Deviation inequalities for martingales

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Definition of martingale differences sequences

Definition

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A sequence $(\mathcal{F}_i)_{i\geqslant 1}$ of sub-σ-algebras of $\mathcal{F}$ is a filtration if for all $i \geqslant 0$, the inclusion $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ holds.
- We say that a sequence of real valued random variables $(X_i)_{i\geqslant 1}$ is a martingale differences sequence with respect to the filtration $(\mathcal{F}_i)_{i\geqslant 0}$ if for all $i \geqslant 1$, the random variable X_i is integrable, $\mathcal{F}_i$-measurable and $\mathbb{E}[X_i \mid \mathcal{F}_{i-1}] = 0$.

Examples

- An independent centered sequence is a martingale differences sequence with respect to the filtration $(\mathcal{F}_i)_{i\geqslant 0}$ defined as $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_i := \sigma(X_k, 1 \leqslant k \leqslant i)$.
- If $(X_i)_{i\geqslant 1}$ is a martingale differences sequence with respect to the filtration $(\mathcal{F}_i)_{i\geqslant 0}$, then for all $i \geqslant$, Y_i is an $\mathcal{F}_{i-1}$-measurable and bounded random variable, then $(X_iY_i)_{i\geqslant 1}$ is a martingale differences sequence with respect to the filtration $(\mathcal{F}_i)_{i\geqslant 0}$.

Goal: obtain deviation inequalities

Deviation inequalities

Let $(X_i)_{i\geqslant 1}$ be a martingale differences sequence with respect to the filtration $(\mathcal{F}_i)_{i\geqslant 0}$. Define $S_n := \sum_{i=1}^n X_i$. Our goal is to find a bound for

$$\mathbb{P}\left\{\max_{1\leqslant m\leqslant n} |S_m| > x\right\}$$

in terms of n, x and the distribution function of $\max_{1\leqslant i\leqslant n}|X_i|$ and $\sum_{i=1}^n \mathbb{E}[|X_i|^p \mid \mathcal{F}_{i-1}]$, $1 < p \leqslant 2$.

We know that there exists a constant C_p such that for all martingale differences sequence $(X_i)_{i\geqslant 1}$,

$$\mathbb{E}[|S_n|^p] \leqslant C_p \sum_{i=1}^n \mathbb{E}[|X_i|^p]. \tag{1}$$

Result (G., [1])

For each $1 < p \leqslant 2$, $q > 0$ and for any martingale differences sequence $(X_i, \mathcal{F}_i)_{i\geqslant 1}$, the following inequality holds for each $n \geqslant 1$ and $x > 0$:

$$\mathbb{P}\left\{\max_{1\leqslant i\leqslant n}|S_i| > x\right\} \leqslant \frac{2^q}{2^q-1} q 2^{-p}\int_0^1 \mathbb{P}\left\{\max_{1\leqslant i\leqslant n}|X_i| > 2^{-1-q/p} C_p^{-1/p} xu\right\} u^{q-1}\mathrm{d}u$$
$$+ \frac{2^q}{2^q-1} q 2^{-p}\int_0^1 \mathbb{P}\left\{\left(\sum_{i=1}^n \mathbb{E}[|X_i|^p \mid \mathcal{F}_{i-1}]\right)^{1/p} > 2^{-1-q/r'} C_p^{-1/p} xu\right\} u^{q-1}\mathrm{d}u,$$

where $S_n = \sum_{i=1}^n X_i$ and C_p is a constant satisfying (1) for any n and any martingale differences sequence.

Remark: Nagaev [2] treated the case $p = 2$.

Application: convergence rates in the law of large numbers

Identically distributed increments (G., [1])

Let $(X_i, \mathcal{F}_i)_{i\geqslant 1}$ be a martingale differences sequence such that:

1. $(|X_i|)_{i\geqslant 1}$ is identically distributed;
2. $\mathbb{E}\left[X_1^2 \log(1 + |X_1|)\right]$ is finite.

Then for all $\alpha \in (1/2, 1]$ and each positive x, the series $\sum_{n=1}^{+\infty} n^{2\alpha-2}\mathbb{P}\{\max_{1\leqslant i\leqslant n}|S_i| > n^\alpha x\}$ converges.

Moments of higher order (G., [1])

Let $p > 2$ and let $(X_i, \mathcal{F}_i)_{i\geqslant 1}$ be a martingale differences sequence such that $(|X_i|)_{i\geqslant 1}$ is identically distributed. Then for all $1/2 < \alpha \leqslant 1$,

- $\sup_{n\geqslant 1} n^{p(\alpha-1/2)}\mathbb{P}\{\max_{1\leqslant i\leqslant n}|S_i| > n^\alpha x\} \leqslant K_p \sup_{t>0} t^p \mathbb{P}\{|X_1| > t\} x^{-p/2-1}$;
- $\sum_{n=1}^{+\infty} n^{p(\alpha-1/2)-1}\mathbb{P}\{\max_{1\leqslant i\leqslant n}|S_i| > n^\alpha x\} \leqslant K_p x^{-p}\mathbb{E}[|X_1|^p]$.

Application: regression model

Assumptions

We consider the stochastic linear regression model given by

$$X_k = \theta\phi_k + \varepsilon_k, \quad 1 \leqslant k \leqslant n,$$

where

- $(X_k)_{1\leqslant k\leqslant n}$ are the observations,
- $(\phi_k)_{1\leqslant k\leqslant n}$ are the regression variables and
- $(\varepsilon_k)_{1\leqslant k\leqslant n}$ the driven noises.

We shall make the following assumptions:

(1) the sequence $(\phi_k)_{1\leqslant k\leqslant n}$ is independent;
(2) the σ-algebra generated by ϕ_k, $1 \leqslant k \leqslant n$ is independent of that generated by ε_k, $1 \leqslant k \leqslant n$;
(3) for each $k \in \{2, \dots, n\}$, $\mathbb{E}[\varepsilon_k \mid \sigma(\varepsilon_i, 1 \leqslant i \leqslant k-1)] = 0$ and $\mathbb{E}[\varepsilon_1] = 0$.

Let θ_n be the least square estimator defined by

$$\theta_n := \frac{\sum_{k=1}^n \phi_k X_k}{\sum_{i=1}^n \phi_i^2}.$$

Result (G., [1])

Suppose that the assumptions (1), (2) and (3) hold. Suppose that there exists constants C_1 and C_2 such that for any $i \in \{1, \dots, n\}$,

$$\mathbb{E}[|\varepsilon_i|^p] \leqslant C_1 \text{ and } \mathbb{E}\left[\varepsilon_i^2 \mid \sigma(\varepsilon_j, 1 \leqslant j \leqslant i-1)\right] \leqslant C_2 \text{ a.s.}$$

Then for any $p > 2$, $q > p$ and any $x > 0$,

$$\mathbb{P}\left\{|\theta_n - \theta|\sqrt{\sum_{i=1}^n \phi_i^2} > x\right\} \leqslant C_1 \frac{2^{q-2}}{2^q-1}\frac{q}{q-p} 2^{p+pq/2} x^{-p} + \frac{2^{q-2}}{2^q-1} q 2^{q+q^2/2} x^{-q} C_2^{q/2}.$$

References

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