Stable limits for Markov chains and contractivity properties of transition operators

Joint work with M. El Machkouri and D. Volný (Rouen)

Probabilistic Limit Theorems for Dynamical Systems
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Motivations

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- Consider an ARCH(1) Markov chain $\{X_n\}$ given by

$$X_{k+1} = \sqrt{\beta + \lambda X_k^2} Z_{k+1}, \ k \geqslant 0,$$

where $\beta, \lambda > 0$ and $\{Z_n\}_{n \in \mathbb{N}}$ is a Gaussian standard i.i.d. sequence that is independent of X_0.

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- The squares of ARCH(1) satisfy a stochastic recurrence equation - see a recent reference Buraczewski, Damek, Mikosch (2016) .

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- If $\beta > 0$ and $\lambda \in (0, 2e^{\gamma})$, then $\{X_k\}_{k\geqslant 0}$ is strictly stationary iff

$$X_0 \sim r_0 \sqrt{\beta \sum_{m=1}^{\infty} Z_m^2 \prod_{j=1}^{m-1} (\lambda Z_j^2)},$$

where r_0 is a Rademacher random variable ($P(r_0 = \pm 1) = 1/2$), independent of $\{Z_n\}$.

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- Then, as $x \to \infty$,
$$P(X_0 > x) \sim \frac{C_{\beta,\lambda}}{2}\, x^{-2\kappa},$$
where
$$C_{\beta,\lambda} = \frac{E\Big[\big(\beta + \lambda X_0^2\big)^{\kappa} - \big(\lambda X_0^2\big)^{\kappa}\Big]}{\kappa\lambda^{\kappa} E\Big[(\lambda Z_1^2)^{\kappa}\ln(\lambda Z_1^2)\Big]} \in (0, +\infty).$$

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- It follows that if $\kappa \in (0,1)$ then the law of X_0 belongs to the domain of strict attraction of some strictly stable law.
- In particular, if $\lambda \in (1, \lambda_0)$ then $2\kappa \in (1,2)$ and $\{X_n\}$ is a stationary sequence of martingale defferences.

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- Consider the normalized partial sums

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{(nC_{\beta,\lambda})^{\frac{1}{2\kappa}}}, \; T_n = \frac{Y_1 + Y_2 + \ldots + Y_n}{(nC_{\beta,\lambda})^{\frac{1}{2\kappa}}},$$

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- If $\exp(-C|\theta|^{2\kappa})$ is the characteristic function of the limit for T_n, then the characteristic function of the limit for S_n is of the form $\exp(-\tau C|\theta|^{2\kappa})$, where $1 > \tau = E\big[|1 + S_\infty|^{2\kappa} - |S_\infty|^{2\kappa}\big] > 0$ and the series
$$S_\infty = \sum_{j=1}^{\infty} \lambda^{j/2} \Big[\prod_{k=1}^{j-1} |Z_k| \Big] Z_j$$
converges a.s.
- The limit for S_n is different from the limit for T_n!
- Unlike in the case of finite variance!

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It is natural to ask about extra conditions that guarantee the same behavior of partial sums of Markov chains and the corresponding i.i.d. sequence.

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- It was important that the limit process was the same as if the summands were independent, to allow for the clear interpretation of the parameters.
- Moreover, modeling with Markov chains provides a physically acceptable solution, while using independent random variables is physically meaningless.

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$$\frac{\Psi(X_1) + \Psi(X_2) + \ldots + \Psi(X_n)}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu_\alpha.$$

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  (Possibly with explicit centering).
- As if $\{\Psi(X_n)\}_{n \geqslant 0}$ were independent!

Stable laws

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- Recall that a *stable* distribution with exponent $\alpha \in (0,2)$ has the characteristic function of the form

$$\hat{\mu}(\theta) = \exp\Big(i\theta a^h + \int \big(e^{i\theta x} - 1 - i\theta x\, \mathbb{1}_{\{|x|\leqslant h\}}\big)\, \nu_{\alpha,c_+,c_-}(dx)\Big),$$

where $a^h \in \mathbb{R}^1$, the Lévy measure ν_{α,c_+,c_-} has the density

$$p_{\alpha,c_+,c_-}(x) = \alpha\Big(c_+ x^{-(\alpha+1)}\, \mathbb{1}_{\{x>0\}} + c_- |x|^{-(\alpha+1)}\, \mathbb{1}_{\{x<0\}}\Big),$$

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- We will consider only the strictly stable limits μ_α of the form:

$$\hat{\mu}_{\alpha,c_+,c_-}(\theta) = \begin{cases} \exp\big(\int (e^{i\theta x} - 1)\nu_{\alpha,c_+,c_-}(dx)\big), & \alpha \in (0,1); \\ \exp\big(\int (e^{i\theta x} - 1)\nu_{1,c,c}(dx)\big), & \alpha = 1; \\ \exp\big(\int (e^{i\theta x} - 1 - i\theta x)\nu_{\alpha,c_+,c_-}(dx)\big), & \alpha \in (1,2). \end{cases}$$

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- Recall also that $\pi \circ \Psi^{-1}$ belongs to the domain of attraction of a stable law μ_{α,c_+,c_-} $(0 < \alpha < 2)$, if

$$\pi(x\,;\,|\Psi(x)| > t) = t^{-\alpha}\ell(t),$$

where $\ell(t)$ is a slowly varying function as $t \to \infty$, and there exist the limits

$$\lim_{t\to\infty} \frac{\pi(x\,;\,\Psi(x) > t)}{\pi(x\,;\,|\Psi(x)| > t)} = \frac{c_+}{c_+ + c_-},$$

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- A suitable choice of the norming constants B_n is

$$\frac{n}{B_n^{\alpha}}\ell(B_n) \to c_+ + c_-.$$

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- Important: there is neither ϕ-mixing nor ultraboundedness!

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- Following Wu (JoFA, 2000) we will say that the transition operator P is uniformly integrable in L^2 (or 2-U.I.) if

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- We believe that 2-U.I. is the proper minimal form for operator contractivity.

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- The transition operator P is said to have an L^2-spectral gap if there is a number $a < 1$ such that

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- For reversible, ψ-irreducible and aperiodic Markov chains the spectral gap property is known to be equivalent to geometric ergodicity, i.e. existence of $0 < \rho < 1$ and $C : \mathbb{S} \to \mathbb{R}^+$ such that

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  where $\|\cdot\|_{TV}$ is the total variance distance.
- If $\{X_n\}$ is irreversible, then the spectral gap property implies the geometric ergodicity, but there are Markov chains that are geometrically ergodic and do not have an L^2-spectral gap.
- Notice that the central limit theorem need not hold for such Markov chains!

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If $\alpha \in (1,2)$, then

$$\frac{\sum_{j=1}^{n} \Psi(X_j) - \mathbb{E}(\Psi(X_j)|\mathcal{F}_{j-1})}{B_n} \xrightarrow[\mathcal{D}]{} \mu_{\alpha,c_+,c_-}.$$

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Corollary

In assumptions of Theorem 1, if $\alpha \in (1,2)$ and

$$\mathbb{E}(\Psi(X_1)\big|\mathcal{F}_0) = 0.$$

i.e. $\Psi(X_1), \Psi(X_2), \ldots$ form a martingale difference sequence, then

$$\frac{\Psi(X_1) + \Psi(X_2) + \ldots + \Psi(X_n)}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu_{\alpha,c_+,c_-}.$$

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In assumptions of Theorem 1 replace
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If $\alpha \in (0,1)$ or $\alpha = 1$ and $c_+ = c_-$ or $\alpha \in (1,2)$ and $\int \Psi(x)\pi(dx) = 0$, then

$$\frac{\Psi(X_1) + \Psi(X_2) + \ldots + \Psi(X_n)}{B_n} \xrightarrow[\mathcal{D}]{} \mu_{\alpha, c_+, c_-}.$$

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- There exists a measurable family of Borel measures $Q(x,dy)$ and a measurable, nonnegative function $p(x,y)$ such that

$$P(x,dy) = p(x,y)\pi(dy) + Q(x,dy) \quad \text{for all } x \in E.$$

$$Q(x,[y : |\Psi(y)| \geqslant \lambda]) \leqslant C \int_{[|\Psi(y)| \geqslant \lambda]} p(x,y)\pi(dy), \lambda \geqslant 0;$$

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- Observe that by the inequality in the second line the term $p(x, y)\pi(dy)$ is essential for properties of the transition operator P.

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- In particular we can ignore the auxiliary random kernel $Q(x, dy)$.

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- For us it is interesting how the PoC gives a theorem on convergence to stable laws, valid for triangular arrays of adapted random variables.
- Let $\{X_{n,j}\,;\, j \in \mathbb{N}, n \in \mathbb{N}\}$ be an array of random variables, which are row-wise adapted to a sequence of filtrations $\{\{\mathcal{F}_{n,j}\,;\, j = 0, 1, \ldots\}\,;\, n \in \mathbb{N}\}$.

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Theorem 3

If $$\max_{1\leqslant j\leqslant k_n} \mathbb{P}\big(|X_{n,j}| > \varepsilon \big| \mathcal{F}_{n,j-1}\big) \underset{\mathcal{P}}{\longrightarrow} 0, \quad \varepsilon > 0;$$

$$\sum_{j=1}^{k_n} \mathbb{P}\big(X_{n,j} > x \big| \mathcal{F}_{n,j-1}\big) \underset{\mathcal{P}}{\longrightarrow} c_+ x^{-\alpha}, \ \sum_{j=1}^{k_n} \mathbb{P}\big(X_{n,j} < -x \big| \mathcal{F}_{n,j-1}\big) \underset{\mathcal{P}}{\longrightarrow} c_- |x|^{-\alpha}, \ x > 0;$$

$$\sum_{j=1}^{k_n} \mathbb{E}\big(X_{n,j}\, \mathbb{1}_{\{|X_{n,j}|\leqslant h\}} \big| \mathcal{F}_{n,j-1}\big) \underset{\mathcal{P}}{\longrightarrow} a^h;$$

$$\sum_{j=1}^{k_n} \mathbb{V}\mathrm{ar}\big(X_{n,j}\, \mathbb{1}_{\{|X_{n,j}|\leqslant h\}} \big| \mathcal{F}_{n,j-1}\big) \underset{\mathcal{P}}{\longrightarrow} \int_{\{|x|\leqslant h\}} x^2\, \nu_{\alpha,c_+,c_-}(dx);$$

then
$$\sum_{j=1}^{k_n} X_{n,j} \underset{\mathcal{D}}{\longrightarrow} \delta_{a^h} * c_h\text{-Poiss}(\alpha, c_+, c_-).$$

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- In fact, one can also replace the summation to constants by summation to stopping times.

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$$\phi_n(\theta) = \prod_{j=1}^{k_n} \mathbb{E}\left(e^{i\theta X_{n,j}} | \mathcal{F}_{n,j-1}\right) \xrightarrow[\mathcal{P}]{} z,$$

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then also $\mathbb{E}\exp(i\theta \sum_{j=1}^{k_n} X_{n,j}) \longrightarrow z.$

In particular, if $\phi_n(\theta) \xrightarrow[\mathcal{P}]{} \hat{\mu}(\theta) \neq 0, \quad \theta \in \mathbb{R}^1$, then

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$$\sum_{j=1}^{k_n} \left|1 - \mathbb{E}\left(e^{i\theta X_{n,j}} | \mathcal{F}_{n,j-1}\right)\right|^2 \xrightarrow[\mathcal{P}]{} 0, \quad \theta \in \mathbb{R}^1.$$

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Let A_n be arbitrary random variables and $\Phi(\theta) \in \mathbb{C}$ be a constant for each $\theta \in \mathbb{R}^1$. The following conditions are equivalent:

$$\left(\sum_{j=1}^{k_n} \left(\mathbb{E}\left(e^{i\theta X_{n,j}} | \mathcal{F}_{n,j-1}\right) - 1\right)\right) - i\theta A_n \xrightarrow[\mathcal{P}]{} \Phi(\theta).$$

$$\left(\prod_{j=1}^{k_n} \mathbb{E}\left(e^{i\theta X_{n,j}} | \mathcal{F}_{n,j-1}\right)\right) e^{-i\theta A_n} \xrightarrow[\mathcal{P}]{} e^{\Phi(\theta)}.$$

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Theorem 5 - continued

In either case we have also

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In either case we have also

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In particular, if $e^{\Phi(\theta)} = \hat{\mu}(\theta)$, $\theta \in \mathbb{R}^1$, for some probability measure μ, then either of the equivalent conditions implies

$$\sum_{j=1}^{k_n} X_{n,j} - A_n \underset{\mathcal{D}}{\longrightarrow} \mu.$$