Levy diffusion of dispersing billiards with flat points

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Outline of the talk

- Searching for billiards with arbitrarily slow mixing rates – a family of billiards with flat points;
- Stable law (with Paul Jung);
- Levy jump-diffusion (with Paul Jung, Francoise Pene)

- Hong-Kun Zhang, Decay of correlations for billiards with flat points II: cusps effect. *Contemporary Mathematics*, 2017.
- Paul Jung and Hong-Kun Zhang, Stable laws for chaotic billiards with cusps at flat points, Annales de l'Institute Henri Poincare, 2018.
- Paul Jung, Francoise Pene, Hong-Kun Zhang, *Convergence to α-stable Lévy motion for chaotic billiards with several cusps at flat points, submitted.*

Statistical properties of chaotic billiards

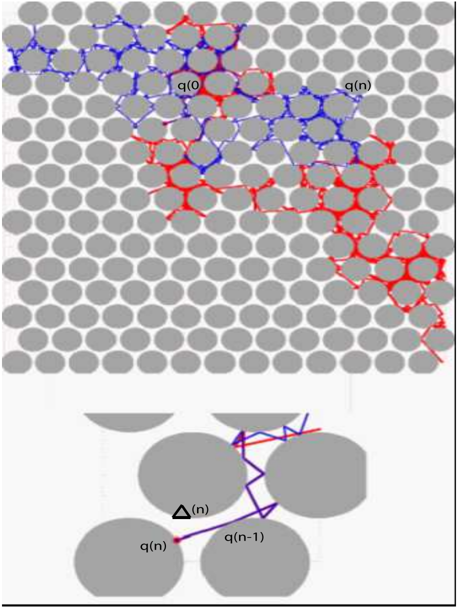
- $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ is the billiard map that preserves μ.
- Let $f: \mathcal{M} \to \mathbb{R}^d$ be a nice observable, say $f = \Delta$ being the displacement function in the configuration space, then

$$X_n := f \circ \mathcal{F}^n$$

  defines a stationary process on $(\mathcal{M}, \mu)$, which are dependent.
- Question: Will the limiting theorems for i.i.d. random variables hold for this stationary process $\{X_n\}$?
- What is $\mathrm{Cov}(X_n, X_0)$? How fast will X_n forget its initial state X_0?
- CLT: $\frac{X_1 + \cdots X_n - n\mu(X_1)}{\sigma\sqrt{n}} \to N(0,1)$ converges in distribution, as $n \to \infty$, provided

$$\sigma^2 = \sum_{k=-\infty}^{\infty} \mathrm{Cov}(X_k, X_0)$$

Diffusion of Lorentz gas with dispersing scatterers



- $\mathbf{q}(n) = q \circ \mathcal{F}^n$ – position vector;
- $\Delta(n) = \Delta \circ \mathcal{F}^n$ – displacement;
- $\mathbf{q}(n) - \mathbf{q}(0) = \Delta(1) + \cdots \Delta(n)$;
- Ergodicity $\Rightarrow \frac{\Delta(1)+\cdots+\Delta(n)}{n} \to \mathbb{E}(\Delta)$
- Isotropic $\Rightarrow \mathbb{E}(\Delta) \equiv 0$;
- $\mathbf{J} := \lim_{n\to\infty} \frac{\mathbf{q}(n)-\mathbf{q}(0)}{n} = 0$;
- CLT/WIP $\Rightarrow$ the diffusion $\{q(n)\}$ is driven by Brownian motion:
  $\mathbf{q}(n) = \mathbf{q}(0) + \sigma B(n) + o(n^{-1/2})$
- Question: What if $\mathrm{Cov}(X_n, X_0) = \mathcal{O}(n^{-a})$ with $a \in (0,1)$?

Various types of diffusion

Investigate the diffusion for systems with slow decay rates of correlations $\mathrm{Cov}(X_n, X_0) = \mathcal{O}(1/n^a)$. And find the diffusion constant σ.

(1) $a > 1$ case, CLT $\Rightarrow q_n \stackrel{\text{dist}}{=} q_0 + \sigma N(0, n) + o(n^{\frac{1}{2}})$;

(2) $a = 1$ case, CLT $\Rightarrow q_n \stackrel{\text{dist}}{=} q_0 + \sigma(\ln n)^{\frac{1}{2}} N(0, n) + o(n^{\frac{1}{2}})$;

(3) $a \in (0, 1)$, CLT $\Rightarrow q_n \stackrel{\text{dist}}{=} q_0 + \sigma n^{1/\alpha} Z + o(n^{1/\alpha})$, where Z has a α-stable law.

- P. Bálint and S. Gouezel, *Limit theorems in the Bunimovich Stadia*, Comm. Math. Phys., **263** (2006), 451–512.
- D. Szasz, T. Varju, Laws and Recurrence for the Planar Lorentz Process with Infinite Horizon. Journal of Statistical Physics, 2007, Volume 129, 59-80.
- N. Chernov and D. Dolgopyat, *Anomalous current in periodic Lorentz gases with infinite horizon*, Russian Mathematical Surveys, **64** (2009), 651–699.
- P. Bálint, N. Chernov, and D. Dolgopyat, *Limit theorems for dispersing billiards with cusps*, *Comm. Math. Phys*. **308** (2011), 479–510.
- Luke Mohr and Hongkun Zhang, *Diffusion constants for nonuniformly hyperbolic systems with singularities*, (2017) submitted.

A long journey to search for billiards with arbitrarily slow mixing rates

- Can one construct a physically meaningful billiard system, with arbitrarily slow decay rates of correlations, of order $\mathcal{O}(n^{-a})$, with $a \in (0, 1)$? and what is the diffusion behavior? (as now CLT fails);
- Main idea: we would like to add some flat points on the boundary of the billiard table, to change the decay rates of correlations.

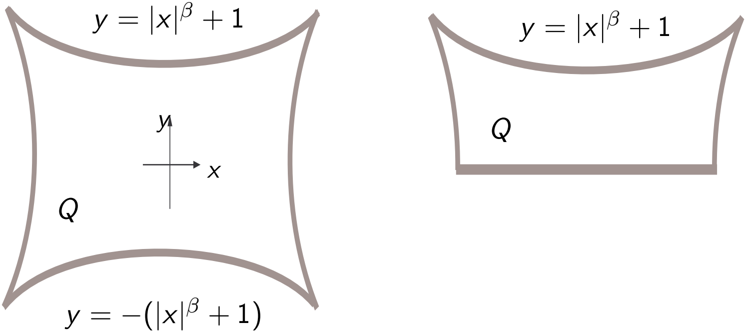
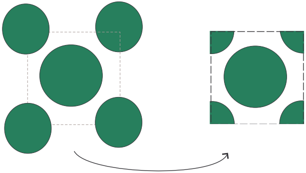
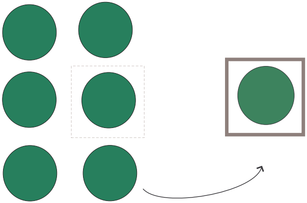


Fig.: Dispersing billiards with walls where the curvature vanishes, $\beta > 2$.



Dispersing billiards with finite horizon



Semi-dispersing (infinite horizon in unfolding spa

Remark: We add symmetric flat points for both tables.

Theorem

For the dispersing billiards with flat points, the correlations for the billiard map $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ and piecewise Hölder continuous functions f, g on $\mathcal{M}$ decay as
(1) (Finite horizon case - Chernov & Zhang (2005))

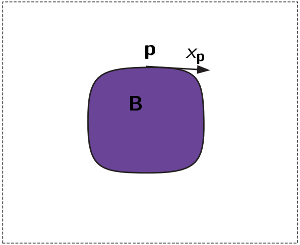
$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq \text{const} \cdot n^{-1-\frac{4}{\beta-2}}, \quad \beta > 2$$

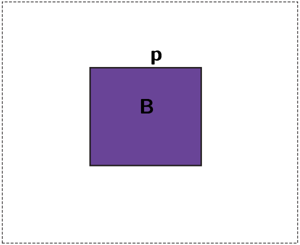
(2) (Semi-diserpsing case - Zhang (2017))

$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq \text{const} \cdot n^{-1}, \quad \beta > 2$$

- If $\beta = 2$, then this is strictly dispersing billiards, which has EDC: $|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq C_{f,g} \cdot \vartheta^n$, for some $\vartheta \in (0, 1)$.
- When $\beta \to \infty$, this is a semi-dispersing billiards, with $\mathcal{C}_n(f, g, \mathcal{F}, \mu) = \mathcal{O}(n^{-1})$.

Semi-dispersing billiards with flat points on a rectangle





(a). Billiards on a rectangle with 4 flat points, for Q_β, the boundaries have zero derivatives up to $\beta - 1$ order at flat points;
(b). The limiting table as $\beta \to \infty$.

- As $\beta \to \infty$, (b) is integrable;
- For $\beta = 2$, the semidispersing billiard has correlation rates $\mathcal{O}(n^{-1})$;
- We guessed that as $\beta \in (2, \infty)$, the decay rates should be $\mathcal{O}(n^{-a})$, with $a \in (0, 1)$?

Billiards with flat points and infinite horizon.

We fix the scatterer **B** and label all other copies of the scatterer in the channel as $\mathbf{B}'$, $\mathbf{B}_1, \cdots, \mathbf{B}_n$, etc. Since the r-coordinate of p is 0, we have $x_0 = (0, \pi/2) \in M$.

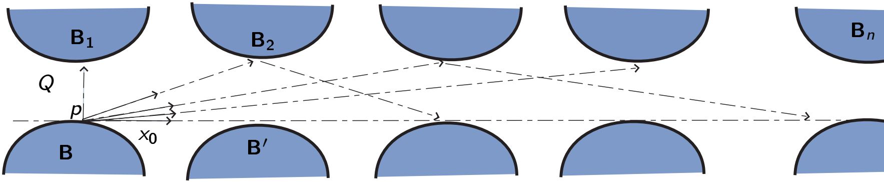


Figure.: Periodic trajectories with collisions only on flat points.

Billiard maps with cusps at flat points

$\exists \varepsilon_0 > 0$, $\beta > 2$, the ε_0-neighborhood of the cusp P satisfies:

$$z_1(s) = \beta^{-1}s^{\beta}, \quad z_2(s) = -\beta^{-1}s^{\beta}, \quad \forall s \in [0, \varepsilon_0] \qquad (1)$$

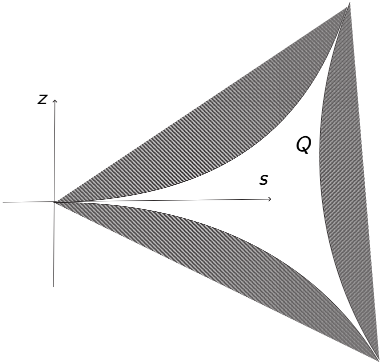


Fig.: A table with 1 cusp at the flat point for $\beta \in (2, \infty)$

Theorem (Zhang, 2017)

Consider the family of billiards with cusps at flat points on Q_β defined above, with $\beta > 2$. Then for any $\gamma \in (0, 1]$, any observables $f, g \in \mathcal{H}(\gamma)$ on $\mathcal{M}$, there exists $C_{f,g} = C(f, g) > 0$, such that

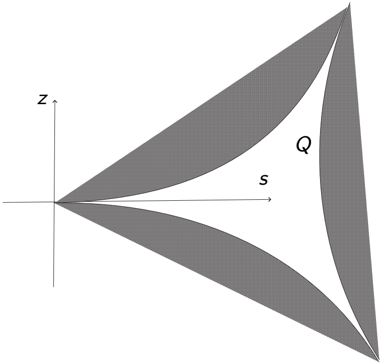
$$|\mu(f \circ \mathcal{F}^n \cdot g) - \mu(f)\mu(g)| \leq C_{f,g} n^{-\frac{1}{\beta-1}} = C_{f,g} n^{-(\alpha-1)},$$

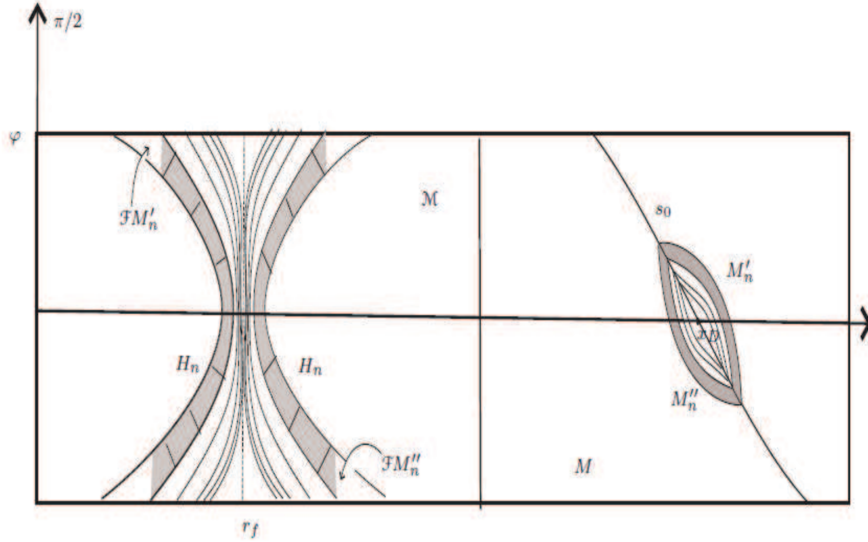
for $n \geq 1$.

- For the case when $\beta = 2$, the system corresponds to the dispersing billiards with cusps and enjoys mixing rates of order $\mathcal{O}(n^{-1})$ (by Chernov, Markarian, 2005)
- Note here $\frac{1}{\beta-1} \in (0, 1)$, which covers all rates slower than $\mathcal{O}(n^{-1})$. Let $\alpha = \frac{\beta}{\beta-1} \in (1, 2)$.

Billiard maps with cusps at a flat point

We define M containing all collisions on the opposite side of the cusp, $R : M \to \mathbb{N}$ as the first return time, and $F : M \to M$ as the induced map.



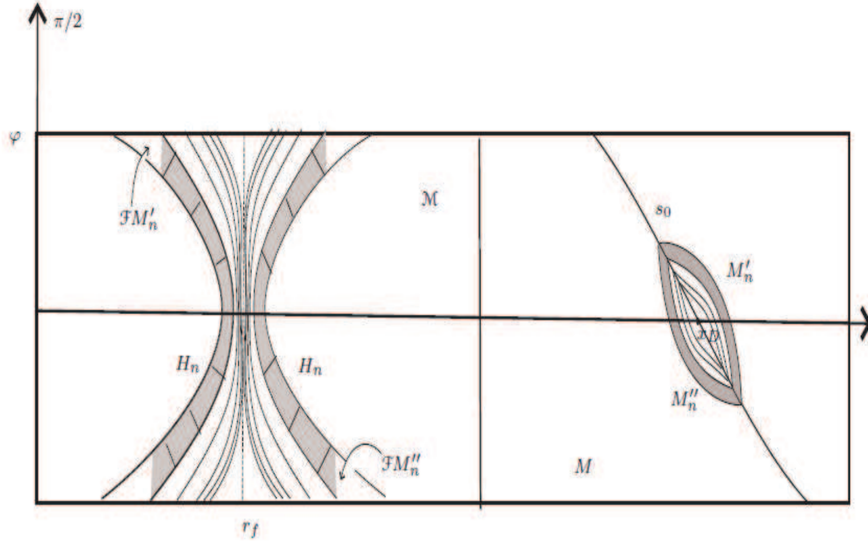


Right: Singularity curves of F near x_D (east wall midpoint)
Left: Same curves under $\mathcal{F}$ near the cusp in $\mathcal{M}$ ($\cos\varphi = \pm Cx^{-\beta}$)

Power law return times via strip estimates

Let $M_n := \{x \in M : \mathcal{R}(x) = n\}$

- M_n has width $\sim n^{-\frac{\alpha^2+\alpha+1}{\alpha+1}}$, length $\sim n^{-\frac{\alpha}{\alpha+1}}$, density ~ 1.
- $\mu(M_n) \sim n^{-1-\alpha}$
- $\mu(x \in M : R > n) \sim n^{-\alpha}$
- $\mu(x \in \mathcal{M} : R > n) \sim n^{-(\alpha-1)} = n^{-\frac{1}{\beta-1}}$,
- We assume $f \in \mathcal{H}_\gamma$ is Holder continuous on $\mathcal{M}$, and there exists a neighborhood of U (resp. U'') containing the cusp $r = r'$ (resp. $r = r''$), such that f has the same sign on $U' \cup U''$.
- $\tilde{f}(x) := f(x) + f(\mathcal{F}x) + \cdots + f(\mathcal{F}^{R(x)-1}x), \quad x \in M.$



Right: Singularity curves of F near x_D (east wall midpoint)
Left: Same curves under $\mathcal{F}$ near the cusp in $\mathcal{M}$ ($\cos\varphi = \pm Cx^{-\beta}$)

Inducing scheme:

For $\{f \circ \mathcal{F}^n\}$ in $(\mathcal{M}, \mu)$, we consider $\{\tilde{f} \circ F^n\}$ in (M, μ_M), with

$$\tilde{f}(x) := f(x) + f(\mathcal{F}x) + \cdots + f(\mathcal{F}^{R(x)-1}x), \quad x \in M.$$

Let $\mathcal{S}_n f = f + \cdots + f \circ \mathcal{F}^{n-1}$, $S_n\tilde{f} = \tilde{f} + \cdots + \tilde{f} \circ F^{n-1}$.

Theorem

If $\{\tilde{f} \circ F^n\}$ satisfies a stable law, then $\{f \circ \mathcal{F}^n\}$ satisfies a stable law too:

$$\frac{S_n\tilde{f}}{\sqrt[\alpha]{n}} \xrightarrow{d} S_{\alpha,\tilde{\sigma}_f} \Rightarrow \frac{\mathcal{S}_n f}{\sqrt[\alpha]{n}} \xrightarrow{d} S_{\alpha,\sigma_f}$$

with $\tilde{\sigma}_f^\alpha \mu(M) = \sigma_f^\alpha$.

Here $S_{\alpha,\sigma}$ is a stable random variable with characteristic function

$$\mathbb{E}\left(e^{iuS_{\alpha,\sigma}}\right) = \exp\left(-|u\sigma|^\alpha\left(1 - i\mathrm{sign}(u)\tan\frac{\pi\alpha}{2}\right)\right). \qquad (2)$$

Stable law for $\tilde{f}$ [Jung & Zhang (2017)]

$$I_f = \frac{1}{4}\int_0^{\pi} (f(r', \varphi) + f(r'', \varphi)) \cos^{\frac{1}{\alpha}} \varphi \, d\varphi$$

$$I_1 = \int_0^{\pi/2} \cos^{\frac{1}{\alpha}} \varphi \, d\varphi.$$

where r', r'' are r coordinates of the cusp point on both boundary.

Theorem

Suppose $I_f > 0$. Then $\tilde{\sigma}_{\tilde{f}} = \frac{I_f}{I_1} \cdot \tilde{\sigma}_R$, and for $n \to \infty$,

$$\frac{S_n \tilde{f}}{\sqrt[\alpha]{n}} \xrightarrow{d} S_{\alpha, \tilde{\sigma}_{\tilde{f}}} \quad (3)$$

Diffusion constant $\tilde{\sigma}_R$ and $\tilde{\sigma}_{\tilde{f}}$.

$$\mu(x \in M\ :\ R > n) \sim n^{-\alpha} \Leftrightarrow \mu(x \in M\ :\ R > n^{\frac{1}{\alpha}}) \sim n^{-1}$$

$$\Leftrightarrow n\mu(x \in M\ :\ R > n^{\frac{1}{\alpha}}) \sim 1$$

Theorem

The diffusion constant satisfies

$$\tilde{\sigma}_R^{\alpha} := \lim_{n\to\infty} n\mu_M\left(x \in M : R(x) > n^{\frac{1}{\alpha}}\right) = \frac{1}{\mu(M)}\frac{2I_1^{\alpha}}{\beta|\partial Q|}$$

$$\tilde{\sigma}_{\tilde{f}}^{\alpha} = \frac{1}{\mu(M)}\frac{2I_f^{\alpha}}{\beta|\partial Q|} \propto \frac{I_f^{\alpha}}{\beta}$$

where
$I_f = \frac{1}{4}\int_0^{\pi}(f(r',\varphi) + f(r'',\varphi))\cos^{\frac{1}{\alpha}}\varphi\, d\varphi,\quad I_1 = \int_0^{\pi/2}\cos^{\frac{1}{\alpha}}\varphi\, d\varphi.$

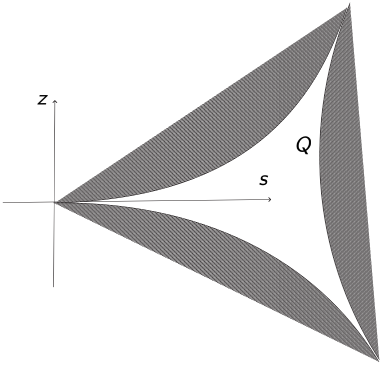


Fig.: A table with 3 identical cusps at flat points for $\beta \in (2, \infty)$

$$I_{f_i} = \frac{1}{4}\int_0^{\pi} (f(r_i', \varphi) + f(r_i'', \varphi)) \cos^{\frac{1}{\alpha}} \varphi \, d\varphi, \quad I_1 = \int_0^{\pi/2} \cos^{\frac{1}{\alpha}} \varphi \, d\varphi.$$

where r_i', r_i'' are r coordinates of the cusp point P_i on both boundary, $i = 1, 2, 3$.

Theorem (3 identical cusps case – Jung, Pene & Zhang (2018))

Suppose $I_{f_i} \neq 0$. Then $\tilde{\sigma}_{\tilde{f},i} = \frac{I_{f,i}}{I_1}\tilde{\sigma}_R$, and for $n \to \infty$,

$$\frac{S_n R - n\mu_M(R)}{\sqrt[\alpha]{n}} \xrightarrow{d} S_{\alpha,\tilde{\sigma}_R}; \qquad \frac{S_n \tilde{f}}{\sqrt[\alpha]{n}} \xrightarrow{d} S_{\alpha,\xi,\tilde{\sigma}_{\tilde{f}}}$$

$$S_{\alpha,\xi,\tilde{\sigma}_{\tilde{f}}} := \sum_{i=1}^{3} S_{\alpha,\xi_i,\tilde{\sigma}_{\tilde{f},i}}$$

is the sum of independent stable variables with
$\mathbf{E}\left(e^{iu\mathcal{S}_{\alpha,\xi,s}}\right) = \exp\left(-|us|^\alpha\left(1 - i\xi\operatorname{sign}(u)\tan\frac{\pi\alpha}{2}\right)\right), \quad u \in \mathbb{R}.$

$$\tilde{\sigma}^\alpha_{\tilde{f},i} := \frac{1}{3}\cdot\frac{2}{\mu(M)|\partial Q|}\cdot\frac{I^\alpha_{f,i}}{\beta} \quad \text{and} \quad \xi_i := \operatorname{sign}(I_{f,i}).$$

$$\tilde{\sigma}^\alpha_{\tilde{f}} = \sum_{i=1}^{3}\tilde{\sigma}^\alpha_{\tilde{f},i}, \quad \xi = \frac{1}{\tilde{\sigma}^\alpha_{\tilde{f}}}\cdot\left(\xi_1\tilde{\sigma}^\alpha_{\tilde{f},1} + \xi_2\tilde{\sigma}^\alpha_{\tilde{f},2} + \xi_3\tilde{\sigma}^\alpha_{\tilde{f},3}\right)$$

Skorohod topologies

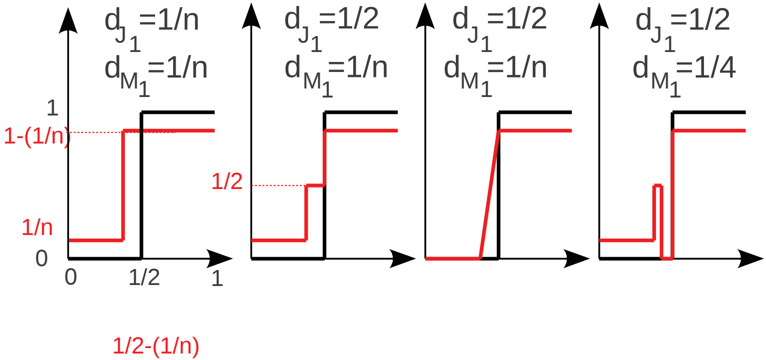
- The Skorokhod space $\mathbb{D} = \mathbb{D}([0,1] : \mathbb{R})$ consists of all catlag functions $\mathbf{x} : [0,1] \to \mathbb{R}$ that has a left limit $\mathbf{x}(t-)$ and right continuous. Remark: Skorohod space consists of path of Levy processes.
- $(\mathbb{D}, \|\cdot\|_\infty)$ is nonseparable space;
- J_1 and M_1 norms can make $\mathbb{D}$ into separable space. J_1 topology is the finest one, which is close to the uniform norm.
- $\mathbf{x}_n \to \mathbf{x}$ in $(\mathbb{D}, J_1)$ topology, if there exists a sequence of increasing homeomorphisms $\lambda_n : [0,1] \to [0,1]$, such that

$$\sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0 \Rightarrow \sup_{t\in[0,1]} |\mathbf{x}(\lambda_n(t)) - \mathbf{x}(t)| \to 0, \text{ as } n \to \infty$$

$$d_{J_1}(\mathbf{x}, \mathbf{y}) = \inf_{\lambda\in\Lambda} | \sup_{t\in[0,1]} |\lambda(t) - t| + \sup_{t\in[0,1]} |\mathbf{x}(\lambda(t)) - \mathbf{x}(t)|$$

- M_1 topology makes the space $\mathbb{D}$ to be complete linear normalized space.
- A complete graph of $\mathbf{x}$ is obtained by connecting $\mathbf{x}(t-)$ and $\mathbf{x}(t)$ using the line segment along the path. Let $\Gamma_{\mathbf{x}} = \{(u,t) : u \in \mathbb{R}, t \in [0,1]\}$ be a parametrization of its complete graph. Let $\Pi_{\mathbf{x}} = \{\Gamma_{\mathbf{x}}\}$ be all parametrization of the complete graph of $\mathbf{x}$.
- $$d_{M_1}(\mathbf{x}_1, \mathbf{x}_2) = \inf_{(u_i,t_i)\in\Pi(\mathbf{x}_i), i=1,2} \max(\|u_1 - u_2\|, \|t_1 - t_2\|)$$

  Then d_{M_1} satisfies the triangle inequality.
- Let the graph of $\mathbf{x}_n \to \mathbf{x}$, such that $\mathbf{x} = \mathbb{I}_{[1/2,1]}$ and $\mathbf{x}_n = \mathbf{x}$ on $[0, 1/2 - n^{-1}] \cup [1/2, 1]$, and is linear continuous in between. Then $\mathbf{x}_n \to \mathbf{x}$ in M_1 but not J_1.


d_{J_1}=1/n
d_{M_1}=1/n
d_{J_1}=1/2
d_{M_1}=1/n
d_{J_1}=1/2
d_{M_1}=1/n
d_{J_1}=1/2
d_{M_1}=1/4
1
1-(1/n)
1/n
0
0
1/2
1
1/2
1/2-(1/n)

$$\mathcal{W}_n(t) := n^{-1/\alpha}\mathcal{S}_{[nt]}f, \quad \tilde{\mathcal{W}}_n(t) := n^{-1/\alpha}S_{[nt]}\tilde{f}$$

Theorem (P.Jung, F.Pene, Zhang (2018))

$$(\tilde{\mathcal{W}}_n)_n = \frac{S_[nt]\tilde{f}}{n^{\frac{1}{\alpha}}} \to W(t)$$

converges in distribution, in the Skorokhod J_1-topology, where $W(t)$ is an α-stable Lévy motion with jumps, and $W(1) = \sum_{i=1}^{3} S_{\alpha,\xi_i,\tilde{\sigma}_{\tilde{f},i}}$, such that $S_{\alpha,\xi_i,\tilde{\sigma}_{f,i}}$ are independent stable random variables with characteristic function

$$\mathbf{E}\left(e^{iu\mathcal{S}_{\alpha,\xi,s}}\right) = \exp\left(-|us|^{\alpha}\left(1 - i\xi sign(u)\tan\frac{\pi\alpha}{2}\right)\right), \quad u \in \mathbb{R}.$$

$$\tilde{\sigma}^{\alpha}_{\tilde{f},i} := \frac{1}{3}\frac{2I^{\alpha}_{f,i}}{\beta\mu(M)|\partial Q|} \quad \textit{and} \quad \xi_i := sign(I_{f,i}).$$

- $$\tilde{f}_0 = (\mathcal{R} - \tilde{\mu}(\mathcal{R})) \sum_{i=1}^{3} I_{f,i} \mathbf{I}_{M^{(i)}}, Z_{n,j} = \frac{\tilde{f}_0 \circ F^j}{n^{1/\alpha}}, \quad \mathcal{Z}_n(t) = \frac{S_{[nt]} \tilde{f}_0}{n^{1/\alpha}}$$
- Define the family of point processes $(N_n)_n$ on $(0, \infty) \times (\mathbb{R} \setminus \{0\})$
$$N_n := \sum_{j \geq 1} \delta_{\left(\frac{j}{n}, \frac{Z_{j-1}}{n^{1/\alpha}}\right)}, \quad \sigma_x^\alpha := \sum_{i=1}^{3} \frac{\frac{2}{3} |I_{f,i}|^\alpha}{\beta \mu(M) |\partial Q|} \mathbf{1}_{\{x I_{f,i} > 0\}}$$
- N is a Poisson point process with intensity measure η having density $d\eta(x,t) = \alpha |x|^{-\alpha-1} \sigma_x^\alpha dxdt$ w.r.p.t. Lebesgue measure on $(0, \infty) \times (\mathbb{R} \setminus \{0\})$.
- $$(N_n)_n \xrightarrow{d} N, \quad \text{as } n \to \infty$$
—- we provided two proofs for this arguments. One follows from Francoise Pene and Benoit Saussol. *Spatio-temporal Poisson processes for visits to small sets, 2018.*

Theorem (Marta Tyran-Kaminska, 2010)

Assume the following two conditions.

Condition I. *(Point process convergence).* $(N_n)_n \xrightarrow{d} N$.

Condition II *(Vanishing small values). For all* $\eta > 0$

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \tilde{\mu}\left[\max_{0\le k\le n}\left|\sum_{j=0}^{k-1}\left(Z_{n,j}\cdot \mathbf{I}_{|Z_{n,j}|<\varepsilon} - \tilde{\mu}\left(Z_{n,1}\cdot \mathbf{I}_{|Z_{n,1}|<\varepsilon}\right)\right)\right| > \eta\right] = 0.$$

Then $\frac{1}{n^{1/\alpha}} S_n \tilde{f}$ *converges in distribution (in the* J_1*-metric) to an* α*-stable Lévy motion* $(\mathcal{W}(t))$ *such that* $\mathcal{W}(1)$ *has the same distribution as* $S_{\alpha,\xi,\sigma}$ *with*

$$\sigma^\alpha := \sum_{i=1}^{3} \frac{\frac{2}{3}|I_{f,i}|^\alpha}{\beta\mu(M)|\partial Q|} \quad \text{and} \quad \xi := \frac{\sum_{i=1}^{3} sign(I_{f,i})\,\frac{\frac{2}{3}|I_{f,i}|^\alpha}{\beta\mu(M)|\partial Q|}}{\sigma^\alpha}.$$

Condition II

- **Condition II** $\Leftarrow$
  Kolmogorov's maximal inequality + EDC of related processes.
- Let $X_1, \cdots, X_n, \cdots$ be i.i.d with zero mean, and finite variance σ^2. Then

$$\mathbf{P}(\max_{1\leq k\leq n} |S_k| \geq \lambda) \leq \frac{1}{\lambda^2}\mathrm{Var}(S_n)$$

In $\mathbb{R}\backslash\{0\}$, we fix a subset $I = I_{c,c'}$ with $c, c' > 0$. Our correlation bounds will depend on sets of the form

$$D_{n,j} := \{x \in M : \frac{1}{\sqrt[\alpha]{n}}(\mathcal{R}(x) - \tilde{\mu}(\mathcal{R})) \circ F^j \in I\}.$$

Lemma (Exponential decay of correlations for q-point marginals)

For every , there is a constant $C > 0$ and $\theta \in (0,1)$ such that

$$|\tilde{\mu}(D^c_{n,1} \cap \cdots \cap D^c_{n,q} \cap D^c_{n,q+k+1} \cap \cdots \cap D^c_{n,2q+k}) - \left(\tilde{\mu}(D^c_{n,1} \cap \cdots \cap D^c_{n,q})\right)^2| \leq C\theta^k$$

for all $k, n, q \in \mathbb{N}$ satisfying $2q + k \leq n$. Also, there exists $\varepsilon > 0$ such that for all $1 \leq i < j \leq n$

$$\tilde{\mu}(D_{n,i} \cap D_{n,j}) \leq o\left(\frac{1}{n^{1+\varepsilon}}\right). \quad (4)$$

Levy diffusion for the induced process

- $$(\tilde{\mathcal{W}}_n)_n = \frac{S_{[nt]}\tilde{f}}{n^{\frac{1}{\alpha}}} \to W(t)$$
  converges in distribution, in the Skorokhod J_1-topology;
- $W(t)$ is an α-stable Lévy motion with jumps.
- $W(1) = \sum_{i=1}^{3} S_{\alpha,\sigma_{f,i},\xi_i}$ is a summation of 3 independent stable variables.
- Question: How to extend it to the process associated to the original map?

Levy convergence for the orginal system

$$\mathcal{W}_n(t) := n^{-1/\alpha}\mathcal{S}_{[nt]}f, \quad \tilde{\mathcal{W}}_n(t) := n^{-1/\alpha}S_{[nt]}\tilde{f}$$

Theorem (Melbourne and Zweimüller(2015))

If $(\tilde{\mathcal{W}}_n(t), t \geq 0)$ converges weakly in the M_1-topology, as $n \to \infty$, to an α-stable Lévy motion $(W(t), t \geq 0)$ and

$$n^{-1/\alpha}\left(\max_{0\leq k\leq n} f^* \circ \mathcal{F}^k\right) \xrightarrow{d} 0. \tag{5}$$

then $(\mathcal{W}_n(s), s \geq 0) \xrightarrow{d} (\mathcal{W}(s\mu(M)), s \geq 0)$ in the M_1-topology.

$$f^*(x) = \left(\max_{0\leq\ell'\leq\ell\leq\mathcal{R}(x)}(\mathcal{S}_{\ell'}f - \mathcal{S}_\ell f)(x)\right) \wedge \left(\max_{0\leq\ell'\leq\ell\leq\mathcal{R}(x)}(\mathcal{S}_\ell f - \mathcal{S}_{\ell'} f)(x)\right).$$

Remark: If f has the same sign in a neighborhood of each cusp, then (5) holds.

Theorem (P.Jung, F.Pene, Zhang (2018)– billiard with 3 identical cusps)

$$(\mathcal{W}_n)_n = \frac{\mathcal{S}_[nt]f}{n^{\frac{1}{\alpha}}} \to W(t)$$

converges in distribution, in the Skorokhod M_1-topology, where $W(t)$ is an α-stable Lévy motion with $W(1) = \sum_{i=1}^{3} S_{\alpha,\xi_i,\sigma_{f,i}}$, such that $S_{\alpha,\xi_i,\sigma_{f,i}}$ are independent stable random variables with characteristic function

$$\mathbf{E}\left(e^{iu\mathcal{S}_{\alpha,\xi,s}}\right) = \exp\left(-|us|^\alpha\left(1 - i\xi sign(u)\tan\frac{\pi\alpha}{2}\right)\right), \quad u \in \mathbb{R}.$$

$$\sigma_{f,i}^\alpha := \frac{1}{3}\frac{2I_{f,i}^\alpha}{\beta|\partial Q|} \quad \text{and} \quad \xi_i := sign(I_{f,i}).$$

Question: Will they converge in J_1 topology?

Non-convergence for the J_1-metric for $f \circ \mathcal{F}^n$

Theorem

Let $(\mathcal{W}_n)(t) := \frac{\mathcal{S}_{[nt]} f}{n^{\frac{1}{\alpha}}}$. *Then* $(\mathcal{W}_n)$ *can not converge to a* α *Levy process* $W(t)$ *with jumps in the Skorokhod* J_1*-topology.*

Proof.

Let $w_n(t)$ be the continuous process obtained by linearization:

$$w_n(t) := W_n(t) + \frac{(nt - \lfloor nt \rfloor) f \circ \mathcal{F}^{\lfloor nt \rfloor}}{n^{1/\alpha}}, \quad t \geq 0.$$

Since f is uniformly bounded, we also have

$$\sup_{t \geq 0} |w_n(t) - W_n(t)| \leq \frac{\|f\|_\infty}{n^{1/\alpha}} \longrightarrow 0, \quad as \quad n \to \infty.$$

$W_n(t) \to W(t)$ in J_1 would imply $w_n(t) \to W(t)$ in J_1. Thus $W(t)$ is a continuous process, which is a contradiction. □

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Thank you!