Rates in almost sure invariance principle for non uniformly expanding maps

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joint work with J. Dedecker, A. Korepanov and F. Merlevède

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Luminy, november 2018

Strong invariance principles

- Let (X, Σ, μ) be a probability space. For $(X_n)_{n\geq 1} \subset L^2(\mu)$ set $S_n = X_1 + \ldots + X_n$. If T is an ergodic transformation preserving μ and $f \in L^2(\mu)$ we shall take $X_n = f \circ T^{n-1}$.

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- Let $(r_n)_{n\geq 1}$ be a non decreasing sequence with $r_n = O((n \log\log n)^{1/2})$. We say that $(S_n)_{n\geq 1}$ satisfies the almost sure invariance principle (ASIP) with rate $(r_n)_{n\geq 1}$, if one can redefine $(S_n)_{n\geq 1}$ without changing its distribution on a (richer) probability space on which there exists a sequence $(Z_i)_{i\geq 1}$ of iid centered Gaussian variables such that

$$\max_{k\leq n} |S_k - B_k| = o(b_n) \text{ almost surely,}$$

where $B_k = \sum_{i=1}^{k} Z_i$.

First results: the iid case

Assume that $(X_n)_{n\geq 1}$ is iid. Then, one has the following ASIPs with rate $(b_n)_{n\geq 1}$.

- Strassen ('64): $b_n = (n \log\log n)^{1/2}$ when $X_1 \in L^2$;
- Major ('76): $b_n = n^{1/p}$ when $X_1 \in L^p$, $2 < p \leq 3$;
- Komlós-Major-Tusnády ('75): $b_n = n^{1/p}$ when $X_1 \in L^p$, $p > 3$;
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When $(X_n)_{n\geq 1}$ is a stationary and ergodic sequence of (reverse) martingale differences and $X_1 \in L^p$, $2 \leq p \leq 4$, the above rates have been obtained up to some logarithmic factors.

A coupling inequality allowing to recover KMT's resullts

- Sakhanenko ('06). Let $(X_i)_{i\geq 1}$ be a sequence of independent, non necessarily identically distibuted, r.v.'s centered and in $\mathbb{L}^2$. Let $r>2$. On a richer probability space, one can construct a sequence $(Z_i)_{i\geq 1}$ of independent centered gaussian r.v.'s with $\mathrm{Var}(Z_n)=\mathrm{Var}(X_n)$ and such that for all $x>0$ and all $n\geq 1$,

$$\mathbb{P}\Big(\max_{1\leq k\leq n}\big|S_k-B_k\big|>c(r)x\Big)\leq\sum_{i=1}^{n}\mathbb{E}\min\Big(\frac{|X_i|^r}{x^r},\frac{|X_i|^2}{x^2}\Big)\,.$$

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- Let $p > 2$. Applying the above result in the iid setting with $n = 2^m$, $x = 2^{m/p}$ and $r > p$, one can prove that for every $\varepsilon > 0$

$$\sum_{m\geq 0} \mathbb{P}\Big(\max_{1\leq k\leq n}\big|S_k - B_k\big| > \varepsilon 2^{m/p}\Big)\,,$$

which implies KMT's results for the corresponding $p > 2$.

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- Sakhanenko ('84) proved a version of the above result for variables with exponential moments that implies KMT's result under an exponential moment.

A first dynamical example: the doubling map

Let $(X, \Sigma, \mu, T) = ([0,1), \mathcal{B}([0,1)), \lambda, T)$ where λ is the Lebesgue measure and $Tx = 2x \mod 1$. Let f be an Hölder observable on X.

- Until 2014, the best available rate in the ASIP has been $b_n = n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}$. This can be proved by mean of martingale approximation.

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  Notice that to study $(f \circ T^{n-1})_{n\geq 1}$ (under λ) it is enough to study $(f(Z_n)_{n\geq 1}$ where $Z_n = \sum_{k\geq 0} \frac{\varepsilon_{k+n}}{2^{k+1}}$ and $(\varepsilon_n)_{n\geq 1}$ is iid with $\mathbb{P}(\varepsilon_0 = 0) = \mathbb{P}(\varepsilon_0 = 1) = 1/2$.

- In 2014, Berkes-Liu-Wu proved the ASIP with rate $o(n^{1/p})$, $p > 2$, when $X_k = g(\varepsilon_k, , \varepsilon_{k+1}, \ldots)$ with $(\varepsilon_k)_{k\geq 1}$ are iid r.v.' s, $\|X_0\|_p < \infty$ and assuming some weak dependence conditions.

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- They assume a polynomial decay of convergence of $\|X_1 - X^*_{1,k}\|_p$ where $X^*_{1,k} = g(\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon^*_k, \varepsilon_{k+1}, \ldots)$ and $(\varepsilon^*_k)_{k\geq 1}$ is an independent copy of $(\varepsilon_k)_{k\geq 1}$.

Extensions of the work of BLW

It follows from the results of BLW that an Hölder observable f on $[0, 1)$, for every $\varepsilon > 0$, $(f \circ T^n)_{n \geq 1}$ satisfies the ASIP with rate $b_n = n^{\varepsilon}$.

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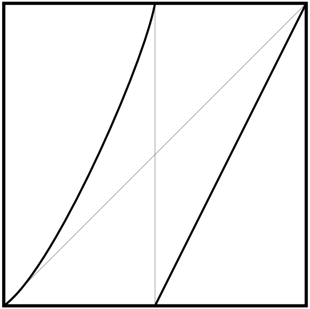
In 2018, C.-Dedecker-Merlevède adapted the ideas developed in BLW to obtain sharp conditions for the ASIP with rate $o(n^{1/p})$. For instance, when $X_k = g(\varepsilon_k, , \varepsilon_{k+1}, \ldots)$ with $(\varepsilon_k)_{k\geq 1}$ they obtained conditions relying on $\|X_1 - \tilde{X}^*_{1,k}\|_p$ where $\tilde{X}_{1,k} = g(\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon^*_k, \varepsilon^*_{k+1}, \ldots)$ and $(\varepsilon^*_k)_{k\geq 1}$ is an independent copy of $(\varepsilon_k)_{k\geq 1}$.

A second dynamical example: the LSV map

Let consider the LSV map (Liverani, Saussol and Vaienti, ('99)):

$$\text{for } 0 < \gamma < 1, \quad T(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$



Graph of f

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- The degree of intermittency is given by the parameter γ and is quantified by choosing an interval away from 0 such as $Y =]1/2, 1]$ and considering the first return time $\tau\colon Y \to \mathbb{N}$,

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- We have $C^{-1} n^{-1/\gamma} \le \mathrm{Leb}\,(\tau \ge n) \le C n^{-1/\gamma}$ (Gouezel'04 or Young'99)

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- If $\gamma < 1/2$, $n^{-1/2} S_n \to^d N(0, c^2)$ with

$$c^2 = \int f^2 \, d\mu + 2 \sum_{n=1}^{\infty} \int f \, f \circ T^n \, d\mu \qquad (*)$$

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- What about rates in the ASIP when $\gamma < 1/2$?

Previous results for the LSV map

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$$S_n - W_n = O(n^{\gamma'}(\log n)^{1/2}(\log\log n)^{(1+\varepsilon)\gamma'})$$

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  for all $\varepsilon > 0$, where $\gamma' = \max\{\gamma, 1/3\}$.
- Using an approximation via reverse martingale difference sequences and an ASIP for reverse MDS due to C.-Merlevède ('15), Korepanov-Kosloff-Melbourne ('18) proved the ASIP with rates

$$S_n(\varphi) - W_n = \begin{cases} o(n^{\gamma+\varepsilon}), & \gamma \in [1/4, 1/2[\\ O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}), & \gamma \in]0, 1/4[\end{cases}$$

  for all $\varepsilon > 0$ (No way to get better bounds with this method!)

Our results

Theorem (C.-Dedecker-Korepanov-Merlevède (submitted))

Let $\gamma \in (0, 1/2)$ and $f : [0, 1] \to \mathbb{R}$ be a Hölder continuous observable with $\int f \, d\mu = 0$. For the LSV map of parameter γ, the random process S_n satisfies the ASIP with variance c^2 given by (*) and rate $o(n^\gamma (\log n)^{\gamma+\varepsilon})$ for all $\varepsilon > 0$.

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Proposition (C-D-K-M. ('18))

There exists a Hölder continuous observable f with $\int f \, d\mu = 0$ such that

$$\limsup_{n\to\infty} (n \log n)^{-\gamma} |S_n - W_n| > 0$$

for all Brownian motions $(W_t)_{t\geq 0}$ defined on the same (possibly enlarged) probability space as $(S_n)_{n\geq 0}$.

A third dynamical example: definition and properties

- Let $\kappa \geq 0$. Let $T\colon [0,1] \to [0,1]$ be defined by

$$T(x) = \begin{cases} x(1 + \frac{c}{|\log x|^\kappa}), & x \leq 1/2 \\ 2x - 1, & x > 1/2 \end{cases} \tag{1}$$

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- There exists $\eta_1, \eta_2 > 0$ such that

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- There exists a unique absolutely continuous invariant probablity measure μ that is bounded and bounded away from 0 on Y.

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Theorem (C-D-K-M, in progress)

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Unfortunately, the obtained rate does not seem to be the best possible. To see that first notice that when $\kappa = 0$ one recovers the doubling map. In that case, our process is f applied to a linear process. Taking $f = \mathrm{Identity}$, it is possible to prove by a direct argument that $b_n = \log n$ in the ASIP.

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Also, let us mention that for Harris recurrent geometrically ergodic Markov chains and (only) bounded observables, Merlevède et Rio obtained the rate $b_n = \log n$.

Main ideas of the proof in the case of the LSV

- The main idea is to construct a stationary Markov chain $(g_n, n \in \mathbb{N})$ on a countable space S and an observable $\psi : \Omega \to \mathbb{R}$ (here $\Omega \subset S^{\mathbb{N}}$) such that $\int \psi \, d\mathbb{P}_\Omega = 0$ and setting

$$X_k = \psi(g_{k-1}, g_k, \ldots) \, , \; k \geq 1,$$

the process $(X_k)_{k\geq 1}$ on the probability space $(\Omega, \mathbb{P}_\Omega)$ has the same law as $(f \circ T^{k-1})_{k\geq 1}$ on $([0,1], \mu)$.

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- The Markov chain is related to the classical Young towers.
- Recall that $Y =]1/2, 1]$ and $\tau \colon Y \to \mathbb{N}$ be the inducing time $\tau(x) = \min\{n \geq 1 \colon f^n(x) \in Y\}$. Let $F : Y \to Y$ be the induced map: $F(x) = f^{\tau(x)}(x)$. Let α be the partition of Y into the intervals where τ is constant.

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- Let $\mathcal{A}$ denote the set of all finite words in the alphabet α, not including the empty word. Denote by $w = a_0 \cdots a_{n-1}$ an element of $\mathcal{A}$. Let also $h \colon \mathcal{A} \to \mathbb{N}$, $h(w) = \tau(a_0) + \cdots + \tau(a_{n-1})$

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- For the LSV maps the constructed Markov chain is aperiodic.

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$$|\psi(a) - \psi(b)| \leq C\theta^{\sum_{k=0}^{N} \mathbf{1}_{\{g_k \in S_0\}}} ,$$

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- With the help of the above property one can prove that there exists a bounded measurable function G_m such that, for any $r \geq 1$,

$$\|X_k - G_m(\varepsilon_{k-m}, \ldots, \varepsilon_{k+m})\|_1 \ll \mathbb{P}(S \geq m) + m^{-r/2}$$

where S is the meeting time

$$S = \inf\{n \geq 0 : g_n = g_n^*\}$$

here g_0^* has distribution ν and is independent of $(g_0, (\varepsilon_k)_{k\geq 1})$ and $g_{n+1}^* = U(g_n^*, \varepsilon_{n+1})$.

- The moments of S can be handled thanks to a result of Lindvall ('79) combined with a result of Korepanov ('18): Since $\text{Leb}\,(\tau \geq n) \leq Cn^{-1/\gamma}$ then $\mathbb{E}(h_{\gamma,\eta}(S)) < \infty$ for any $\eta > 1$ where $h_{\beta,\eta}(x) = x^{(1-\gamma)/\gamma}(log(1+x))^{-\eta}$.

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- Their proof consists first in providing a conditional Gaussian approximation by freezing some part of the $(\varepsilon_k)_k$, making suitable blocks and applying Sakhanenko's '06 result, and after of proceeding to a unconditional Gaussian approximation.

Thank you for your attention!