The extremal index and the cluster size

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Consider a stationary stochastic process $X_0, X_1, X_2, \dots$ with marginal d.f. F and let $\bar{F} = 1 - F$.

In the Extreme Value Theory (EVT) we study the distributional properties of the maximum

$$M_n = \max\{X_0, \dots, X_{n-1}\} \tag{1}$$

as $n \to \infty$.

Definition

We say that we have an Extreme value law (EVL) for M_n if there is a non-degenerate d.f. $H : \mathbb{R} \to [0, 1]$ (with $H(0) = 0$) and for all $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$ such that

$$nP(X_0 > u_n) \to \tau \ \text{ as } n \to \infty, \tag{2}$$

and for which the following holds:

$$P(M_n \leq u_n) \to \bar{H}(\tau) \ \text{ as } n \to \infty. \tag{3}$$

.

The independent case

In the case $X_0, X_1, X_2, \dots$ are i.i.d. r.v. then since

$$P(M_n \le u_n) = (F(u_n))^n$$

we have that if (2) holds, then (3) holds with $\bar{H}(\tau) = e^{-\tau}$:

$$P(M_n \le u_n) = (1 - P(X_0 > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \to e^{-\tau} \ \text{ as } \ n \to \infty,$$

and vice-versa.

When the random variables $X_0, X_1, X_2, \dots$ are not independent but satisfy some mixing condition $D(u_n)$ introduced by Leadbetter then something can still be said about H.

Condition $D(u_n)$ from Leadbetter

Let $F_{i_1,\dots,i_n}$denote the joint d.f. of $X_{i_1},\dots,X_{i_n}$, and set $F_{i_1,\dots,i_n}(u) = F_{i_1,\dots,i_n}(u,\dots,u)$.

Condition ($D(u_n)$)

We say that $D(u_n)$ holds for the sequence $X_0, X_1, \dots$ if for any integers $i_1 < \dots < i_p$ and $j_1 < \dots < j_k$ for which $j_1 - i_p > t$, and any large $n \in \mathbb{N}$,

$$\left| F_{i_1,\dots,i_p,j_1,\dots,j_k}(u_n) - F_{i_1,\dots,i_p}(u_n) F_{j_1,\dots,j_k}(u_n) \right| \leq \gamma(n,t),$$

where $\gamma(n,t_n) \xrightarrow[n\to\infty]{} 0$, for some sequence $t_n = o(n)$.

Extremal Index

Theorem ([C81], see also [LLR83])

If $D(u_n)$ holds for $X_0, X_1, \ldots$ and the limit (3) *exists for some $\tau > 0$ then there exists $0 \leq \theta \leq 1$ such that $\bar{H}(\tau) = e^{-\theta\tau}$ for all $\tau > 0$.*

Definition

We say that $X_0, X_1, \ldots$ has an *Extremal Index* (EI) $0 \leq \theta \leq 1$ if we have an EVL for M_n with $\bar{H}(\tau) = \mathrm{e}^{-\theta\tau}$ for all $\tau > 0$.

Stationary stochastic processes arising from chaotic dynamics

Consider a discrete dynamical system

$$(\mathcal{X}, \mathcal{B}, P, f),$$

where

$\mathcal{X}$ is a topological space,

$\mathcal{B}$ is the Borel σ-algebra,

$f : \mathcal{X} \to \mathcal{X}$ is a measurable map,

P is an f-invariant probability measure.

In this context, we consider the stochastic process $X_0, X_1, \ldots$ given by

$$X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N}, \tag{4}$$

where $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ is an observable (achieving a global maximum at $\zeta \in \mathcal{X}$), of the form

$$\varphi(x) = g\,(\text{dist}(x, \zeta))\,, \tag{5}$$

where $\zeta \in \mathcal{X}$, “dist” denotes a Riemannian metric in $\mathcal{X}$ and the function $g : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ has a global maximum at 0 and is a strictly decreasing bijection for a neighborhood V of 0.

.

So, if at time $j \in \mathbb{N}$ we have an exceedance of the level u sufficiently large, i.e. $X_j(x) > u$, then we have an entrance of the orbit of x in the ball $B_{g^{-1}(u)}(\zeta)$ at time j, i.e. $f^j(x) \in B_{g^{-1}(u)}(\zeta)$.

The behaviour of $1 - F(u)$, as $u \to u_F$, depends on the form of g^{-1}.

Assuming $D(u_n)$ holds, let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{6}$$

Condition ($D'(u_n)$)

We say that $D'(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ satisfying (6) and such that

$$\limsup_{n\to\infty} n \sum_{j=1}^{[n/k_n]} P\{X_0 > u_n \text{ and } X_j > u_n\} = 0. \tag{7}$$

Theorem (Leadbetter)

Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D(u_n)$ and $D'(u_n)$ hold. Then

$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

Assuming $D(u_n)$ holds, let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that

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Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D(u_n)$ and $D'(u_n)$ hold. Then

$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

Motivated by the work of Collet [C01] we considered:

Condition ($D_2(u_n)$)

We say that $D_2(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if for any integers ℓ, t and n

$$|P\{X_0 > u_n \cap \max\{X_t, \ldots, X_{t+\ell-1} \leq u_n\}\} - P\{X_0 > u_n\}P\{M_\ell \leq u_n\}| \leq \gamma(n,t),$$

where $\gamma(n,t)$ is nonincreasing in t for each n and $n\gamma(n,t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$.

Theorem ([FF08a])

Let $\{u_n\}$ be such that $n(1-F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D_2(u_n)$ and $D'(u_n)$ hold. Then

$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

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Theorem ([FF08a])

Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D_2(u_n)$ and $D'(u_n)$ hold. Then

$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

Periodic points

Now, we assume that:

(R) $\zeta \in \mathcal{X}$ is a repelling periodic point of period $p \in \mathbb{N}$. The periodicity of ζ implies that for all u sufficiently large, $\{X_0 > u\} \cap \{X_p > u\} \neq \emptyset$ and $\{X_0 > u\} \cap \{X_j > u\} = \emptyset$ for all $j = 1, \ldots, p-1$.
We also suppose that we have backward contraction implying that there exists $0 < \theta < 1$ so that $\{X_0 > u\} \cap \{X_p > u\}$ is another ball of smaller radius around ζ with

$$P\left(\{X_0 > u\} \cap \{X_p > u\}\right) \sim (1-\theta)P(X_0 > u),$$

for all u sufficiently large.

Under this assumption, $D'(u_n)$ does not hold since

$$n \sum_{j=1}^{[n/k_n]} P(X_0 > u_n, X_j > u_n) \geq nP(X_0 > u_n, X_p > u_n) \to (1-\theta)\tau$$

Define the events

$$U(u) = \{X_0 > u\} \text{ and } A_{p,0}(u) := \{X_0 > u, X_p \leq u\}.$$

Observe that for u sufficiently large, $A_{p,0}(u)$ corresponds to an annulus centred at ζ.

Define the events: $A_{p,i}(u) := \{X_i > u, X_{i+p} \leq u\}$, and

$\mathcal{Q}_{p,s,\ell}(u) = \bigcap_{i=s}^{s+\ell-1} A_{p,i}^c(u)$.

Theorem (F, Freitas, Todd - [FFT12])

Let $(u_n)_{n\in\mathbb{N}}$ be such that $nP(X_0 > u_n) \to \tau$, for some $\tau \geq 0$. Suppose $X_0, X_1, \ldots$ is as in (4). Then

$$\lim_{n\to\infty} P(M_n \leq u_n) = \lim_{n\to\infty} P(\mathcal{Q}_{p,0,n}(u_n)) \tag{8}$$

- First observe that $\{M_n \leq u_n\} \subset \mathcal{Q}_{p,0,n}(u_n)$.
- It follows by stationarity that

$$P(\mathcal{Q}_{p,0,n}(u_n) \setminus \{M_n \leq u_n\}) \leq pP\left(X_0 > u_n\right) = p\frac{\tau}{n} \xrightarrow[n\to\infty]{} 0.$$

.

Condition ($\text{Д}_p(u_n)$)

We say that $\text{Д}_p(u_n)$ holds for$X_0, X_1, \ldots$ if for any ℓ, t and n

$$\left|P\left(A_{p,0}(u_n) \cap \mathscr{Q}_{p,t,\ell}(u_n)\right) - P(A_{p,0}(u_n))P(\mathscr{Q}_{p,0,\ell}(u_n))\right| \leq \gamma(n,t),$$

where $\gamma(n,t)$ is nonincreasing in t for each n and $n\gamma(n,t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$.

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that $k_n \to \infty$ and $k_n t_n = o(n)$.

Condition ($\text{Д}'_p(u_n)$)

We say that $\text{Д}'_p(u_n)$ holds for the sequence $X_0, X_1, X_2, \ldots$ if

$$\lim_{n\to\infty} n \sum_{j=1}^{[n/k_n]} P(A_{p,0}(u_n) \cap A_{p,j}(u_n)) = 0. \tag{9}$$

Theorem (F, Freitas, Todd - [FFT12])

Let $(u_n)_{n\in\mathbb{N}}$ *be such that* $nP(X_0>u_n)\to\tau$, *for some* $\tau\geq 0$. *Suppose* $X_0, X_1, \ldots$ *is as in* (4) *and (R) holds. Assume further that conditions* $Д_p(u_n)$ *and* $Д'_p(u_n)$ *hold. Then*

$$\lim_{n\to\infty} P(M_n\leq u_n)=\lim_{n\to\infty} P(\mathcal{Q}_{p,0,n}(u_n))=e^{-\theta\tau}, \qquad (10)$$

where $\theta=\lim_{n\to\infty}\frac{P(A_{p,0}(u_n))}{P(U(u_n))}$.

Consider the *Rare Event Point Process* (REPP) by counting the number of exceedances (or hits to $U(u_n)$) up to time nt:

$$N_n(t) := \sum_{j=0}^{[nt]} \mathbf{1}_{\{X_j > u_n\}}. \tag{11}$$

Consider the events

$$U^{(0)}(u) = U(u) \quad \text{and} \quad A_p^{(0)}(u) = \{X_0 > u, X_1 \leq u, ..., X_p \leq u\}.$$

Now let

$$U^{(k)}(u) = U^{(k-1)}(u) - A_p^{(k-1)}(u),$$
$$A_p^{(k)}(u) := U^{(k)}(u) \cap \bigcap_{i=1}^{p} f^{-i}\left((U^{(k)}(u))^c\right).$$

Condition ($Д_p(u_n)^*$)

We say that $Д_p(u_n)^*$ holds for the sequence $X_0, X_1, X_2, \ldots$ if for any integers $t, \kappa_1, \ldots, \kappa_\varsigma$, n and any $J = \cup_{i=2}^{\varsigma} I_j \in \mathcal{R}$ with $\inf\{x : x \in J\} \geq t$,

$$\left| P\left(A_p^{(\kappa_1)}(u_n) \cap \left(\cap_{j=2}^{\varsigma} \mathscr{N}_{u_n}(I_j) = \kappa_j\right)\right) - P\left(A_p^{(\kappa_1)}(u_n)\right) P\left(\cap_{j=2}^{\varsigma} \mathscr{N}_{u_n}(I_j) = \kappa_j\right) \right| \leq \gamma(n,t),$$

where $\mathscr{N}_{u_n}(I_j) = \sum_{i \in \mathbb{N} \cap I_j} \mathbf{1}_{\{X_j > u_n\}}$. for each n we have that $\gamma(n,t)$ is nonincreasing in t and $n\gamma(n,t_n) \to 0$ as $n \to \infty$, for some sequence $t_n = o(n)$.

Assuming $Д_p(u_n)^*$ holds, let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers such that $k_n \to \infty$ and $k_n t_n = o(n)$.

Condition ($Д'_p(u_n)^*$)

We say that $Д'_p(u_n)^*$ holds for the sequence $X_0, X_1, X_2, \ldots$ if

$$\lim_{n\to\infty} n \sum_{j=1}^{[n/k_n]} P(A_p^{(0)}(u_n) \cap \{X_j > u_n\}) = 0. \tag{12}$$

Theorem (F, Freitas, Todd - [FFT13])

Let $(u_n)_{n\in\mathbb{N}}$ be such that $nP(X_0 > u_n) \to \tau$, for some $\tau \geq 0$. Suppose $X_0, X_1, \ldots$ is as in (4) and (R) holds. Assume that conditions $Д_p(u_n)^*$, $Д'_p(u_n)^*$ hold.
Then, the REPP N_n converges in distribution to a compound Poisson process N with intensity $\theta\tau$ and multiplicity d.f. π given by $\pi(\kappa) = \theta(1-\theta)^{\kappa-1}$, for every $\kappa \in \mathbb{N}$, where the extremal index θ is given by $\theta = \lim_{n\to\infty} \frac{P(A_p^{(0)}(u_n))}{P(U^{(0)}(u_n))}$.

If $p = 0$, we obtain the result of [FFT10]: under a condition $D_3(u_n)$ and $D'(u_n)$, the REPP N_n converges in distribution to a (simple) Poisson process.

Systems to which we can apply directly the above result are:

- uniformly expanding maps on the circle/interval;
- piecewise expanding maps, like Rychlik maps;
- higher dimensional expanding maps studied by Saussol (2000) ([S00]).

For these type of systems, we then have the following:

- if the point ζ is non periodic, then the REPP N_n converges in distribution to a Poisson process.

- if the point ζ is periodic, then the REPP N_n converges in distribution to a compound Poisson process.

In [FFTV16] we studied the limiting process for the REPP N_n in the case of a simple non-uniformly hyperbolic dynamical system, the Manneville-Pomeau (MP) map equipped with an absolutely continuous invariant probability measure.

The form for such map that we studied is the one considered in [LSV99], and given by

$$f = f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{for } x \in [0, 1/2) \\ 2x - 1 & \text{for } x \in [1/2, 1] \end{cases}$$

for $\alpha \in (0, 1)$.

In [FFTV16] we have proved that for this map and for $\zeta \in (0, 1]$:

- if the point ζ is non periodic, then the REPP N_n converges in distribution to a Poisson process.

- if the point ζ is periodic, then the REPP N_n converges in distribution to a compound Poisson process with intensity $\theta\tau$ for $\theta = 1 - |D(f^{-p})(\zeta)|$ and multiplicity distribution function π given by $\pi_\kappa = \theta(1-\theta)^{\kappa-1}$, for every $\kappa \in \mathbb{N}$.

We recall that if a r.v. $D \sim Ge(\theta)$ then $\mathbb{E}(D) = 1/\theta$ and so $\theta = 1/\mathbb{E}(D)$.

Even in more general cases, typically, the extremal index coincides with the inverse of the mean of the limiting cluster size distribution.

In a very recent paper ([AFF18]) we built a counterexample for that.

The idea was to make a balanced mixture of a behaviour associated with an extremal index equal to 0 with the behaviour of an extremal index different from 0.

For that, we considered the LSV map and assumed that the observable φ was maximized at two points. One of them was $\zeta_1 = 0$ and for the other one we considered two cases:

1) $\zeta_2 \in]1/2, 1]$ such that $f^j(\zeta_2) \notin \{\zeta_1, \zeta_2\}, \forall j \in \mathbb{N}$,

2) $\zeta_2 \in]1/2, 1]$ such that for some $p \in \mathbb{N}$, $f^j(\zeta_2) = \zeta_2$ and $f^j(\zeta_2) \neq \zeta_2, \forall j \in \{1, \ldots, p-1\}$.

Study of case 1)

We start by noting that if the observable was maximized at the single point $\zeta_1 = 0$, then the extremal index would be equal to 0.

And if the observable was maximized at a single non periodic point ζ_2, then the extremal index would be equal to 1.

Here we consider the case where the chosen observable is maximized at the two points ζ_1 and ζ_2.

Study of case 1)

Theorem (Abadi, F, Freitas - 2018)

Consider the LSV map for some $0 < \alpha < \sqrt{5} - 2$. *Let* $(u_n)_{n\in\mathbb{N}}$ *be such that* $nP(X_0 > u_n) \to \tau$, *for some* $\tau \geq 0$. *Suppose* $X_0, X_1, \ldots$ *is as in* (4) *for an observable* φ *conveniently chosen and maximized at the points* ζ_1 *and* ζ_2.
Then, this process admits an EI $\theta = 1/2$. *Moreover the REPP* N_n *converges in distribution to a Poisson process N with intensity* $\theta\tau$.

So, in this case, $\theta = 1/2$.

The multiplicity distribution is given by $\pi(1) = 1$ and $\pi(\kappa) = 0$ for $\kappa \geq 2$ and so the corresponding mean is equal to 1.

Then, this is an example for which θ does not coincide with the inverse of the mean of the limiting cluster size distribution.

Study of case 2)

In this case, if the observable was maximized at the single point $\zeta_1 = 0$, then the extremal index would be equal to 0.

If the observable was maximized at a single periodic point ζ_2 of period p, then the extremal index would be equal to a certain $0 < \theta_2 < 1$.

Here we consider the case where the chosen observable is maximized at the two points ζ_1 and ζ_2.

Study of case 2)

Theorem (Abadi, F, Freitas - 2018)

Consider the LSV map for some $0 < \alpha < \sqrt{5} - 2$. Let $(u_n)_{n\in\mathbb{N}}$ be such that $nP(X_0 > u_n) \to \tau$, for some $\tau \geq 0$. Suppose $X_0, X_1, \ldots$ is as in (4) for an observable φ conveniently chosen and maximized at the points ζ_1 and ζ_2.
Then, this process admits an EI $\theta = \frac{1}{2}(1 - \gamma^{-1})$, where $\gamma = Df^p(\zeta_2)$. Moreover the REPP N_n converges in distribution to a compound Poisson process N with intensity $\theta\tau$ and multiplicity distribution given by $\pi(\kappa) = (1 - \gamma^{-1})(\gamma^{-1})^{\kappa-1}$, $\forall\kappa \in \mathbb{N}$.

So, in this case, $\theta = \frac{1}{2}(1 - \gamma^{-1})$ and the multiplicity follows a geometric distribution with parameter $1 - \gamma^{-1}$ (that is, with mean $(1 - \gamma^{-1})^{-1}$)

Then, the extremal index does not coincide with the inverse of the mean of the limiting cluster size distribution is the parameter.

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Decay of correlations implies $D_2(u_n)$

Suppose that there exists a nonincreasing function $\gamma : \mathbb{N} \to \mathbb{R}$ such that for all $\phi : \mathcal{X} \to \mathbb{R}$ with bounded variation and $\psi : \mathcal{X} \to \mathbb{R} \in L^\infty$, there is $C > 0$ independent of ϕ, ψ and n such that

$$\left| \int \phi \cdot (\psi \circ f^t) d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C \mathrm{Var}(\phi) \|\psi\|_\infty \gamma(t), \quad \forall t \geq 0, \tag{13}$$

where Var(ϕ) denotes the total variation of ϕ and $n\gamma(t_n) \to 0$, as $n \to \infty$ for some sequence $t_n = o(n)$.

Taking $\phi = \mathbf{1}_{\{X > u_n\}}$ and $\psi = \mathbf{1}_{\{M_\ell \leq u_n\}}$, then

$$(13) \Rightarrow D_2(u_n),$$

(with $\gamma(n, t) = C\mathrm{Var}(\mathbf{1}_{\{X > u_n\}}) \|\mathbf{1}_{\{M_\ell \leq u_n\}}\|_\infty \gamma(t) \leq C' \gamma(t)$ and for the sequence $\{t_n\}$ such that $t_n/n \to 0$ and $n\gamma(t_n) \to 0$ as $n \to \infty$).

Decay of correlations against L^1 implies $Д'_p(u_n)$

Suppose that there exists a nonincreasing function $\gamma : \mathbb{N} \to \mathbb{R}$ such that for all $\phi : \mathcal{X} \to \mathbb{R}$ with bounded variation and $\psi : \mathcal{X} \to \mathbb{R} \in L^1$, there is $C > 0$ independent of ϕ, ψ and n such that

$$\left| \int \phi \cdot (\psi \circ f^t) d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C \mathrm{Var}(\phi) \|\psi\|_1 \gamma(t), \quad \forall t \geq 0, \tag{14}$$

where $\mathrm{Var}(\phi)$ denotes the total variation of ϕ and $n\gamma(t_n) \to 0$, as $n \to \infty$ for some sequence $t_n = o(n)$.

Taking $\phi = \mathbf{1}_{A_p(u_n)}$ and $\psi = \mathbf{1}_{A_p(u_n)}$, then

$$(14) \Rightarrow Д'_p(u_n),$$

$P(A_{p,0}(u_n) \cap A_{p,j}(u_n)) \leq P(A_{p,0}(u_n))^2 + C' P(A_{p,0}(u_n))\gamma(j) \lesssim (\tau/n)^2 + C'(\tau/n)\gamma(j)$.

So, $n \sum_{j=R_n}^{n/k_n} P(A_{p,0}(u_n) \cap A_{p,j}(u_n)) \lesssim \frac{n^2}{k_n}(\tau/n)^2 + n \sum_{j=R_n}^{\infty} C'(\tau/n)\gamma(j) = \tau^2/k_n + C'\tau \sum_{j=R_n}^{\infty} \gamma(j) \to_{n\to\infty} 0$ if we check that $\lim_{n\to\infty} R_n = +\infty$ (for non-periodic points this is true if for example the map f is continuous at every point of the orbit of ζ; for periodic points it is enough to be a repelling periodic point which implies the existence of the derivative ...) - R_n is the first return time of the set to itself.

Proof of the extreme value law

This proof is for the case of no clustering. In the case of clustering we just have to replace balls by annulli.
Let k be the number of big blocks, let $\ell = \ell_n = [\frac{n}{k}]$ be the approximate size of each block where $[\frac{n}{k}]$ is the integer part of $\frac{n}{k}$ and let t be the size of the small blocks. We begin by replacing $P(M_n \leq u_n)$ by $P(M_{k(\ell+t)} \leq u_n)$ for some $t > 1$. We have

$$\left|P(M_n \leq u_n) - P(M_{k(\ell+t)} \leq u_n)\right| \leq ktP(X > u_n). \tag{15}$$

We now estimate recursively $P(M_{i(\ell+t)} \leq u_n)$ for $i = 0, \ldots, k$. Using a Lemma and stationarity, we have for any $1 \leq i \leq k$

$$\left|P(M_{i(\ell+t)} \leq u_n) - (1 - \ell P(X > u_n))P(M_{(i-1)(\ell+t)} \leq u_n)\right| \leq \Gamma_{n,i},$$

where

$$\Gamma_{n,i} = \left|\ell P(X > u_n)P(M_{(i-1)(\ell+t)} \leq u_n) - \sum_{j=0}^{\ell-1} P\left(\{X_j > u_n\} \cap \{M_{\ell+t,(i-1)(\ell+t)} \leq u_n\}\right)\right|$$
$$+ tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P\left(\{X > u_n\} \cap \{X_j > u_n\}\right).$$

Using stationarity, $D(u_n)$ and, in particular, that $\gamma(n,t)$ is nonincreasing in t for each n we conclude

$$\Gamma_{n,i} \leq \sum_{j=0}^{\ell-1} \left|P(X_0 > u_n)P(M_{(i-1)(\ell+t)} \leq u_n) - P\left(\{X_0 > u_n\} \cap \{M_{\ell+t-j,(i-1)(\ell+t)} \leq u_n\}\right)\right|$$
$$+ tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P\left(\{X > u_n\} \cap \{X_j > u_n\}\right)$$
$$\leq \ell\gamma(n,t) + tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P\left(\{X > u_n\} \cap \{X_j > u_n\}\right).$$

Define $\Upsilon_n = \ell\gamma(n,t) + tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P\left(\{X > u_n\} \cap \{X_j > u_n\}\right)$. Then for every $1 < i \le k$ we have

$$\left|P(M_{i(\ell+t)} \le u_n) - (1 - \ell P(X > u_n))P(M_{(i-1)(\ell+t)} \le u_n)\right| \le \Upsilon_n$$

and for $i = 1$

$$\left|P(M_{(\ell+t)} \le u_n) - (1 - \ell P(X > u_n))\right| \le \Upsilon_n.$$

Assume that k and n are large enough in order to have $\ell P(X > u_n) < 2$, which implies that $\left|1 - \ell P(X > u_n)\right| < 1$. A simple inductive argument allows to conclude

$$\left|P(M_{k(\ell+t)} \le u_n) - (1 - \ell P(X > u_n))^k\right| \le k\Upsilon_n.$$

Then we have

$$\left|P(M_n \le u_n) - (1 - \ell P(X > u_n))^k\right| \le ktP(X > u_n) + k\Upsilon_n. \tag{16}$$

Since $nP(X > u_n) = n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \ge 0$, we have

$$\lim_{k\to\infty} \lim_{n\to\infty} \left(1 - [\tfrac{n}{k}]P(X > u_n)\right)^k = \lim_{k\to\infty} (1 - \tfrac{\tau}{k})^k = e^{-\tau}.$$

Now, observe that $nP(X > u_n) = n(1 - F(u_n)) \to \tau$ is equivalent to $P(\hat{M}_n \le u_n) = (F(u_n))^n \to e^{-\tau}$, where the limits are taken when $n \to \infty$ and $\tau \ge 0$ (see [LLR83], Theorem 1.5.1] for a proof of this fact). Hence,

$$\lim_{k\to\infty} \lim_{n\to\infty} (1 - [\tfrac{n}{k}]P(X > u_n))^k = \lim_{n\to\infty} P(\hat{M}_n \le u_n). \tag{17}$$

It is now clear that, according to (16) and (17), M_n and $\hat{M}_n$ share the same limiting distribution if

$$\lim_{k\to\infty} \lim_{n\to\infty} (ktP(X > u_n) + k\Upsilon_n = 0,$$

that is

$$\lim_{k\to\infty} \lim_{n\to\infty} 2ktP(X > u_n) + n\gamma(n,t) + 2n\sum_{j=1}^{\ell} P\left(\{X > u_n\} \cap \{X_j > u_n\}\right) = 0. \tag{18}$$

Assume that $t = t_n$ where $t_n = o(n)$ is given by Condition $D(u_n)$. Then, for every $k \in \mathbb{N}$, we have $\lim_{n\to\infty} kt_nP(X > u_n) = 0$, since $nP(X > u_n) \to \tau \ge 0$. Finally, we use $D(u_n)$ and $D'(u_n)$ to obtain that the two remaining terms in (18) also go to 0.

Convergence in distribution of a point process

A point process N_n converges in distribution to a point process N if and only if for any s disjoint intervals $I_1, \ldots, I_s$ ($I_j = [a_j, b_j)$), the joint distribution of N_n over these intervals converge to the joint distribution of N over the same intervals, i.e.

$$(N_n(I_1), \ldots, N_n(I_s)) \to^D (N(I_1), \ldots, N(I_s))$$

that is,

$$P(N_n(I_1) = k_1, \ldots, N_n(I_s) = k_s) \to P(N(I_1) = k_1, \ldots, N(I_s) = k_s).$$

This is equivalent to show that the joint moment function of $N_n(I_1), \ldots, N_n(I_s)$ converge to the joint moment generating function of $N(I_1), \ldots, N(I_s)$.

Definition of a compound Poisson process

Definition

We say that $\{N(t)\}_{t \geq 0}$ is a compound Poisson process of intensity θ and multiplicity d.f. π if we may write

$$N(t) = \sum_{i=1}^{M(t)} D_i$$

where $\{M(t)\}_{t \geq 0}$ is a Poisson process of intensity θ and $D_1, D_2, \ldots$ is a sequence of i.i.d. r.v.'s with d.f. π, which are independent of $M(t)$.

In our case, D_i corresponds to the size of the cluster i and $M(t)$ to the number of clusters observed up to time t.

Theorem

Let $X_0, X_1, \ldots$ satisfy conditions $Д_q(u_n)^*$ and $Д'_q(u_n)^*$, where $(u_n)_{n\in\mathbb{N}}$ is such that $nP(X_0 > u_n) \to \tau$, for some $\tau > 0$. Assume that the limit $\theta = \lim_{n\to\infty} \theta_n$ exists, where $\theta_n = \frac{P(A_q^{(0)}(u_n))}{P(U(u_n))}$, and moreover that for each $\kappa \in \mathbb{N}$, the following limit also exists

$$\lim_{n\to\infty} \pi_n(\kappa) = \lim_{n\to\infty} \frac{P(A_q^{(\kappa-1)}(u_n)) - (A_q^{(\kappa)}(u_n))}{P(A_q^{(0)}(u_n))}. \tag{19}$$

Then the REPP N_n converges in distribution to a compound Poisson process with intensity $\theta\tau$ and multiplicity distribution π given by (19).

Convergence of the REPP for the intermittent map

The method was to use inducing techniques, extending a result of [HWZ14] (for hitting times).

We proved that if for the first return time induced map the REPP converges to a certain limiting point process, then for the original system the REPP converges N_n to the same limiting point process.

Definition

Given a sequence of sets $(U_n)_{n\in\mathbb{N}}$ so that $P(U_n) \to 0$, the system has *RTS* $\tilde{G}$ for $(U_n)_{n\in\mathbb{N}}$ if for all $t \geq 0$

$$P_{U_n}\left(r_{U_n} \leq \frac{t}{P(U_n)}\right) \to \tilde{G}(t) \text{ as } n \to \infty. \tag{20}$$

and the system has *HTS* G for $(U_n)_{n\in\mathbb{N}}$ if for all $t \geq 0$

$$P\left(r_{U_n} \leq \frac{t}{P(U_n)}\right) \to G(t) \text{ as } n \to \infty, \tag{21}$$

We say that the system has HTS G for balls centred at ζ if we have HTS G for $(U_n)_n = (B_{\delta_n}(\zeta))_n$, for any sequence $(\delta_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that $\delta_n \to 0$ as $n \to \infty$.

Theorem

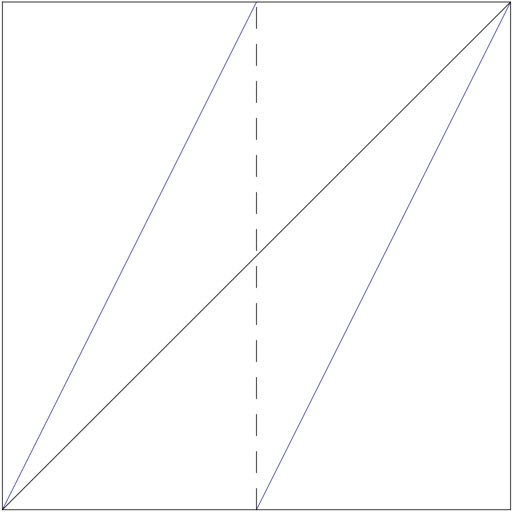
Consider an unperturbed map $T_\beta(x) = \beta x \bmod 1$ *for* $\beta > 1 + c$*, with* $c > 0$*, with invariant absolutely continuous probability* $\mu = \mu_\beta$ *with respect to Lebesgue measure m. Consider a sequential system acting on the unit circle and given by* $\mathcal{T}_n = T_n \circ \cdots \circ T_1$*, where* $T_i = T_{\beta_{i-1}}$*, for all* $i = 1, \ldots, n$ *and* $|\beta_n - \beta| \le n^{-\xi}$ *holds for some* $\xi > 1$*. Let* $X_1, X_2, \ldots$ *be as before, where the observable function* φ*, given by* (5)*, achieves a global maximum at a chosen* $\zeta \in [0, 1]$*. Let* $(u_n)_{n\in\mathbb{N}}$ *be such that* $n\mu(X_0 > u_n) \to \tau$*, as* $n \to \infty$ *for some* $\tau \ge 0$*. Then, there exists* $0 < \theta \le 1$ *such that*

$$\lim_{n\to\infty} m(X_0 \le u_n, X_1 \le u_n, \ldots, X_{n-1} \le u_n) = e^{-\theta\tau}.$$

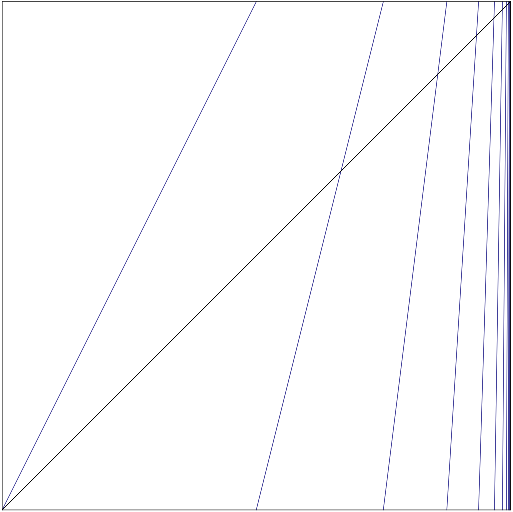
The value of θ *is determined by the behaviour of* ζ *under the original dynamics* T_β*, namely,*

- *If the orbit of* ζ *by* T_β *never hits* $0 \sim 1$ *and* ζ *is periodic of prime period p then* $\theta = 1 - \beta^{-p}$*;*
- *If the orbit of* ζ *by* T_β *never hits* $0 \sim 1$ *and* ζ *is not periodic then* $\theta = 1$*.*

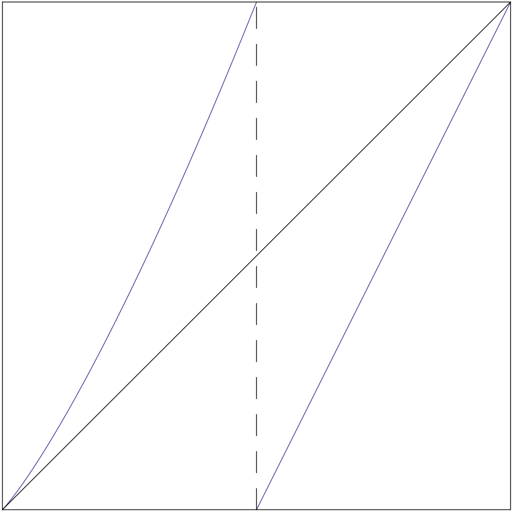
Doubling map



Rychlik map



Intermittent map



Benedicks-Carleson maps