MULTIPLICATIVE ERGODICITY OF LAPLACE TRANSFORMS FOR ADDITIVE FUNCTIONAL OF MARKOV CHAINS WITH APPLICATION TO AGE-DEPENDENT BRANCHING PROCESS.

Sana Louhichi

University Grenoble Alpes

Based on common works with *Bernard Ycart (2015)* and *Loïc Hervé* & *Françoise Pène (2017)*

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Introduction: classical model

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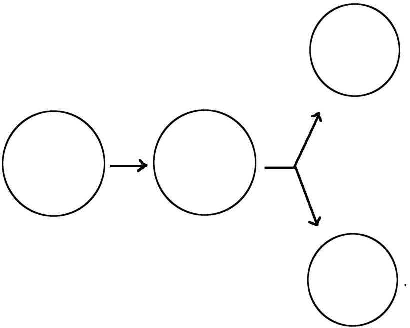
3. The time of birth of a child coincides with the time of death of the parent.



The process continue.

N_t: the number of objects present at time t.

$(N_t)_{t\geq 0}$: age-dependent branching process.

Age-dependent branching processes are mathematical models for

1. **Cell division:** mitosis
2. **Microbiology (growth of bacteria):** Certain phases of the multiplication of colonies of bacteria
3. **Models for the reproduction**

Purpose: asymptotic behavior of the process $(N_t)_{t\geq 0}$

We summarize the model, as follows,

(A) to each cell v, is associated a parameter $x_v \in \mathbb{X}$, called its characteristics, (with $(\mathbb{X}, \mathcal{X})$ a measurable space) which determines its lifetime $\xi(x_v)$ and the number of new cells $\kappa(x_v)$ in which the cell splits at the end of its lifetime (where ξ and κ are two measurable functions with values in $[0, +\infty)$ and in $\mathbb{Z}_+$ respectively);

(B) there exists a process $(X_n)_n$ with values in $\mathbb{X}$ such that, for each line $(v_n)_{n\geq 0}$ of cells, the characteristics along this line is given by a copy of $(X_n)_{n\geq 0}$ (these copies are not assumed to be mutually independent);

(C) $\kappa(x) \geq 2$ for any x, i.e. each cell gives birth to more than two children.

Classical assumptions and results about (N_t)

- **Classical assumptions: Bellman and Harris(1952). when the lifetimes are modeled by a sequence of i.i.d. random variables independent of the random numbers of the news cells which are also assumed to be i.i.d.**
- **Results:** N_t is a.s. asymptotically exponential:

$$N_t \sim \mathbf{C} e^{\nu_0 t} \mathbf{W} \text{ as } t \text{ tends to infinity.}$$

ν_0 which determines the exponential rate of growth called *Malthusian parameter.*

Thomas Robert **Malthus** (1766-1834): a British economist. *In 1798 published the "Essay on the Principle of Population," which argued that population multiplies exponentially or geometrically and food arithmetically. Therefore, the population will outstrip the food supply.* Contributions : Malthusian growth model

- The growth rate ν_0 (the Malthusian parameter) was defined, in this context, as the positive root of the equation,

$$\mathbb{E}[\kappa(X_1)]\,\mathbb{E}\left[e^{-\nu_0\xi(X_1)}\right] = 1, \tag{1}$$

as soon as the distribution of $\xi(X_1)$ is not lattice.

- The constant C equals to

$$C = \frac{\mathbb{E}[\kappa(X_1)] - 1}{\nu_0\left(\mathbb{E}[\kappa(X_1)]\right)^2 \mathbb{E}\left[\xi(X_1)e^{-\nu_0\xi(X_1)}\right]}$$

- W is a positive random variable with finite second moment and $\mathbb{E}(W) = 1$. (Harris (1963))

Explicit calculations for the Malthusian parameter in the iid case

- if $\kappa(x) = 2$ for any x, then

$$E(e^{-\nu_0 \xi(X_1)}) = \frac{1}{2}, \;\; C = \frac{1}{4\nu_0 E(\xi(X_1) e^{-\nu_0 \xi(X_1)})}.$$

- if $\kappa(x) = 2$ for any x and in the exponential case: $\xi(X_1) \sim \mathcal{E}(\lambda)$, then

$$\nu_0 = \lambda, \;\; C = 1$$

Louhichi and Ycart (2015) extend some results of Harris to the case where the lifetimes are a sequence of dependent random variables and when each cell is divided, after a random lifetime, into two cells: $(X_n)_n$ is a stationary process and $\kappa(x) = 2$ for any x: ν_1 is expressed in terms of the Laplace transform of S_n

$$\mathbf{S_n} := \sum_{\mathbf{k=0}}^{\mathbf{n}} \xi(\mathbf{X_k}) \tag{2}$$

which models the birth date of the $(n+1)$-th individual of a same line. More precisely,

$$\nu_1 = \inf\left\{\gamma > 0,\ \sum_{n\geq 0} 2^n \mathbb{E}\left[e^{-\gamma S_n}\right] < \infty\right\}. \tag{3}$$

$$\lim_{\gamma\to 0} \frac{\gamma}{\gamma+\nu_1} \sum_{n=1}^{\infty} 2^{n-1}\mathbb{E}(e^{-(\gamma+\nu_1)S_n}) =: \mathbf{C}_{\nu_1} < \infty.$$

Extension to random dependent variables $(\kappa(X_n))_n$ (I)- Calculation of $\mathbb{E}(N_t)$ in a general setting

We suppose $\kappa(x) \geq 2$ for any x and only the stationary assumption: for each line of cells $(0, v_1, \cdots, v_{n-1})$ (n-sequence of the form parent-child),

$$(X_0, X_{v_1}, \cdots, X_{v_{n-1}}) \sim (X_0, X_1, \cdots, X_{n-1})$$

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Proposition (Hervé, Louhichi & Pène (2017))

Let $t > 0$ be fixed. If $\sum_{n\geq 0} \mathbb{E}\left[\left(\prod_{j=0}^{n} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t\}}\right] < \infty$, then $\mathbb{E}[N_t] < \infty$ and

$$\mathbb{E}[N_t] = 1 + \sum_{n\geq 0} \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right) (\kappa(X_n) - 1)\, \mathbf{1}_{\{S_n \leq t\}}\right] \qquad (4)$$

(with the usual convention $\prod_{j=0}^{-1} \kappa(X_j) = 1$).

we obtain the following exponential behavior in mean of $\mathbb{E}[N_t]$ in a very general setting of dependence with the use of the function G given by

$$G(\gamma) := \sum_{n\geq 0} g_n(\gamma), \tag{5}$$

$$g_n(\gamma) = \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j)\right)(\kappa(X_n) - 1)e^{-\gamma S_n}\right].$$

$$\nu = \inf\{\gamma > \mathbf{0},\ \mathbf{G}(\gamma) < \infty\}.$$

$$\mathbf{C}_\nu := \lim_{\gamma\to\mathbf{0}} \frac{\gamma}{\gamma+\nu}\mathbf{G}(\nu+\gamma).$$

Corollary

Assume the stationary assumption and that $\nu < \infty$ and that the following limit exists

$$C_\nu := \lim_{\gamma\to 0} \frac{\gamma}{\gamma+\nu} G(\nu+\gamma)\,. \tag{6}$$

Then,
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-\nu s}\mathbb{E}[N_s]ds = C_\nu. \tag{7}$$

Proof. Let $\tilde{A}_\nu$ be the Laplace transform of A_ν. As a particular case of Feller (1971):

$$\lim_{\gamma\searrow 0} \gamma\tilde{A}_\nu(\gamma) = C \iff \lim_{t\to+\infty} \frac{1}{t}\int_0^t A_\nu(s)\,\mathrm{d}s = C\ .$$

Here $A_\nu(s) = e^{-\nu s}\mathbb{E}[N_s]$.

Outline of the proof of the calculation on $\mathbb{E}(N_t)$

Harris's approach in the iid case. The result follows from the renewal equation, denoting by $m(t) = \mathbb{E}(N_t)$,

$$m(t) = \mathbb{P}(\xi(X_0) > t) + \mathbb{E}(\kappa(X_0)) \int_0^t m(t-u) dF_{\xi(X_0)}(u)$$

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Proof. Based on a control of the generating function $F(s,t) = \mathbb{E}(s^{N_t}),\ T_0 = \xi(X_0)$,

$$N_t \mathbb{I}_{T_0<t} = \sum_{i=1}^{\kappa(X_0)} N_i(t) \mathbb{I}_{T_0<t},\ (N_i(t)\mathbb{I}_{T_0<t})_i \ \ iid \ \ \sim N_{t-T_0}.$$

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$$\mathbb{E}(s^{N_t} \mathbb{I}_{T_0<t}) = \mathbb{E}\left[\left(\mathbb{E}(s^{N_{t-T_0}} \mathbb{I}_{T_0<t}) \right)^{\kappa(X_0)} \right].$$

$$F(s,t) = s\mathbb{P}(T_0 > t) + \int_0^t \mathbb{E}[(F(s,t-u))^{\kappa(X_0)}] dF_{T_0}(u),\ \ |s| \le 1,\ \ t \ge 0.$$

Direct calculations. For every $n \geq 0$, we write $\Sigma_n(t)$ for the number of cells of generation n alive at time t. Observe that $\mathbb{E}[\Sigma_0(t)] = \mathbb{P}(\xi(X_0) > t)$ and that, for every $n \geq 1$ (with the convention $k_0 = 0$),

$$\mathbb{E}[\Sigma_n(t)] = \mathbb{E}\left[\sum_{k_1=1}^{D_0}\sum_{k_2=1}^{D_{0,k_1}} .. \sum_{k_n=1}^{D_{0,k_1,\cdots,k_{n-1}}}\right.$$

$$\left.\mathbf{1}_{\{T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_{n-1}}\leq t< T_0+T_{0,k_1}+\cdots+T_{0,k_1,\cdots,k_n}\}}\right]$$

$$= \mathbb{E}\left[D_0 D_{0,1}\cdots D_{0,1^{n-1}}\mathbf{1}_{\{T_0+T_{0,1}+\cdots+T_{0,1^{n-1}}\leq t< T_0+T_{0,1}+\cdots+T_{0,1^n}\}}\right]$$

$$= \mathbb{E}\left[\left(\prod_{j=0}^{n-1}\kappa(X_j)\right)\left(\mathbf{1}_{\{S_{n-1}\leq t\}} - \mathbf{1}_{\{S_n\leq t\}}\right)\right].$$

In order to study the function $G(\cdot)$, and so ν and C_ν, we adapt the notion of "multiplicative ergodicity", as introduced in Kontoyiannis & Meyn (2003-2005) to our context.

Definition

Let $\gamma_1 > 0$. We say that $(S_n, \kappa(X_n))_n$ is **multiplicatively ergodic** on $J = [0, \gamma_1)$ if there exist two continuous maps A and ρ from J to $(0, +\infty)$ such that, for every compact subset K of $(0, \gamma_1)$, there exist $M_K > 0$ and $\theta_K \in (0, 1)$ such that, for every $n \geq 1$,

$$\forall \gamma \in K, \quad |g_n(\gamma) - A(\gamma)(\rho(\gamma))^n| \leq M_K(\rho(\gamma)\theta_K)^n. \qquad (8)$$

When $\kappa(\cdot)$ is constant, we will simply say that $(S_n)_n$ is multiplicatively ergodic on J.

Remark

Assume that $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $J = [0, \gamma_1)$. Then

- *For every $\gamma \in J$ we have:*
  $G(\gamma) = \sum_{n \geq 0} g_n(\gamma) < \infty \iff \rho(\gamma) < 1.$
- *For every compact subset K of J, we obtain that*

$$\forall \gamma \in K \cap (\nu, +\infty), \quad \left| G(\gamma) - \frac{A(\gamma)}{1 - \rho(\gamma)} \right| \leq \frac{M_K}{1 - \rho(\gamma)\theta_K}.$$

- $\nu < \gamma_1$ means that

$$\nu = \inf\{\gamma \in \mathbf{J} \ : \ \rho(\gamma) < \mathbf{1}\} < \gamma_{\mathbf{1}}.$$

- If moreover ρ is differentiable at ν with $\rho(\nu) = 1$ and $\rho'(\nu) \neq 0$, then (6) follows with

$$\mathbf{C}_\nu = -\frac{\mathbf{A}(\nu)}{\nu\rho'(\nu)}.$$

Multiplicative ergodicity property and additive functional of Markov chains

The multiplicative ergodicity property is specially adapted for additive functional of Markov chains, that is: $X = (X_n)_n$ is a Markov chain on $(\mathbb{X}, \mathcal{X})$ with Markov kernel $P(x, dy)$, invariant probability π, and initial distribution μ (i.e. μ is the distribution of X_0).

• *The Laplace kernel associated with* (P, ξ, κ). We assume that, for every $n \geq 1$, the random variable $\prod_{j=0}^{n} \kappa(X_j)$ is integrable. We set $h_{\kappa,\gamma} := \big(\kappa - 1\big)\, e^{-\gamma\xi}$. Let $\gamma \in (0, +\infty)$. For $n \geq 1$,

$$\begin{aligned} g_n(\gamma) &= \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j) e^{-\gamma\xi(X_j)}\right) h_{\kappa,\gamma}(X_n)\right] \\ &= \mathbb{E}\left[\left(\prod_{j=0}^{n-1} \kappa(X_j) e^{-\gamma\xi(X_j)}\right) (Ph_{\kappa,\gamma})(X_{n-1})\right], \end{aligned}$$

with $(Ph)(x) := \int_{\mathbb{X}} h(y)\, P(x, dy)$.

If $n \geq 2$, we continue and obtain

$$g_n(\gamma) = \mathbb{E}\left[\left(\prod_{j=0}^{n-2} \kappa(X_j) e^{-\gamma \xi(X_j)}\right)(P_\gamma(Ph_{\kappa,\gamma}))(X_{n-2})\right],$$

with $P_\gamma h := P(h \kappa e^{-\gamma\xi})$. An easy induction gives

$$\forall n \geq 1, \quad g_n(\gamma) = \mu\left(\kappa\, e^{-\gamma\xi}\, P_\gamma^{n-1}\,(Ph_{\kappa,\gamma})\right). \qquad (9)$$

The multiplicative ergodicity property can be proved in the case when the Laplace kernels P_γ satisfy some nice spectral properties on a suitable Banach space $\mathcal{B}$:

$$\left\| P_\gamma^n f - r(\gamma)^n \Pi_\gamma f \right\|_{\mathcal{B}} \leq M_K \left(\theta_K \, r(\gamma)\right)^n \|f\|_{\mathcal{B}}.$$

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$$\left\| P_\gamma^n f - r(\gamma)^n \Pi_\gamma f \right\|_{\mathcal{B}} \leq M_K \left(\theta_K \, r(\gamma) \right)^n \|f\|_{\mathcal{B}}.$$

Then ν is proved to be finite and given by

$$\nu = \inf\{\gamma > \mathbf{0},\ \mathbf{r}(\gamma) < \mathbf{1}\},$$

where $r(\gamma)$ denotes the spectral radius of P_γ on $\mathcal{B}$.

More precisely, the following assertions hold:

(i) If the functions $\gamma \mapsto r(\gamma)$ and $\gamma \mapsto B(\gamma) := \mu\left(\kappa\, e^{-\gamma\xi}\Pi_\gamma(Ph_{\kappa,\gamma})\right)$ are continuous from J_0 to $(0,+\infty)$, then $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on J_0 with $A(\gamma) := \frac{B(\gamma)}{r(\gamma)}$ and $\rho(\gamma) = r(\gamma)$.

(ii) If moreover $\inf_{\gamma\in J_0} r(\gamma) < 1 < \sup_{\gamma\in J_0} r(\gamma)$, then ν is finite and

$$\nu = \inf\{\gamma > 0 \,:\, r(\gamma) < 1\}. \tag{10}$$

(iii) If furthermore the functions $r(\cdot)$ and $B(\cdot)$ are C^1-smooth on J_0, and if $r'(\nu) \neq 0$, then the constant C_ν is well defined and finite.

Theorem (Linear autoregressive model)

Let $\mathbb{X} := \mathbb{R}$ and $X_n = \alpha X_{n-1} + \vartheta_n$ for $n \geq 1$, where X_0 is a real-valued random variable, $\alpha \in (-1, 1)$, and $(\vartheta_n)_{n\geq 1}$ is a sequence of i.i.d. real-valued random variables independent of X_0. Let $r_0 > 0$. We assume that ϑ_1 has a continuous Lebesgue probability density function $p > 0$ on $\mathbb{X}$ satisfying the following condition: for all $x_0 \in \mathbb{R}$, there exist a neighbourhood V_{x_0} of x_0 and a non-negative function $q_{x_0}(\cdot)$ such that $y \mapsto (1 + |y|)^{r_0}\, q_{x_0}(y)$ is Lebesgue-integrable and such that

$$\forall y \in \mathbb{R},\ \forall v \in V_{x_0},\ p(y + v) \leq q_{x_0}(y). \tag{11}$$

Assume that the initial distribution μ is either the stationary probability measure π or δ_x for some $x \in \mathbb{R}$. Let N_0 be a positive integer. Assume that κ is bounded, that $\lim_{|x|\to+\infty} \xi(x) = +\infty$, that the Lebesgue measure of the set $[\xi = 0]$ is zero, and that $\sup_{x\in\mathbb{R}} \frac{\xi(x)}{(1+|x|)^{r_0}} < \infty$.

Then $(S_n, \kappa(X_n))_n$ is multiplicatively ergodic on $J = [0, +\infty)$ with $\lim_\gamma \rho(\gamma) \geq 2$ and $\lim_{\gamma \to +\infty} \rho(\gamma) = 0$.

Thus ν is well defined (and is independent of the choice of the initial distribution μ).

If moreover there exists $\tau > 0$ such that $\sup_{x \in \mathbb{R}} \frac{\xi(x)^{1+\tau}}{(1+|x|)^{r_0}} < \infty$, then the constant C_ν is well defined in $(0, +\infty)$.

(II) Second moment: behaviour of $\mathbb{E}[N_t N_{t+\tau}]$ and the a.s. convergence of $(e^{-\nu t} N_t)_{t \geq 0}$

we need an additional assumption involving the characteristics for lines of cells coinciding up to the k-th generation.

Hypothesis (Second assumption of stationarity)

The first stationary assymption holds true. Moreover, for each $k \in \mathbb{N}$, *there exists a process* $X^{(k)} = (X_n^{(k)})_{n \geq 0}$ *such that*

$$\begin{cases} (X_n^{(k)})_{0 \leq n \leq k} = (X_n)_{0 \leq n \leq k} & a.s. \\ (X_n^{(k)})_{n \geq 0} = (X_n)_{n \geq 0} & in\ law, \end{cases} \tag{12}$$

and such that, for every couple of sequences of positive integers $(m_i)_{i \geq 1}$ *and* $(\ell_i)_{i \geq 1}$ *such that* $m_1 = \ell_1, \dots, m_k = \ell_k$ *and* $\ell_{k+1} \neq m_{k+1}$, $((X_{0,m_1,\dots,m_n})_n, (X_{0,\ell_1,\dots,\ell_n})_n)$ *has the same distribution as* $(X, X^{(k)})$.

Now define, for any integers $n \geq 1, m \geq 1$ and $\min(n, m) - 1 \geq k \geq 0$ the random variables $A_{n,m,k}$ as follows:

$$A_{n,m,k} = \left(\prod_{i=0}^{n-2} \kappa(X_i)\right) \left(\prod_{j=\min(k+1,n-1)}^{m-2} \kappa(X_j^{(k)})\right)$$

$$\left(\prod_{j\in\{k\}\setminus\{n-1,m-1\}} (\kappa(X_j) - 1)\right) (\kappa(X_{n-1}) - 1) \left(\kappa(X_{m-1}^{(k)}) - 1\right),$$

with the usual convention $\prod_{i=k+1}^{\ell} \cdots = 1$ if $\ell \leq k$. Define also $S_n^{(k)} := \sum_{j=0}^{n} \xi(X_j^{(k)})$. The main result of this section is the following proposition.

Proposition (2)

Assume that the second assumption of stationarity holds. Let $t > 0$ and $\tau \geq 0$ be fixed. If $\sum_{n\geq 0} \mathbb{E}\left[\left(\prod_{j=0}^{n} \kappa(X_j)\right) \mathbf{1}_{\{S_n \leq t+\tau\}}\right] < \infty$, then

$$\mathbb{E}[N_t N_{t+\tau}] = \mathbb{E}[N_t] + \mathbb{E}[N_{t+\tau}] - 1$$
$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left[A_{n,m,k} \mathbf{1}_{\{S_{n-1} \leq t, S_{m-1}^{(k)} \leq t+\tau\}}\right].$$

Convergence a.s of $e^{-\nu t}N_t$

Corollary

Assume that the assumptions of Proposition 2 are satisfied, that $\nu < \infty$, that $\limsup_{t\to\infty} e^{-\nu t}\mathbb{E}[N_t] < \infty$ and that there exists $K > 0$ such that

$$\lim_{t\to\infty} \sup_{\tau\geq 0}$$

$$\left| e^{-\nu(2t+\tau)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left[A_{n,m,k} \mathbf{1}_{\{S_{n-1}\leq t, S^{(k)}_{m-1}\leq t+\tau\}} \right] - K \right| = 0$$

Then there exists a square integrable random variable W such that $e^{-\nu t}N_t$ converges in quadratic mean to W as t tends to infinity.

Corollary

Assume that the assumptions of Proposition 2 are satisfied, that $\nu < \infty$, that $\limsup_{t\to\infty} e^{-\nu t}\mathbb{E}[N_t] < \infty$ and that there exists $K > 0$ such that

$$\lim_{t\to\infty} \sup_{\tau\geq 0} \left| e^{-\nu(2t+\tau)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\min(n,m)-1} \mathbb{E}\left[A_{n,m,k} \mathbf{1}_{\{S_{n-1}\leq t, S^{(k)}_{m-1}\leq t+\tau\}}\right] - K \right| = 0$$

Then there exists a square integrable random variable W such that $e^{-\nu t} N_t$ converges in quadratic mean to W as t tends to infinity. If moreover the above convergence is exponentially fast and if $W > 0$ then $e^{-\nu t} N_t$ converges almost surely to W as t tends to infinity.

Proof of Corollary 2.

$$\mathbb{E}\left[\left(e^{-\nu t}N_t - e^{-\nu(t+\tau)}N_{t+\tau}\right)^2\right]$$
$$= e^{-2\nu t}\mathbb{E}\left[N_t^2\right] + e^{-2\nu(t+\tau)}\mathbb{E}\left[N_{t+\tau}^2\right] - 2e^{-2\nu t-\nu\tau}\mathbb{E}\left[N_t N_{t+\tau}\right].$$

Now Proposition 2 gives,

$$e^{-2\nu t-\nu\tau}\mathbb{E}[N_t N_{t+\tau}]$$
$$= e^{-2\nu t-\nu\tau}\mathbb{E}[N_t] + e^{-2\nu t-\nu\tau}\left(\mathbb{E}[N_{t+\tau}] - 1\right)$$
$$+e^{-2\nu t-\nu\tau}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\min(n,m)-1}\mathbb{E}\left[A_{n,m,k}\mathbf{1}_{\{S_{n-1}\le t, S_{m-1}^{(k)}\le t+\tau\}}\right].$$

Thanks to the assumptions of Corollary 2, the two first terms of the right hand side of the last equality tends to 0 as t tends to infinity. While the third term tends to K. Those three limits hold for any $\tau \geq 0$ and uniformly in τ. □

$$\lim_{t\to\infty} \mathbb{E}\left[\left(e^{-\nu t}N_t - e^{-\nu(t+\tau)}N_{t+\tau}\right)^2\right] = K + K - 2K = 0,$$

for any $\tau \geq 0$, uniformly in τ. The Cauchy criterion ensures then the convergence in quadratic mean of $e^{-\nu t}N_t$ as t tends to infinity to a random variable W with finite second moment.

$$\lim_{t\to\infty}\mathbb{E}\left[\left(e^{-\nu t}N_t - e^{-\nu(t+\tau)}N_{t+\tau}\right)^2\right] = K + K - 2K = 0,$$

for any $\tau \geq 0$, uniformly in τ. The Cauchy criterion ensures then the convergence in quadratic mean of $e^{-\nu t}N_t$ as t tends to infinity to a random variable W with finite second moment. For the last point, we deduce from Proposition 2 that $\int_0^\infty \mathbb{E}\left[(e^{-\nu t}N_t - W)^2\right] dt < \infty$. This yields (arguing as for the proof of Theorem 21.1 in [2]) the almost sure convergence, as t tends to infinity, of $e^{-\nu t}N_t$ to W.

For further results, we will make the following stronger assumption involving some independence assumptions.

Hypothesis

The sequence of "Children number" is a sequence of i.i.d. square integrable random variables of expectation κ_1, and is independent of the sequence of life-time length. Moreover, for all $k \in \mathbb{N}$, $(X_n^{(k)})_{n \geq k+1}$ and $(X_n)_{n \geq k+1}$ are independent given X_k. Finally the number ν satisfies

$$\forall x \in \mathbb{X}, \quad \nu = \inf \left\{ \gamma > 0, \sum_{n \geq 0} \kappa_1^n \mathbb{E}\left[e^{-\gamma S_{n+1}} | X_0 = x \right] < \infty \right\} < \infty. \tag{13}$$

We set $\kappa_2 := \mathbb{E}[\kappa(X_1)(\kappa(X_1) - 1)]$.

Remark

Observe that under this Hypothesis,

$$\mathbb{E}\left[\left(\prod_{j=0}^{n}\kappa(X_j)\right)\mathbf{1}_{\{S_n\le t\}}|X_0\right] =$$

$$\kappa_1^n\kappa(X_0)\mathbb{E}\left[\mathbf{1}_{\{S_n\le t\}}|X_0\right] \le \kappa_1^{n+1}\kappa(X_0)\mathbb{E}\left[e^{-\gamma(S_n-t))}|X_0\right].$$

Hence, Proposition 1 applies and (4) *can be rewritten*

$$\mathbb{E}[\mathbf{N_t}] = \mathbf{1} + \sum_{\mathbf{n}\ge\mathbf{0}}\kappa_1^{\mathbf{n}}(\kappa_1 - \mathbf{1})\mathbb{P}\left(\mathbf{S_n} \le \mathbf{t}\right).$$

Technical lemma (Harris, 1963)

Let f be a function and $\tilde{f}$ its Laplace transform. Suppose that there exist two positive reals δ and ϵ such that:

1. $\tilde{f}$ is analytic in $\{z = x + \mathrm{i}y\,,\ |x| < \delta + \epsilon\} \setminus \{0\}$,
2. **$\tilde{f}$ has a simple pole at 0, with residue C,**
3. $$\int_{-\infty}^{\infty} |\tilde{f}(\delta + \mathrm{i}y)|\,\mathrm{d}y < \infty\,,$$
4. $$\lim_{y \to \pm\infty} \tilde{f}(x + \mathrm{i}y) = 0\,,$$
   uniformly in $x \in [-\delta, \delta]$,
5. $$\psi := \int_{-\infty}^{\infty} |\tilde{f}(-\delta + \mathrm{i}y)|\,\mathrm{d}y < \infty\,.$$

Technical lemma (Harris, 1963)

Let f be a function and $\tilde{f}$ its Laplace transform. Suppose that there exist two positive reals δ and ϵ such that:

1. $\tilde{f}$ is analytic in $\{z = x + \mathrm{i}y\,,\ |x| < \delta + \epsilon\} \setminus \{0\}$,
2. **$\tilde{f}$ has a simple pole at 0, with residue C,**
3. $$\int_{-\infty}^{\infty} |\tilde{f}(\delta + \mathrm{i}y)|\,\mathrm{d}y < \infty\,,$$
4. $$\lim_{y\to\pm\infty} \tilde{f}(x + \mathrm{i}y) = 0\,,$$
   uniformly in $x \in [-\delta, \delta]$,
5. $$\psi := \int_{-\infty}^{\infty} |\tilde{f}(-\delta + \mathrm{i}y)|\,\mathrm{d}y < \infty\,.$$

Then, for all $t > 0$,

$$|\mathbf{f(t) - C}| \leqslant \frac{\psi}{\mathbf{2}\pi}\,\mathrm{e}^{-\delta\mathbf{t}}\,.$$

Define $f_{x,0}(t) = (\kappa_1 - 1)e^{-\nu t} \sum_{n \geq 0} \kappa_1^n \mathbb{P}\left(S_{n+1} - S_0 \leq t | X_0 = x\right)$.
We will make the following assumption involving the Laplace transform $\tilde{f}_{x,0}$ of $f_{x,0}$: $\forall \gamma > 0$,

$$\begin{aligned}\tilde{f}_{x,0}(\gamma) &= \int_0^{\infty} e^{-\gamma t} f_{x,0}(t) dt \\ &= \frac{\kappa_1 - 1}{\gamma + \nu} \sum_{n \geq 0} \kappa_1^n \mathbb{E}\left[e^{-(\gamma+\nu)(S_{n+1} - S_0)} | X_0 = x\right].\end{aligned}$$

Hypothesis

Suppose that there exist two positive reals $\delta < \nu$ and ϵ such that, for any x, the Laplace transform $\tilde{f}_{x,0}$, extended on the complex plane, satisfies the following conditions:

1. $\tilde{f}_{x,0}$ *has a simple pole at 0, with residue* $\tilde{C}_0(x)$,
2. $\Psi_0(x) := \int_{-\infty}^{+\infty} |\tilde{f}_{x,0}(-\delta + iy)| dy < \infty$.

Lemma

Assume Hypothesis of the technical lemma. Then, for any $t > 0$,

$$\left| e^{-\nu t} \sum_{n \geq 0} \kappa_1^n (\kappa_1 - 1) \mathbb{P}(S_{1,n+1} \leq t | X_0 = x) - \tilde{C}_0(x) \right|$$

$$\leq \frac{\Psi_0(x)}{2\pi} e^{-\delta t}.$$

Proposition

If, moreover, $\mathbb{E}[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)] < \infty$ and if $\mathbb{E}(e^{-(\nu-\delta)\xi(X_0)}\Psi_0(X_0)) < \infty$, then $\mathbb{E}[N_t] < \infty$ and there exists $\epsilon_1 > 0$ such that

$$\mathbb{E}[N_t] = e^{\nu t}\kappa_1\mathbb{E}[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)](1 + O(e^{-\epsilon_1 t})), \; as \; t \to \infty\,.$$

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Proposition

Suppose, moreover, that $(X_n)_n$ a Markov process and also that $\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}[\tilde{C}_0^2(X_k)e^{-2\nu S_k}] < \infty$, then, for any $t > 0$, $\tau \geq 0$, , $\mathbb{E}[N_t N_{t+\tau}] < \infty$ and as $t \to \infty$

$$\mathbb{E}[N_t N_{t+\tau}] = e^{\nu(2t+\tau)}\kappa_2\sum_{k=0}^{\infty}\kappa_1^k\mathbb{E}\left[\tilde{C}_0^2(X_k)e^{-2\nu S_k}\right](1 + ae^{-\epsilon_1 t}),$$

where a and ϵ_1 are positive constants independent of t and τ.

Theorem

Assume Hypotheses of the two previous propositions (with $(X_n)_n$ a Markov process) then there exists a square integrable random variable W such that $e^{-\nu t}N_t$ converges in quadratic mean to W as t tends to infinity, with

$$\mathbb{E}[W] = \kappa_1 \mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right]$$

$$\mathrm{Var}(W) = \kappa_2 \sum_{k=0}^{\infty} \kappa_1^k \mathbb{E}\left[\tilde{C}_0^2(X_k) e^{-2\nu S_k}\right]$$

$$-\kappa_1^2 \left(\mathbb{E}\left[e^{-\nu\xi(X_0)}\tilde{C}_0(X_0)\right]\right)^2.$$

If, moreover, $W > 0$ almost surely then $e^{-\nu t}N_t$ converges almost surely to W.

Remark.

Recall that, $S_{1,n+1} = \sum_{i=1}^{n+1} \xi(X_i)$. Suppose that

$$\mathbb{E}\left[e^{-\gamma S_{1,n+1}} | X_0 = x\right] = \alpha(\gamma, x) L^{n+1}(\gamma) + r_{n+1}(\gamma, x) \qquad (14)$$

for suitable non-negative functions α, L and $(r_n)_n$ satisfying

(a) $\forall\ x \in \mathbb{X}$, $\alpha(\cdot, x)$, L and $r_n(\cdot, x)$ can be extended to analytic functions in $\{z = u + iy, |u| \leq \delta + \epsilon < \nu, y \in \mathbb{R}\}$

(b) L is positive and non-increasing on $\mathbb{R}_+^*$. The equation $\kappa_1 L(z) = 1$, has a unique positive solution in $\mathbb{C}$, denoted by ν.

(c) The series $\sum_{n>0} \kappa_1^n r_n(\gamma, x)$ converges uniformly in γ in a neighborhood of ν uniformly in x.

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(c) The series $\sum_{n>0} \kappa_1^n r_n(\gamma, x)$ converges uniformly in γ in a neighborhood of ν uniformly in x.

Lemma

The assumptions of the technical lemma are satisfied under Conditions (a)-(b)-(c) with

$$\tilde{C}_0(x) = -\frac{(\kappa_1 - 1)}{\kappa_1^2 \nu L'(\nu)} \alpha(\nu, x)$$

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