}} \xi_{\mathbf{k}-\mathbf{j}}$$

Examples

Example (Volterra field)

Let $(\xi_{\mathbf{n}})_{\mathbf{n}\in Z^d}$ be a random field of independent random variables identically distributed centered and $E\left(|\xi_{\mathbf{0}}|^{2+\delta}\right) < \infty$. For $\mathbf{k} \geqslant \mathbf{0}$, define

$$X_{\mathbf{k}} = \sum_{(\mathbf{u},\mathbf{v})\geqslant(\mathbf{0},\mathbf{0})} a_{\mathbf{u},\mathbf{v}}\xi_{\mathbf{k}-\mathbf{u}}\xi_{\mathbf{k}-\mathbf{v}}.$$

where $a_{\mathbf{u},\mathbf{v}}$ are real coefficients with $a_{\mathbf{u},\mathbf{u}} = 0$ and $\sum_{\mathbf{u},\mathbf{v}\geqslant\mathbf{0}} a^2_{\mathbf{u},\mathbf{v}} < \infty$. In addition, assume that

$$\sum_{\mathbf{k}\geqslant\mathbf{1}} \frac{1}{|\mathbf{k}|^{1/(2+\delta)}} \left(\sum_{\substack{(\mathbf{u},\mathbf{v})\geqslant(\mathbf{k}-\mathbf{1},\mathbf{k}-\mathbf{1}) \\ \mathbf{u}\neq\mathbf{v}}} a^2_{\mathbf{u},\mathbf{v}} \right)^{1/2} < \infty. \tag{15}$$

(a) If $\delta = 0$, then the quenched CLT holds for $S_{n,\cdots,n}/n^{d/2}$.
(b) If $\delta > 0$, the quenched CLT holds for $S_{\mathbf{n}}/\sqrt{|\mathbf{n}|}$.

Sketch of Proof: (Using decoupling inequality for U-statistics)

$$E_1(X_{\mathbf{k}}) = \sum_{(\mathbf{u},\mathbf{v})\geqslant(\mathbf{k}-\mathbf{1},\mathbf{k}-\mathbf{1})} a_{\mathbf{u},\mathbf{v}}\xi_{\mathbf{k}-\mathbf{u}}\xi_{\mathbf{k}-\mathbf{v}}$$

Rothsenthal's inequality

For convenience, we mention one classical inequality for martingales, see Theorem 2.11, p. 23, Hall and Heyde (1980) and also Theorem 6.6.7 ch. 6, p. 322, de la Peña and Giné (1999).

Theorem (Rothsenthal's Inequality)

Let $p \geqslant 2$. Let $M_n = \sum_{k=1}^{n} X_k$ where $\{M_n, \mathcal{F}_n\}$ is a martingale with martingale differences X_i, then there are constants $0 < c_p, C_p < \infty$ such that

$$c_p \left\{ \sum_{k=1}^{n} E|X_k|^p + E\left[\left(\sum_{k=1}^{n} E(X_k^2|\mathcal{F}_{k-1}) \right)^{p/2} \right] \right\}$$

$$\leqslant \|M_n\|_p^p \leqslant C_p \left\{ E\left[\left(\sum_{k=1}^{n} E(X_k^2|\mathcal{F}_{k-1}) \right)^{p/2} \right] + \sum_{k=1}^{n} E|X_k|^p \right\}.$$

Decoupling Inequality

The following is a decoupling result for U-statistics, which can be found on p. 99, Theorem 3.1.1, de la Peña and Giné (1999).

Theorem (Decompling inequlity)

Let $(X_i)_{1\leqslant i\leqslant n}$ *be* n *independent random variables and let* $(X_i^k)_{1\leqslant i\leqslant n}$, $k=1,\cdots,m$, *be* m *independent copies of this sequences. For each* $(i_1,i_2,\cdots,i_m)\in I_n^m$, *let* $h_{i_1,\cdots,i_m}: R^m\to R$ *be a measurable function such* $E|h_{i_1,\cdots,i_m}(X_{i_1},\cdots,X_{i_m})|<\infty$. *Let* $f:[0,\infty)\to[0,\infty)$ *be a convex non-decreasing function such that* $Ef(|h_{i_1,\cdots,i_m}(X_{i_1},\cdots,X_{i_m})|)<\infty$ *for all* $(i_1,i_2,\cdots,i_m)\in I_n^m$, *where* $I_n^m=\{(i_1,\cdots,i_m): i_j\in\mathbb{N}, 1\leqslant i_j\leqslant n, i_j\neq i_k,\ \text{if } j\neq k\}$. *Then there exists* $C_m>0$ *such that*

$$Ef(|\sum_{I_n^m} h_{i_1,\cdots,i_m}(X_{i_1},\cdots,X_{i_m})|)\leqslant Ef(C_m|\sum_{I_n^m} h_{i_1,\cdots,i_m}(X_{i_1}^1,\cdots,X_{i_m}^m)|).$$