Quenched limit theorems via complex cone contraction

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Nagaev-Guivarc'h method

The Nagaev Guivarc'h method is a powerful tool to obtain limit theorem for dynamical systems. **Setting**([1]): A Lipschitz expanding map $T : [0,1] \to [0,1]$, a Hölder observable f, L_T its transfer operator. The *twisted transfer operator* L_t is defined by

$$L_t(\phi) := L_T(e^{itf}\phi)$$

Goal: Study the probabilistic behavior of the Birkhoff sums

$$S_n f := \sum_{k=0}^{n-1} f \circ T^k$$

Three major steps:

- **Representation** of the Birkhoff sums characteristic function $\mathbb{E}\left[e^{itS_n(f)}\right] = \int_0^1 L_t^n(1)dm$.
- Establish the **spectral gap** property for the twisted transfer operator on Hölder spaces.
- Study the **regularity** of the maximal eigenvalue map $t \mapsto \lambda_t$.

Random NG method

Setting([2]): $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, a driving map $\tau : \Omega \to \Omega$ ($\mathbb{P}$ preserving and ergodic), consider the cocycle above (Ω, τ) of piecewise C^2, interval expanding maps

$$T_\omega^{(n)} := T_{\tau^{n-1}\omega} \circ \cdots \circ T_\omega$$

Let $g \in L^\infty(\Omega, BV([0,1]))$. Introduce the *twisted cocycle of transfer operator*

$$L_{\omega,t}(\phi) := L_\omega(e^{itg_\omega}\phi)$$

Goal: Study the behavior of the (random) Birkhoff sums

$$S_n g(\omega) = \sum_{k=0}^{n-1} g_{\tau^k\omega} \circ T_\omega^{(k)}$$

Steps (under *centering* condition on g, *admissibility* assumptions on L_ω):

- **Representation** of random Birkhoff sum characteristic function as an integral of n^{th} random composition of twisted transfer operator.
- **Gap** of the twisted transfer operator cocycle **Lyapunov spectrum**.
- **Regularity** of the top Lyapunov exponent χ and top Oseledets space.

Conclusion

The method allows to show a variety of (quenched) limit theorems for (random) dynamical systems ([1, 2]):
• Central limit theorem
• Large deviations estimates
• Local limit theorem
• Invariance principle
• **Berry-Esseen** estimates ?

A generalization of Birkhoff cones: complex cones

$C \subset E$ is a complex cone if $C \neq \emptyset$, and $\mathbb{C}^* C = C$.
C is *proper* if $\overline{C}$ contains no complex planes, *inner regular* if $\text{Int}(C) \neq \emptyset$ and *outer regular* if there is $\ell \in E'$, $K > 0$ s.t $|\langle \ell, x\rangle| \geq K\|x\|_E$ for each $x \in C$. It is **linearly convex** if through each $x \notin C$ passes a hyperplane not intersecting C.
Let $E(x,y) = \{z \in \mathbb{C}, zx - y \notin C\}$. The *complex gauge*(projective distance) $d_C(x,y)$ is $\log(b(x,y)/a(x,y))$, where $b(x,y) = \sup|E(x,y)|$ and $a = \inf|E(x,y)|$.
Theorem([3, 4]) If C is linearly convex, $(\mathcal{C}, d_C)$ is a complete metric space, with $\mathcal{C}$ the projectivised of C.

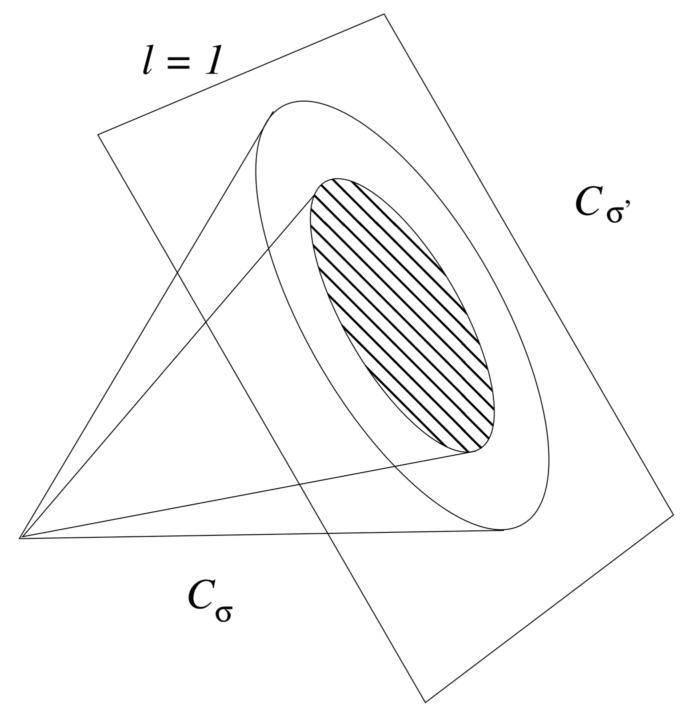

Figure 1: A linearly convex, regular $\mathbb{C}$-cone and a cone contraction.

$L \in \mathcal{L}(E)$ is a *cone contraction* if $L(C) \subset C$. It is *strict* if $\Delta := \sup_{x,y\in C} d_C(Lx, Ly) < \infty$.
Theorem:([5, 3]) A strict contraction of a regular $\mathbb{C}$-cone admits a spectral gap.

Random product of complex cone contractions

Definition: $C \subset E$ a regular and linearly convex $\mathbb{C}$-cone. Let

$$M_C(\Delta, \rho) := \{L \in \mathcal{L}(E), L(C) \subset C, \sup_{x,y\in C} d_C(Lx, Ly) \leq \Delta < \infty, B(L\phi, \rho\|L\phi\|) \subset C\}$$

Theorem(**Step 2** in the random NG method): Let $L : \Omega \to M_C(\Delta, \rho)$ an operator cocycle above τ. There exists measurables $\lambda : \Omega \to \mathbb{C}$, $h : \Omega \to E$, $\mu : \Omega \to E'$ such that

- $L_\omega h_\omega = \lambda_\omega h_{\tau\omega}$, $h \in L^\infty(\Omega, E)$ and $\langle \ell, h\rangle = 1$.
- $L_\omega^* \mu_{\tau\omega} = \lambda_\omega \mu_\omega$, $\mu \in L^\infty(\Omega, E')$, $\langle \mu, h\rangle = 1$.
- $\log|\lambda| \in L^1(\Omega, \mathbb{P})$ and $\chi = \mathbb{E}[\log(|\lambda|)]$.
- There exists $0 < \eta < 1$, such that $\forall \phi \in E$, $\left\| \frac{L_\omega^{(n)}\phi}{\lambda_\omega^{(n)}} - h_{\tau^n\omega}\langle \mu_\omega, \phi\rangle \right\|_E \leq C\eta^{n-1}\|\phi\|_E$

Applications:

- $T_\omega : [0,1] \to [0,1]$ piecewise C^2, expanding maps, $g \in L^\infty(\Omega, BV([0,1]))$. (Under uniform bounds on expansion, distortion) $L_{\omega,t}$ is a random product of strict contractions of the **complexification** of
$$C_a := \{f \in BV([0,1]), f \geq 0, |f|_{BV} \leq a \int_0^1 f dm\}$$
- $T_\omega : \mathbb{S}^1 \to \mathbb{S}^1$ uniformly expanding, $g \in L^\infty(\Omega, Lip(\mathbb{S}^1))$. (Under uniform bounds on expansion) $L_{\omega,t}$ is a random product of strict contractions of the **complexification** of
$$C_L := \{f \in Lip(\mathbb{S}^1), f \geq 0, f(x) \leq f(y)e^{Ld(x,y)}\}$$

Step 3([5, Thm 10.2]): If $z \in \mathbb{D} \mapsto L_z \in L^\infty(\Omega, M_C(\Delta, \rho))$ is analytic, then $z \in \mathbb{D} \mapsto h_z \in L^\infty(\Omega, E)$, $z \in \mathbb{D} \mapsto \chi_z$ are analytic.

References

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