Mixing and the local central limit theorem for hyperbolic dynamical systems

Péter Nándori
University of Maryland
based on joint work with Dmitry Dolgopyat

Probabilistic limit theorem for dynamical systems, CIRM

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Mixing of infinite measures

Local limit theorem (LLT) for iid random variables

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$$\mathbb{P}(S_n - v\sqrt{n} \in [A, B]) \sim \frac{1}{\sqrt{n}} \mathfrak{g}_\sigma(v)(B - A),$$

where $\mathfrak{g}_\sigma$ is the centered Gaussian density with variance σ^2.

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   where $v_n \in na + b\mathbb{Z}$ and $v_n \sim v\sqrt{n}$, $A, B \notin b\mathbb{Z}$ and u is the b times the counting measure on $b\mathbb{Z}$.

Mixing and LLT (MLLT) for dynamical systems

Setup:

- (X, d) metric space
- ν a Borel probability measure
- $T : X \to X$ preserves ν
- $f : X \to \mathbb{R}^d$ observable, $S_n(f) = \sum_{i=0}^{n-1} f \circ T^i$

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$$\int_X \mathfrak{z}(S_n(f,x) - n\nu(f) - v_n)d\nu(x)$$
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$$\int_X \mathfrak{x}(x)\mathfrak{y}(T^n x)\mathfrak{z}(S_n(f,x) - n\nu(f) - v_n)d\nu(x)$$

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Suspension (semi)flows

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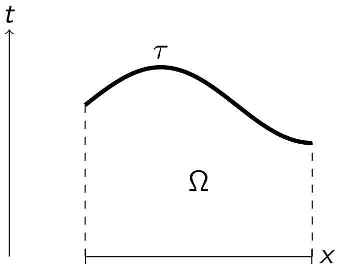

G^t suspension (semi)flow on Ω, preserves $\mu = \frac{1}{\nu(\tau)} \nu \otimes Leb$

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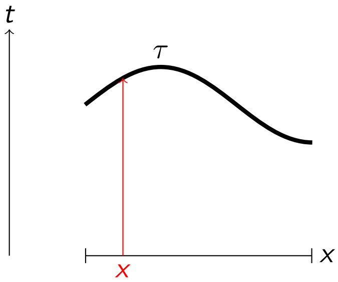

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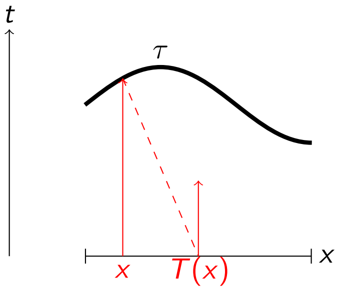

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Mixing

Theorem 1(Dolgopyat, N. '18 ETDS) *Assume* (T, τ) *satisfies MLLT and certain moderate deviation estimates. Then*

$$G^t \text{ is mixing iff } G^t \text{ is weakly mixing iff } \overline{< M(\tau), r(\tau) >} = \mathbb{R}$$

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Question: MLLT for G^t?
Let $\alpha : \Omega \to \mathbb{R}$
MLLT for (G^t, α): replace n by t, $S_n(f)$ by $\int_0^t \alpha(G^s(y))ds$. Let $\hat{M}, \hat{r}$ be the group and the translation

MLLT for suspension flows

Theorem 2(Dolgopyat, N. '18 ETDS) *Assume* $(T, (\int_0^{\tau(x)} \alpha(x,s)ds, \tau))$ *satisfies the MLLT with group* M *and translation* r *and certain moderate deviation estimates.*

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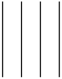
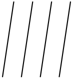
$$\overline{< M, r >} = \mathbb{R}^2$$ *or*  *or* 

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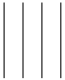
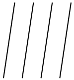
$$\overline{< M, r >} = \mathbb{R}^2 \text{ or}$$  *or* 

Remarks

- In case 1 $\hat{M} = \mathbb{R}$, no coboundary term is needed
- In case 2 $\hat{M} = c\mathbb{Z}$, $\hat{r} = 0$, coboundary term is needed (generalized MLLT)
- In case 3, $\hat{M} = c\mathbb{Z}, \hat{r} = 1/c$, coboundary term is needed
- In case $\overline{< M, r >}$ is a lattice, we have convergence along subsequences.

Counterexample for MLLT: a renewal reward process

(X_i, τ_i) iid uniform on
$\{z_1 = (-1, 2-\sqrt{2}), z_2 = (0,1), z_3 = (1, \sqrt{2})\}$

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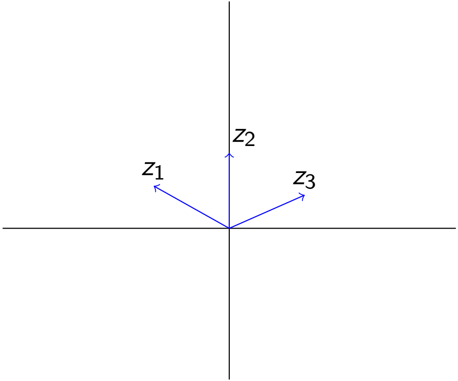

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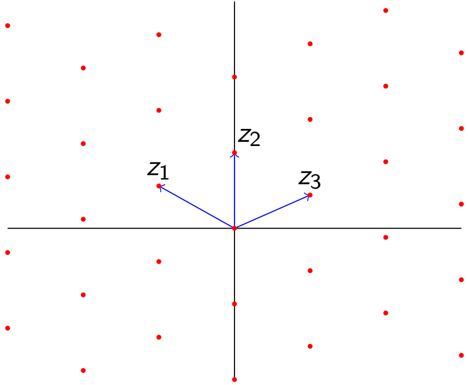

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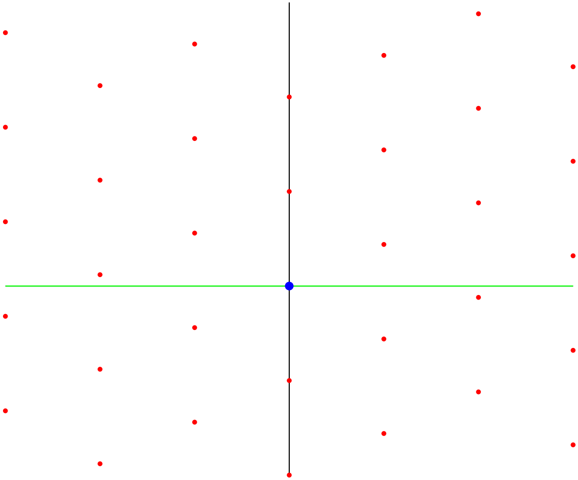

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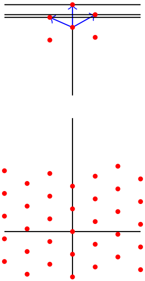

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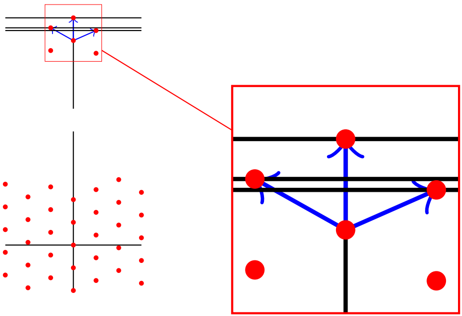

(M)LLT for hyperbolic systems

- Guivarc'h, Hardy '88: Anosov diffeomorphisms
- Aaronson, Denker '99: Stable MLLT for Gibbs Markov maps
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Examples: Sinai billiard flows, Axiom A flows, suspensions over LSV maps, geometric Lorenz attractor.

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Mixing of infinite measures

Krickeberg-Hopf mixing I

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G^t suspension flow on Ω, $\mu = \nu \otimes Leb$ σ-finite invariant measure.

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Theorem 4(Dolgopyat, N.' 18+ BLMS) *Assume* $\nu(\tau > t) \sim t^{-\alpha}$ *with* $\alpha \in (0, 1)$. *Assume furthermore that* (T, τ) *satisfies a stable version of the MLLT and some moderate deviation estimates. Then*

$$\mu(fg \circ G^t) \sim ct^{\alpha - 1} \mu(f) \mu(g)$$

for f, g *a.e. continuous and compactly supported on the fiber iff* $\overline{< M(\tau), r(\tau) >} = \mathbb{R}$.

Krickeberg-Hopf mixing II

Remarks:

- If $\alpha \in (0, 1/2)$, then $t^{-\alpha} \gg t^{\alpha-1}$. Only assuming $\nu(\tau > t) \sim t^{-\alpha}$, no reasonable system satisfies the conditions, not even iid (Garsia, Lamperti '62).
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Example: $\tilde{T} : [0,1] \to [0,1]$, LSV map with infinite invariant measure ($r > 1$), $\tilde{\tau}$ piecewise Hölder roof function. If $\tilde{\tau}$ is not cohomologous to a function supported on $b\mathbb{Z}$, then Krickeberg mixing holds for f, g with $supp f, supp g \subset [\varepsilon, 1]$.

Global mixing (Marco Lenci)

Informal Definitions Let $T : X \to X$ preserve an infinite measure ν. Fix a set $\mathbf{G} \subset L^\infty(X)$ so that

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T is *local global (l-g) mixing* if $\forall \phi \in L^1(X), \forall \Phi \in \mathbf{G}$

$$\lim_{n\to\infty} \int \phi(x)\Phi(T^n x) d\nu = \left(\int \phi d\nu \right) \bar{\Phi}.$$

T is *global global (g-g) mixing* if $\forall \Phi_1, \Phi_2 \in \mathbf{G}$

$$\lim_{n\to\infty} \liminf_{\nu(V)\to\infty} \left(\limsup_{\nu(V)\to\infty} \right) \frac{1}{\nu(V)} \int_V \Phi_1(x)\Phi_2(T^n x) d\nu = \bar{\Phi}_1 \bar{\Phi}_2.$$

$\mathbb{Z}^d$-extensions

- $X = M \times \mathbb{Z}^d$, M is a locally compact metric space
- $x = (y, z) \in X$ and $T(y, z) = (F(y), z + \kappa(y))$, F preserves a probability measure ν_0.
- $\nu = \nu_0 \times$ counting measure
- **Remark**: MLLT for (F, κ) implies that T is Krickeberg-Hopf mixing
- $\mathbf{G}_0$ set of bounded uniformly continuous functions whose averages over boxes containing $z = 0$ converge
- $\mathbf{G}_U$ set of bounded uniformly continuous functions whose averages over arbitrary boxes converge

MLLT implies global mixing

Theorem 5(Dolgopyat, N.' 18+) *If* (F, κ) *satisfies the MLLT, then l-g and g-g mixing holds for both* $\mathbf{G}_0, \mathbf{G}_U$.

E.g.: Lorentz gas (map & flow)

Theorem 6(Dolgopyat, N.' 18+) *If* $\tilde{T}$ *is very well (well) approximated by* T *at infinity and* T *is g-g (l-g) mixing with respect to* $\mathbf{G}_U$, *then so is* $\tilde{T}$.

Examples

	g-g mixing	l-g mixing
Locally perturbed Lorentz gas (Dolgopyat, Szász, Varjú '08)	map:$\mathbf{G}_0$ flow:$\mathbf{G}_0$	map:$\mathbf{G}_0$ flow:$\mathbf{G}_0$
Lorentz gas in potential fields (Chernov '11)	asymp. vanishing potential map:$\mathbf{G}_0$ flow:$\mathbf{G}_0$	small potential map:$\mathbf{G}_U$ flow:$\mathbf{G}_U$
Galton board (Chernov, Dolgopyat '09)	map:$\mathbf{G}_0$ flow:$\mathbf{G}_0$	large energy map:$\mathbf{G}_U$ flow:$\mathbf{G}_U$
Fermi-Ulam pingpong (De Simoi, Dolgopyat '12)	map:$\mathbf{G}_U$ flow:$\mathbf{G}_U$	?
Bouncing ball in gravity field (Zhou '18+)	map:$\mathbf{G}_U$ flow: NOT	?