Limit theorem for random fields 1 - weak

Davide Giraudo

Ruhr Universität Bochum

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Probabilistic Limit Theorems for Dynamical Systems

Contents

Setting

- Let $(\Omega, \mathcal{A}, \mu)$ be a probability space.
- For $q \in \{1, 2\}$, let $T_q : \Omega \to \Omega$ be a bijective, bi-measurable measure preserving map.
- We assume that $T_1 \circ T_2 = T_2 \circ T_1$ and denote $T^{i_1,i_2} = T_1^{i_1} \circ T_2^{i_2}$, $(i_1, i_2) \in \mathbb{Z}^2$.
- We are interested in the asymptotic behavior of the partial sums over rectangles, that is,

$$S_{n_1,n_2}(f) := \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} f \circ T^{i_1,i_2}$$

where $f : \Omega \to \mathbb{R}$.

Invariance principle

For $i_1, i_2 \geqslant 1$, we consider the unit cube with upper corner at (i_1, i_2) that is,

$$R_{i_1,i_2} := (i_1 - 1, i_1] \times (i_2 - 1, i_2].$$

For $f : \Omega \to \mathbb{R}$, $t_1, t_2 \in [0, 1]$, $n_1, n_2 \in \mathbb{N}$, the partial sum process is

defined by

$$W_{n_1,n_2}(f, t_1, t_2) := \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \lambda \left(([0, n_1 t_1] \times [0, n_2 t_2]) \cap R_{i_1.i_2} \right) U^{i_1,i_2} f,$$

where λ denotes the Lebesgue measure on $\mathbb{R}^2$ and $U^{i_1,i_2} f(\omega) = f\left(T^{i_1,i_2}\omega\right)$.

Note that $W_{n_1,n_2}\left(f, \frac{i_1}{n_1}, \frac{i_2}{n_2}\right) = S_{i_1,i_2}(f)$ and $(t_1, t_2) \mapsto W_{n_1,n_2}(f, t_1, t_2)$ is continuous.

Invariance principle

Definition

We say that f satisfies the invariance principle if $\left(W_{n_1,n_2}(\cdot)/(n_1 n_2)^{1/2}\right)_{n_1,n_2 \geqslant 1}$ converges in distribution in $C\left([0,1]^2\right)$ endowed with the uniform norm, that is, if there exists a processes W with paths in $C\left([0,1]^2\right)$ such that

$$\lim_{\min\{n_1,n_2\}\to+\infty} \mathbb{E}\left[g\left(W_{n_1,n_2}(\cdot)/(n_1 n_2)^{1/2}\right)\right] = \mathbb{E}\left[g(W)\right]$$

for each continuous and bounded function $g\colon C\left([0,1]^2\right) \to \mathbb{R}$.

If $\left(f \circ T^{i_1,i_2}\right)_{i_1,i_2\in\mathbb{Z}}$ is a centered, square integrable i.i.d. random field, then f satisfies the invariance principle.

In dimension one, we only have an application $T\colon \Omega \to \Omega$. If $\left(f \circ T^i, T^{-i}\mathcal{F}_0\right)_{i\in\mathbb{Z}}$ is a martingale differences sequence, then f satisties the invariance principle.

Martingale approximation

In dimension one, martingale approximation is a useful tool for establishing the invariance principle. The idea dates back to Gordin (1969).

If f is such that there exists $m\colon \Omega \to \mathbb{R}$ such that $\left(m \circ T^i, T^{-i}\mathcal{F}_0\right)_{i\in\mathbb{Z}}$ is a stationary martingale differences sequence and

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \left\| \max_{1\leqslant i\leqslant n} \left|S_i\left(f-m\right)\right| \right\| = 0,$$

then f satisfies the invariance principle.

Question

1. What plays the role of martingales in dimension greater than one?
2. How to develop a satisfactory theory of martingale approximations in the multidimensional setting?

Commuting filtrations

Let $\mathcal{F}_{0,0}$ be a sub-σ-algebra of $\mathcal{A}$ such that $T_q \mathcal{F}_{0,0} \subset \mathcal{F}_{0,0}$, $q = 1, 2$. Then defining $\mathcal{F}_{i_1,i_2} := T^{-i_1,-i_2}\mathcal{F}_{0,0}$ yields a filtration. Define

$$\mathrm{Proj}_{i_1,i_2}(Y) := \mathbb{E}\left[Y \mid \mathcal{F}_{i_1,i_2}\right], \quad Y \in \mathbb{L}^1.$$

We assume that the filtration $(\mathcal{F}_{i_1,i_2})_{i_1,i_2 \in \mathbb{Z}}$ is commuting in sense that

$$\mathrm{Proj}_{i_1,i_2} \circ \mathrm{Proj}_{j_1,j_2} = \mathrm{Proj}_{j_1,j_2} \circ \mathrm{Proj}_{i_1,i_2} = \mathrm{Proj}_{\min\{i_1,j_1\},\min\{i_2,j_2\}}.$$

Examples

1. T is the shift on $\mathbb{R}^{\mathbb{Z}^2}$ and $\mathcal{F}_{i_1,i_2} := \sigma\left(\varepsilon_{u_1,u_2}, u_1 \leqslant i_1, u_2 \leqslant i_2\right)$, where $(\varepsilon_{u_1,u_2})_{u_1,u_2 \in \mathbb{Z}}$ is an i.i.d. random field.

2. Assume that $\left(e_j^{(q)}\right)_{j \in \mathbb{Z}}$, $q \in \{1,2\}$ is i.i.d. and the sequence $\left(e_j^{(1)}\right)_{j \in \mathbb{Z}}$ is independent of $\left(e_j^{(2)}\right)_{j \in \mathbb{Z}}$. Let $\mathcal{F}_{i_1,i_2} := \sigma\left(e_{j_q}^{(q)}, j_q \leqslant i_q, 1 \leqslant q \leqslant 2\right)$.

Definition of orthomartingale, first properties

Definition

Let $\left(\mathcal{F}_{i_1,i_2}\right)_{i_1,i_2\in\mathbb{Z}} = \left(T^{-i_1,-i_2}\mathcal{F}_{0,0}\right)_{i_1,i_2\in\mathbb{Z}}$ be a commuting filtration and $m\colon \Omega \to \mathbb{R}$.

We say that $\left(m \circ T^{i_1,i_2}\right)_{i_1,i_2\in\mathbb{Z}}$ is an orthomartingale differences random field if the function m is $\mathcal{F}_{0,0}$-measurable and for each $q \in \{1,2\}$, $\mathbb{E}\left[m \mid T_q\mathcal{F}_{0,0}\right] = 0$.

- For any $N_1 \in \mathbb{N}$, the sequence $\left(S_{N_1,n}(m)\right)_{n\geqslant 1}$ is a martingale.
- If $m \in \mathbb{L}^2$, then the family

$$\left\{\frac{1}{n_1 n_2}\max_{\substack{1\leqslant i_1\leqslant n_1\\ 1\leqslant i_2\leqslant n_2}}\left|S_{i_1,i_2}(m)\right|^2, n_1, n_2 \geqslant 1\right\}$$

is uniformly integrable (Volný, Wang, 2014).

Definition of orthomartingale approximation

If T_1 or T_2 is ergodic, then any stationary orthomartingale differences random field satisfies the invariance principle (Volný, 2015). In this case, an orthomartingale approximation leads to the invariance principle.

Definition

We say that the function f admits an orthomartingale approximation if there exists a square integrable function m such that $\left(m \circ T^{i_1,i_2}\right)_{i_1,i_2 \in \mathbb{Z}}$ is an orthomartingale differences random field such that

$$\lim_{N\to\infty} \sup_{n_1,n_2 \geqslant N} \frac{1}{\sqrt{n_1 n_2}} \left\| \max_{\substack{1 \leqslant i_1 \leqslant n_1 \\ 1 \leqslant i_2 \leqslant n_2}} \left| S_{i_1,i_2}(f-m) \right| \right\| = 0.$$

We need a sufficient condition for orthomartingale approximation.

Contractions

If f is an $\mathcal{F}_{0,0}$-measurable function and $i_1, i_2 \in \mathbb{N}$, we define

$$P^{i_1,i_2} f = \mathbb{E}\left[U^{i_1,i_2} f \mid \mathcal{F}_{0,0}\right],$$

$$P_1^{i_1} f = \mathbb{E}\left[U^{i_1,0} f \mid \mathcal{F}_{0,0}\right] \text{ and}$$

$$P_2^{i_2} f = \mathbb{E}\left[U^{0,i_2} f \mid \mathcal{F}_{0,0}\right].$$

Then

$$P^{i_1,i_2} \circ P^{j_1,j_2} f = P^{i_1+j_1,i_2+j_2} f$$

$$P^{i_1,i_2} U^{-i_1,-i_2} f = f.$$

If $\left(m \circ T^{i_1,i_2}\right)_{i_1,i_2 \in \mathbb{Z}}$ is an orthomartingale differences random field, then

for all $i_1,i_2 \geqslant 1$,

$$P^{i_1,i_2} m = 0.$$

Orthomartingale-coboundary decomposition

Proposition

Let f be an $\mathcal{F}_{0,0}$-measurable function. Assume that there is an $\mathcal{F}_{0,0}$-measurable and square integrable function h such that $f = (I - P_1)(I - P_2)h$. Then there exists square integrable functions m, m_1, m_2 and g such that

$$f = m + \left(I - U^{1,0}\right) m_2 + \left(I - U^{0,1}\right) m_1 + \left(I - U^{1,0}\right)\left(I - U^{0,1}\right) g,$$

$\left(m \circ T^{i_1,i_2}\right)_{(i_1,i_2)\in\mathbb{Z}^2}$ *is an orthomartingale differences random field and* $\left(m_q \circ T_q^i\right)_{i\in\mathbb{Z}}$, $q \in \{1;2\}$ *are martingale differences sequences.*

Idea of proof: write

$$f = \left(I - U_1^{-1}P_1 + U_1^{-1}P_1 - P_1\right)\left(I - U_2^{-1}P_2 + U_2^{-1}P_2 - P_2\right) h$$

and expand.

A sufficient condition for orthomartingale approximation

Definition

Let $f : \Omega \to \mathbb{R}$ be a measurable function. The plus semi-norm, denoted by $\|\cdot\|_+$, is defined by

$$\|f\|_+ := \lim_{N \to +\infty} \sup_{n_1, n_2 \geqslant N} \frac{1}{(n_1 n_2)^{1/2}} \left\| \max_{\substack{1 \leqslant i_1 \leqslant n_1 \\ 1 \leqslant i_2 \leqslant n_2}} |S_{i_1, i_2}(f)| \right\|.$$

Definition

Let $f : \Omega \to \mathbb{R}$ be an $\mathcal{F}_{0,0}$-measurable function and let k be an integer greater or equal to 1. Let

$$A_k(f) := \frac{1}{k^2} \sum_{i_1, i_2 = 1}^{k} \left(I - P_1^{i_1}\right) \left(I - P_2^{i_2}\right)(f)$$

and $R_k(f) := f - A_k(f)$ (so that $f - R_k(f) = A_k(f)$).

Sufficient condition

Theorem (G., 2018+)

Assume that f is such that $\left\|R_k(f)\right\|_+ \to 0$ as k goes to infinity. Then f admits an orthomartingale approximation.

Idea of proof: write

$$f = m_k + (I - U_1)\, m'_k + (I - U_2)\, m''_k + (I - U_1)(I - U_2)\, g_k + R_k(f),$$

where $\left(m_k \circ T^{i_1,i_2}\right)_{(i_1,i_2)\in\mathbb{Z}^2}$ is an orthomartingale differences random field and $\left(m''_k \circ T_1^i\right)_{i\in\mathbb{Z}}$, $\left(m'_k \circ T_2^i\right)_{i\in\mathbb{Z}}$ are martingale differences sequences and $g_k \in \mathbb{L}^2$.

If $\left\|R_k(f)\right\|_+ \to 0$, then the sequence $(m_k)_{k\geqslant 1}$ is Cauchy in $\mathbb{L}^2$ and converges in $\mathbb{L}^2$ to some m.

Maxwell and Woodroofe condition

Theorem (G., 2018+)

Let T_1, T_2 be commuting measure preserving maps and let $\mathcal{F}_{0,0}$ be a sub-σ-algebra such that $\left(T^{-i_1,-i_2}\mathcal{F}_{0,0}\right)_{i_1,i_2\in\mathbb{Z}}$ is a commuting filtration. Let f be an $\mathcal{F}_{0,0}$-measurable square integrable function such that

$$\sum_{n_1,n_2\geqslant 1}\frac{1}{n_1^{3/2}n_2^{3/2}}\left\|\mathbb{E}\left[S_{n_1,n_2}(f)\mid\mathcal{F}_{0,0}\right]\right\|<+\infty.$$

Then there exists a function m such that $\left(m\circ T^{i_1,i_2}\right)_{i_1,i_2\in\mathbb{Z}}$ is an orthomartingale differences random field and

$$\limsup_{\min\{n_1,n_2\}\to+\infty}\frac{1}{\sqrt{n_1 n_2}}\left\|\max_{\substack{1\leqslant i_1\leqslant n_1\\ 1\leqslant i_2\leqslant n_2}}\left|S_{i_1,i_2}(f-m)\right|\right\|=0.$$

Remarks

Recall

$$\sum_{n_1, n_2 \geqslant 1} \frac{1}{n_1^{3/2} n_2^{3/2}} \left\| \mathbb{E} \left[S_{n_1, n_2} (f) \mid \mathcal{F}_{0,0} \right] \right\| < +\infty. \tag{C}$$

- Peligrad and Zhang (2017) obtained a central limit theorem under (C) (plus ergodicity of T_1).
- Like in dimension one (Peligrad, Utev, 2005), the weight $(n_1 n_2)^{-3/2}$ cannot be replaced by $a_{n_1} (n_1 n_2)^{-3/2}$, where $a_{n_1} \to 0$.
- The quantity $\left\| \mathbb{E} \left[S_{n_1, n_2} (f) \mid \mathcal{F}_{0,0} \right] \right\|$ involved in condition (C) can be estimated in the case of linear random fields of Volterra random fields.

Prospects

1. Marcinkievicz-Sygmund law of large numbers;
2. Law of the iterated logarithms;
3. U-statistics: $U_n = \sum_{1 \leqslant i < j \leqslant n} h(X_i, X_j)$, where $h \colon \mathbb{R}^2 \to \mathbb{R}$ and $(X_i)_{i \in \mathbb{Z}}$ is a stationary sequence.

   Denker and Gordin's paper (2014) uses tensor products and a kind of orthomartingale-coboundary decomposition.