On the quenched CLT for stationary random fields

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Introduction

Central Limit Theorems (CLTs) are at the heart of probability theory. A big part of this field of study consist in studying and trying to show these theorems under different dependence conditions. Recently a new type of CLTs has been extensively studied : the quenched CLTs (also called CLTs started at a point). When studying and proving such theorems, the main problem that arises is that the previously stationary process loose this property when started from a fixed trajectory. It is therefore required to find a method which does not utilize the stationarity of the process.

Another subject of interest for CLTs is spatial data (i.e. data indexed by multi-dimensional indexes). As far as we know, quenched CLTs were seldom researched for stationary random fields. In addition to the lack of stationarity, another difficulty arises when analyzing the asymptotic properties of random fields : the future and the past do not have a unique interpretation. To compensate for the lack of ordering of the filtration, mathematicians make use of the notion of commuting filtrations.

A fruitful approach to prove the limit theorems for general random fields is via martingale approximations, which were introduced by Rosenblatt [3]. Using this method, in combination with the newly proved results concerning the quenched CLT for orthomartingales by Peligrad and Volný [1] allows us to state some interesting results concerning more general random fields under a projective condition.

Bi-dimensional case

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\epsilon_i)_{i\in\mathbb{Z}^2}$ be a stationary random field. We assume that **the filtration** $\mathcal{F}_{i,j} = \sigma(\epsilon_{u,v}, u \leq i, v \leq j)$ **is commuting**, meaning that $E_{a,b}E_{u,v}X = E_{u\wedge a, v\wedge b}X$.

Let T and S be respectively the horizontal and vertical shift defined on $\mathbb{R}^{\mathbb{Z}^d}$. And let $\mathcal{P}_{0,0}$ be the projection operator.

Let φ be defined for every $x \in \mathbb{R}$ by $\varphi(x) = x^2\log(1+|x|)$, and $||\cdot||_\varphi$ be its associated Luxembourg norm.

We denote, for every $\omega \in \Omega$, by E^ω the expected value with respect to the probability $P^\omega = P(\cdot|\mathcal{F}_{0,0})(\omega)$. Let $(X_{i,j})_{i,j\in\mathbb{Z}}$ be a stationary random field defined, for every $i,j \in \mathbb{Z}$, by $X_{i,j} = f(T^uS^v(\epsilon_k)_{k\in\mathbb{Z}^d})$.

In this work, we investigated **the asymptotic behavior, under the probability measure** P^ω**, of the quantity** $S_{n,m} = \sum_{u=1}^n \sum_{v=1}^m X_{u,v}$.

The two main results established in [2] are the following quenched central limit theorems :

Theorem 1. *Assume that T (or S) is ergodic and in addition*

$$\sum_{u,v\geq 0} ||\mathcal{P}_{0,0}(X_{u,v})||_2 < \infty. \tag{1}$$

Then, for almost all $\omega \in \Omega$,

$$\frac{1}{n}(S_{n,n} - R_{n,n}) \Rightarrow N(0,\sigma^2) \text{ under } P^\omega \text{ when } n \to \infty.$$

where $R_{n,n} = E_{n,0}(S_{n,n}) + E_{0,n}(S_{n,n}) - E_{0,0}(S_{n,n})$.

Theorem 2. *Furthermore, assume now that (1) is reinforced to*

$$\sum_{u,v\geq 0} ||\mathcal{P}_{0,0}(X_{u,v})||_\varphi < \infty.$$

Then, for almost all $\omega \in \Omega$,

$$\frac{1}{(nm)^{1/2}}(S_{n,m} - R_{n,m}) \Rightarrow N(0,\sigma^2) \text{ under } P^\omega \text{ when } n\wedge m \to \infty.$$

where $R_{n,m} = E_{n,0}(S_{n,m}) + E_{0,m}(S_{n,m}) - E_{0,0}(S_{n,m})$.

d-dimensional case, $d \geq 3$

These two results can be extended to a higher dimension field. A consequence of these multi-dimensional results is the following corollary : Let d be a positive integer and let $(X_\mathbf{i})_{\mathbf{i}\in\mathbb{Z}^d}$ be a stationary random field. Suppose that **the filtration** $\mathcal{F}_\mathbf{i} = \sigma(\epsilon_\mathbf{u}, \mathbf{u} \leq \mathbf{i})$ **is commuting**.

Assume there is an integer i, $1 \leq i \leq d$, such that T_i is ergodic and, in addition, assume that for $\delta > 0$,

$$\sum_{\mathbf{u}\geq \mathbf{1}} \frac{||E_\mathbf{1}(X_\mathbf{u})||_{2+\delta}}{|\mathbf{u}|^{1/(2+\delta)}} < \infty.$$

(a) If $\delta = 0$, then the quenched CLT holds for $\frac{1}{n^{d/2}}S_{n,\cdots,n}$.

(b) If $\delta > 0$, then the quenched CLT holds for $\frac{1}{\sqrt{|\mathbf{n}|}}S_\mathbf{n}$.

Applications of the theorems

In this section we will present two results derived from these theorems.

Example 1. Let $(\xi_\mathbf{n})_{\mathbf{n}\in Z^d}$ be a random field of independent, identically distributed random variables which are centered and $E\left(|\xi_\mathbf{0}|^{2+\delta}\right) < \infty$ with $\delta > 0$. For $\mathbf{k} \geq \mathbf{0}$, we define a linear field $X_\mathbf{k} = \sum_{\mathbf{j}\geq\mathbf{0}} a_\mathbf{j}\xi_{\mathbf{k}-\mathbf{j}}$. Assume that

$$\sum_{\mathbf{k}\geq\mathbf{1}} \frac{1}{|\mathbf{k}|^{1/(2+\delta)}} \left(\sum_{\mathbf{j}\geq\mathbf{k}-\mathbf{1}} a_\mathbf{j}^2\right)^{\frac{1}{2}} < \infty.$$

Then for $\frac{1}{\sqrt{|\mathbf{n}|}}S_\mathbf{n}$, a quenched CLT holds.

Example 2. Let $(\xi_\mathbf{n})_{\mathbf{n}\in Z^d}$ be a random field of independent random variables identically distributed centered and $E\left(|\xi_\mathbf{0}|^{2+\delta}\right) < \infty$. For $\mathbf{k} \geq \mathbf{0}$, we define a Volterra field $X_\mathbf{k} = \sum_{(\mathbf{u},\mathbf{v})\geq(\mathbf{0},\mathbf{0})} a_{\mathbf{u},\mathbf{v}}\xi_{\mathbf{k}-\mathbf{u}}\xi_{\mathbf{k}-\mathbf{v}}$ where $a_{\mathbf{u},\mathbf{v}}$ are real coefficients with $a_{\mathbf{u},\mathbf{u}} = 0$ and $\sum_{\mathbf{u},\mathbf{v}\geq\mathbf{0}} a_{\mathbf{u},\mathbf{v}}^2 < \infty$. In addition, assume that

$$\sum_{\mathbf{k}\geq\mathbf{1}} \frac{1}{|\mathbf{k}|^{1/(2+\delta)}} \left(\sum_{\substack{(\mathbf{u},\mathbf{v})\geq(\mathbf{k}-\mathbf{1},\mathbf{k}-\mathbf{1}) \\ \mathbf{u}\neq\mathbf{v}}} a_{\mathbf{u},\mathbf{v}}^2\right)^{1/2} < \infty.$$

Then for $\frac{1}{\sqrt{|\mathbf{n}|}}S_\mathbf{n}$, a quenched CLT holds.

References

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