FCLT for toral automorphisms along the orbits of random walks

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Let (ζ_k) be $\mathbb{Z}^2$-valued i.i.d. random variables on $(\Omega, \mathbf{P})$, with $\mathbb{E}(\zeta_0) = 0$ and $\mathbb{E}|\zeta_0|^2 < \infty$. Let (Z_n) be the associate random walk, defined by $Z_0 = (0,0)$ and $Z_n = \zeta_0 + \cdots + \zeta_{n-1}$ with covariance matrix Σ. Assume the vector space generated by $(\underline{\ell} : \mathbf{P}(\zeta_0 = \underline{\ell}) > 0)$ is $\mathbb{R}^2$.

A result of Bolthausen [Bo89]: Let $X(\underline{\ell})$, $\underline{\ell} \in \mathbb{Z}^2$ be i.i.d. $\mathbb{R}$-valued centered random variables with a finite positive variance σ^2 defined on a space (X, μ), which are independent of the (ζ_i). It is shown in [Bo89] that

$$S_n(\omega, x) = \sum_{i=1}^{n} X(Z_i(\omega))(x)$$

satisfies the CLT w.r.t. $(\mathbf{P} \times \mu)$. Moreover, if we consider in $D(0,1)$ the random functions

$$Y_n(\omega, x, t) = \sqrt{\pi}(\det \Sigma)^{\frac{1}{4}} S_{\lfloor nt \rfloor}(\omega, x)/(\sigma\sqrt{n \log n}),$$

Bolthausen proved a functional central limit theorem (FCLT), i.e., a weak convergence in law to the Wiener measure of Y_n (with the distribution of (Y_n) taken with respect to the product measure $(\mathbf{P} \times \mu)$).

Let $\mathbb{Z}^2 \ni \underline{\ell} \mapsto A^{\underline{\ell}}$ be a totally ergodic $\mathbb{Z}^2$-action by automorphisms on $(\mathbb{T}^\rho, \mu)$, $\rho > 1$, with μ the Lebesgue measure (hence, defined by commuting $\rho \times \rho$ matrices A_1, A_2 with integer entries, determinant ± 1 such that the eigenvalues of $A^{\underline{\ell}} = A_1^{\ell_1} A_2^{\ell_2}$ are $\neq 1$, if $\underline{\ell} = (\ell_1, \ell_2) \neq (0, 0)$.

Explicit examples can be computed like the example below from the book of Henri Cohen on computational algebraic number theory.

$$A_1 = \begin{pmatrix} -3 & -3 & 1 \\ 10 & 9 & -3 \\ -30 & -26 & 9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 11 & 1 & -1 \\ -10 & -1 & 1 \\ 10 & 2 & -1 \end{pmatrix}.$$

Our aim is to replace the i.i.d. variables $(X(\underline{\ell}), \underline{\ell} \in \mathbb{Z}^2)$ in Bolthausen's model by

$$X(\underline{\ell}) = A^{\underline{\ell}} f = f \circ A^{\underline{\ell}}, \quad \underline{\ell} \in \mathbb{Z}^2,$$

generated by an observable f defined on the torus $\mathbb{T}^\rho$.

For this model, we would like to prove an annealed FCLT and a 'quenched' FCLT. It means that we fix ω and we look, for $\mathbf{P}$-a.e. ω, at the distribution with respect to the Lebesgue measure μ.

The method should be able to provide a quenched FCLT in the model studied by Bolthausen (see also [Guillotin et al] for a quenched result in this model), as well as in the algebraic action (which includes Ledrappier's example of a non mixing $\mathbb{Z}^d$-action of all orders) studied in the unpublished paper [CohCo16].

We present the following quenched FCLT:

Theorem 1. *For a real function f on $\mathbb{T}^\rho$, put*

$$S_n(f,\omega) := \sum_{k=0}^{n-1} A^{Z_k(\omega)} f.$$

If $f \in AC_0(\mathbb{T}^\rho)$ has a non zero asymptotic variance, the FCLT holds for

$$\Big(\frac{1}{\sqrt{n \log n}} S_{\lfloor nt \rfloor}(f,\omega)\Big)_{t\in[0,1]}$$

for a.e. ω.

The usual quenched CLT ($t = 1$) for this model was proved in [CohCo17]. The proof uses an algebraic result that we will describe later.

Steps of the proof.

As usual, we need to consider the finite dimensional distributions. That is, we need to show that for every $0 = t_0 < t_1 < \ldots < t_r \leq 1$,

$$(Y_n(t_1) - Y_n(t_0), ..., Y_n(t_r) - Y_n(t_{r-1})) \underset{n\to\infty}{\Longrightarrow} (W_{t_1}, ..., W_{t_r - t_{r-1}}). \qquad (1)$$

Also, we need to show tightness:

$$\forall \varepsilon > 0, \lim_{\delta\to 0} \limsup_n \mathbb{P}(\sup_{|t-s|\leq\delta} |Y_n(t) - Y_n(s)| \geq \varepsilon) = 0.$$

By Billingsley, the tightness can be checked as follows: If $\min_{1<i<r}(t_i - t_{i-1}) \geq \delta$, then

$$\mathbf{P}(\sup_{|s-t|\leq\delta} |Y_n(s) - Y_n(t)| \geq 3\varepsilon) \leq \sum_{i=1}^{r} \mathbf{P}(\sup_{t_{i-1}\leq s\leq t_i} |Y_n(s) - Y_n(t_{i-1})| \geq \varepsilon) \qquad (2)$$

We put

$$R_n(\underline{\ell}) = \sum_{j=1}^{n} \mathbf{1}_{Z_j=\underline{\ell}} \quad \text{for } \underline{\ell} \in \mathbb{Z}^2.$$

We denote by φ_f is the spectral density of f w.r.t. the action - it will be defined shortly. Let $a_1, \ldots, a_r \in \mathbb{R}$ and let $d_n = \sqrt{n \log n}/(\sqrt{\pi}(\det \Sigma)^{\frac{1}{4}})$. For the finite dimensional distributions (1) we may use the Cramer-Wold device, we have to show, for a.e. ω, convergence in distribution of

$$\sum_{j=1}^{r} a_j \Big(Y_n(t_j) - Y_n(t_{j-1}) \Big) =$$

$$\sum_{j=1}^{r} \sum_{\underline{\ell} \in \mathbb{Z}^2} a_j \Big(R_{\lfloor nt_j \rfloor}(\underline{\ell}) - R_{\lfloor nt_{j-1} \rfloor}(\underline{\ell}) \Big) X(\underline{\ell})/d_n$$

to a normal law $\mathcal{N}\Big(0, \varphi_f(\underline{0}) \sum_{j=1}^{r} a_j^2 (t_j - t_{j-1})\Big)$. For the i.i.d. case, Bolthausen proved the above convergence in distribution of the annealed model, that is, distributions w.r.t. $(\mathbf{P} \times \mu)$.

Equivalently, we have to show:

$$d_n^{-1}\Big(\sum_{j=1}^{r} a_j^2(t_j - t_{j-1})\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{r} a_j \sum_{k=\lfloor nt_{j-1}\rfloor}^{\lfloor nt_j\rfloor} A^{Z_k(\omega)} f(.)\Big)$$

$$\xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \varphi_f(\underline{0})).$$

Here φ_f is the spectral density of f w.r.t. the action, that is

$$\langle A^{\underline{\ell}} f, f\rangle = \int_{\mathbb{T}^2} \mathrm{e}^{2\pi i\langle \underline{\ell}, \underline{t}\rangle} \varphi_f(\underline{t}) d\underline{t} := \hat{\varphi}_f(\underline{\ell}), \quad \underline{\ell} \in \mathbb{Z}^2,$$

which is well defined and continuous, since

$$\sum_{\underline{\ell}\in\mathbb{Z}^2} |\langle A^{\underline{\ell}} f, f\rangle| \le \sum_{\underline{\ell}\in\mathbb{Z}^2} \sum_{\underline{k}\in\mathbb{Z}^\rho} |\hat{f}(\underline{k})||\hat{f}(A^{\underline{\ell}}\underline{k})| =$$

$$\sum_{\underline{k}\in\mathbb{Z}^\rho} |\hat{f}(\underline{k})| \Big(\sum_{\underline{\ell}\in\mathbb{Z}^2} |\hat{f}(A^{\underline{\ell}}\underline{k})|\Big) \le \Big(\sum_{\underline{k}\in\mathbb{Z}^\rho} |\hat{f}(\underline{k})|\Big)^2 = \|f\|^2_{AC_0} < \infty.$$

In the previous inequality, we used the fact that by total ergodicity the vectors $A^{\underline{\ell}}\underline{k}, \underline{\ell} \in \mathbb{Z}^2$, are pairwise distinct for each $\underline{k} \in \mathbb{Z}^\rho$.

The problem of the quenched CLT was considered in [CohCo17]. We introduced the notion of a "summation sequence", i.e., a sequence $w = (w_n)_{n \geq 1}$ of functions from $\mathbb{Z}^d$ to $\mathbb{R}$ with

$$0 < \sum_{\underline{\ell} \in \mathbb{Z}^d} |w_n(\underline{\ell})| < \infty, \ \forall n \geq 1.$$

We studied there the *asymptotic behaviour in distribution of the self-normalized sums*

$$\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell})\, T^{\underline{\ell}} f./\| \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n(\underline{\ell})\, T^{\underline{\ell}} f.\|_2.$$

The random walk can be considered by taking

$$w_n(\underline{\ell}) = \#\{k < n : Z_k(\omega) = \underline{\ell}\}.$$

The result in [CohCo17] can be generalized for

$$w_n^{a_1,...,a_s}(\underline{\ell}) = a_1 w_n^1(\underline{\ell}) + ... + a_s w_n^s(\underline{\ell}),$$

where $a_1, ..., a_s$ are reals and w_n^i are s summation sequences.

In the next proposition we assume the existence of $\sigma_{w_i}^2(f) =$

$$\lim_n \Big(\sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^i(\underline{\ell})^2 \Big)^{-1} \| \sum_{\underline{\ell} \in \mathbb{Z}^d} w_n^i(\underline{\ell}) T^{\underline{\ell}} f \|_2^2, \ i = 1, ..., s.$$

Proposition. *If the following conditions are satisfied:*

$$\lim_n \frac{\sum_{\underline{\ell},\underline{\ell}'\in\mathbb{Z}^d} w_n^i(\underline{\ell})\, w_n^j(\underline{\ell}')\langle T^{\underline{\ell}} f, T^{\underline{\ell}'} f\rangle}{\sum_{\underline{\ell}\in\mathbb{Z}^d}(w_n^i(\underline{\ell})^2 + (w_n^j(\underline{\ell})^2)} = 0, i \neq j,$$

$$\sum_{i_1,\ldots,i_r\in\{1,\ldots,s\}^r} w_n^{i_1}(\underline{\ell}_1)\ldots w_n^{i_r}(\underline{\ell}_r)\, c(X_{\underline{\ell}_1},\ldots,X_{\underline{\ell}_r})$$

$$= o(\sum_{\underline{\ell}\in\mathbb{Z}^d} [w_n^1(\underline{\ell})^2 + \ldots + w_n^s(\underline{\ell})^2])^{r/2}, \forall r \geq 3,$$

then the process $\sum_{\underline{\ell}\in\mathbb{Z}^d} w_n^{a_1,\ldots,a_s}(\underline{\ell})\, T^{\underline{\ell}} f$ *after normalization satisfies the CLT:*

$$\frac{\sum_{\underline{\ell}\in\mathbb{Z}^d} w_n(\underline{\ell})^{a_1,\ldots,a_s} T^{\underline{\ell}} f}{(a_1^2 \sum_{\underline{\ell}\in\mathbb{Z}^d} w_n^1(\underline{\ell})^2 + \ldots + a_s^2 \sum_{\underline{\ell}\in\mathbb{Z}^d} w_n^s(\underline{\ell})^2)^{\frac{1}{2}}}$$

$$\Rightarrow \mathcal{N}(0, \sum_{i=1}^s a_i^2\, \sigma_{w_i}^2(f) / \sum_{i=1}^s a_i^2).$$

Variance and asymptotic orthogonality of the cross terms.

For any intervals $I, J \subset [1, n]$ and $\underline{p} \in \mathbb{Z}^2$, put

$$V_{I,J,\underline{p}}(\omega) :=$$

$$\int \Big(\sum_{u \in I} e^{2\pi i \langle Z_u, \underline{t} \rangle}\Big) \Big(\sum_{v \in J} e^{-2\pi i \langle Z_v, \underline{t} \rangle}\Big) e^{-2\pi i \langle \underline{p}, \underline{t} \rangle} \, d\underline{t}$$

$$= \#\{(u, v) \in I \times J : \, Z_u - Z_v = \underline{p}\} \geq 0.$$

For $I = J$, we write simply $V_{I,\underline{p}}(\omega)$.

Proposition 1. *For* $0 < A < B < C < D < 1$, *we have for a.e.* ω *and for every* $\underline{p} \in \mathbb{Z}^2$

$$V_{[nA,nB],[nC,nD],\underline{p}} = o(n \log n).$$

Remarks. Bolthausen proved the above for the probability of the cross intersections. In [CohCo17] we proved $\lim V_{[1,n],\underline{p}}/E(V_{[1,n],\underline{p}}) = 1$ a.e. and [Lew93] proved $\mathbb{E}(V_{[1,n],\underline{p}}) \sim cn \log n$.

Sketch of proof.

We will use the following elementary lemma:

Lemma 1.

Let $(y(j), j \geq 1)$ *be a sequence with values in* $\{0, 1\}$ *such that* $\lim_n \frac{1}{n} \sum_{j=1}^{n} y(j) = a$, *with* $0 < a \leq 1$. *Let* (k_n) *be the sequence of successive times such that* $y(k_n) = 1$, *then, for every* $\delta > 0$, *there is* $N(\delta)$ *such that, for* $N \geq N(\delta)$,

$$k_{n+1} - k_n \leq \delta N, \forall\, n \in [1, N].$$

If $z_n := \max(j \leq n : y(j) = 1)$, then $n - z_n = o(n)$.

We use this lemma for the successive returns of a point ω into specific sets under the iterates of the shift θ. It will be useful for the finite dimensional distributions part and also for the tightness part.

Proof of Proposition 1

We omit $\underline{p}$ as it is fixed. So we write $V_{[1,n]}(\omega)$ instead of $V_{[1,n],\underline{p}}(\omega)$. Usually we omit also ω. For a fixed $A \in (0,1)$, put:

$$U(n, A, \omega) := V_{[1,n]}(\omega) - V_{[1,nA]}(\omega) - V_{[nA,n]}(\omega).$$

Claim: $U(n, A, \omega) := o(n \log n)$ a.e.

The result for the cross term will follow from this claim. Indeed we have:

$$V_{[1,n]} = V_{[1,nA]} + V_{[nA,n]} + V_{[1,nA],[nA,n]} + V_{[nA,n],[1,nA]}.$$

Each of the quantities $V_{[1,nA],[nA,n]}$ and $V_{[nA,n],[1,nA]}$ is positive and less than $U(n, A)$. Hence the result.

Put $\varphi_n(\omega, \underline{p}) := V_{n,\underline{p}}(\omega)/Cn \ln n$ with C some absolute constant. We have mentioned that, for any fixed $A \in [0, 1]$:

$$V_{[1,nA],\underline{p}}(\omega) \sim CnA \ln n, \ \lim_n \varphi_n(\omega, \underline{p}) = 1, \text{ a.s.}$$

Put

$$A(L, \delta) := \{\omega : \varphi_n(\omega) \in [1 - \delta, 1 + \delta], \forall n \geq L\}.$$

We have $\lim_{L\uparrow\infty} \mathbb{P}(A(L, \delta)) = 1$.

Using the pointwise ergodic theorem we obtain the existence of a set Ω_0 of full measure such that, for $\omega \in \Omega_0$, for all $L \geq 1$, all $M \geq 1$

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} 1_{A(L,M^{-1})}(\theta^i \omega) = \mathbb{P}(A(L, M^{-1})).$$

For $\delta > 0$, let M be an integer such that $\frac{1}{M} \leq \delta$. Take $L = L(M)$ such that for $\omega \in A(L, M^{-1})$ and $n \geq L$, we have $\varphi_n(\omega) = 1 + \delta_n$, with $\delta_n \in [-\delta, +\delta]$.

Since we are working with a.e. fixed ω's, taking if necessary L bigger, we can assume that $\omega \in A(L(M), M^{-1}) \bigcap \Omega_0$.

We apply the elementary lemma to the sequence of times of visits of $\theta^i\omega$ to $A(L, \frac{1}{M})$). Let n_A, n'_A be the visit times of $\theta^k\omega$ to $A(L(M), \frac{1}{M}))$ defined by $n_A \leq nA < n'_A$. By Lemma 1, for n big enough, we have $0 < n'_A - n_A \leq \delta n$ and we can write

$$n_A = nA\,(1 - \rho_n),\;\; n'_A = nA\,(1 - \rho'_n),$$

with $0 \leq \rho_n, \rho'_n \leq \delta$.

$U(n, A, \omega)$ can be written up to the absolute constant C:

$$\varphi(n, \omega)\, n \ln n - \varphi(nA, \omega)\, nA \ln(nA) -$$

$$\varphi(n(1 - A), \theta^{nA}\omega)\, n(1 - A) \ln(n(1 - A)).$$

It suffices to show that, for a constant C,

$$\frac{V_{[nA,n]}(\omega)}{n(1-A)\ln(n(1-A))} \in [1-C\delta, 1+C\delta]. \tag{3}$$

As $V_J(\omega)$ increases if we increase the set J, we have by the choice of n_A and n'_A:

$$V_{[n'_A,n]}(\omega) \le V_{[nA,n]}(\omega) \le V_{[n_A,n]}(\omega)$$

and $(1-\delta'_n)(n-n'_A)\ln(n-n'_A) \le V_{[n'_A,n]}(\omega)$, $V_{[n_A,n]}(\omega) \le (1+\delta_n)(n-n_A)\ln(n-n_A)$.

This shows (3), if n is big enough and $\max(\rho_n, \delta_n, \delta'_n, \delta''_n) \le$ constant δ.

About the tightness.

We prove the property when the observable f is a character. Since we are taking $f \in AC_0(\mathbb{T}^\rho)$ and the bounds will be independent of the specific character, we may conclude for $f \in AC_0(\mathbb{T}^\rho)$.

We use the following algebraic result (S-unit theorem) in its version given by Evertse, J.-H., Schlickewei, H. P., Schmidt, W. M (2002)):

Theorem. *Let K be an algebraically closed field of characteristic 0 and let r be a natural number. Let Γ_r be a subgroup of the multiplicative group $(K^*)^r$ of finite rank ρ. For any $(a_1, ..., a_r) \in (K^*)^r$, the number $A(a_1, ..., a_r, \Gamma_r)$ of solutions $x = (x_1, ..., x_r) \in \Gamma_r$ of the equation*

$$a_1 x_1 + ... + a_r x_r = 1,$$

such that no proper subsum of $a_1 x_1 + ... + a_r x_r$ vanishes, satisfies the estimate

$$A(a_1, ..., a_r, \Gamma_r) \leq A(r, \Gamma_r) = \exp((6r)^{3r}(\rho + 1)).$$

Moments of order 4

To show the tightness, we will use Móricz's maximal inequality. We need first to bound the moments of order 4. As remarked, it suffices to consider functions reduced to characters.

Let us consider on the torus $\mathbb{T}^\rho$ a character χ_v, $x \to \exp(2\pi i \langle v, x\rangle)$, where $v \in \mathbb{Z}^\rho \setminus \{0\}$.

Let $\alpha_{u,j}, u = 1, ..., \rho'$, be the set of distinct eigenvalues of A_j, $j = 1, 2$.

We write $\alpha_{\underline{u}}^{\underline{\ell}}$ for $\alpha_{u_1,1}^{\ell^1} \alpha_{u_2,2}^{\ell^2}$, if $\underline{u} = (u_1, u_2)$ and $\underline{\ell} = (\ell^1, \ell^2)$.

We first compute

$$\| \sum_{i=N_1}^{N_2} A^{Z_i}\chi\|^4 =$$
$$\#\{(i_1,i_2,i_3,i_4)\in[N_1,N_2]^4 : (A^{Z_{i_1}}-A^{Z_{i_2}}+A^{Z_{i_3}}-A^{Z_{i_4}})v=0\}(4)$$

There is a decomposition $\mathbb{C}^\rho = \oplus_k E_k$, with E_k simultaneously invariant by $\tilde{A}_j$, $j=1,2$, such that there is a basis B_k in which A_i restricted to E_k is represented in a triangular form with eigenvalues of $\tilde{A}_i$ on the diagonal. The number in (4) is bounded by

$$\#\{(i_1,i_2,i_3,i_4)\in[N_1,N_2]^4 :\ (\alpha_{\underline{u}}^{Z_1}-\alpha_{\underline{u}}^{Z_2}+\alpha_{\underline{u}}^{Z_3}-\alpha_{\underline{u}}^{Z_4})v_0=0\},$$

where v_0 is some non zero component of v and $\alpha_{\underline{u}}=\alpha_{u_1}\alpha_{u_2}$, with α_{u_1} (resp. α_{u_2}) some eigenvalue of A_1 (resp. A_2).

This number is less than $f_1^2(N_1,N_2-N_1)+f_2(N_1,N_2-N_1)$, where

$$f_1(N_1,N_2-N_1) := \#\{(i_1,i_2)\in[N_1,N_2]^2 :\ \alpha_{\underline{u}}^{Z_{i_1}}-\alpha_{\underline{u}}^{Z_{i_2}}=0\},$$
$$f_2(N_1,N_2-N_1) := \#\{(i_1,i_2,i_3,i_4)\in[N_1,N_2]^4 :\ \alpha_{\underline{u}}^{Z_{i_1}}-\alpha_{\underline{u}}^{Z_{i_2}}+\alpha_{\underline{u}}^{Z_{i_3}}-\alpha_{\underline{u}}^{Z_{i_4}} = 0,\quad \text{without vanishing proper sub-sum}\}.$$

The total ergodicity of the action implies that, if $\alpha_{\underline{u}}^{Z_{i_1}} = \alpha_{\underline{u}}^{Z_{i_2}}$, then $Z_{i_1} = Z_{i_2}$. Therefore we have:

$$f_1^2(N_1, N_2 - N_1) = (\#\{i_1, i_2 \in [N_1, N_2] : Z_{i_1} = Z_{i_2}\})^2,$$

which is the square of the self-intersections of the random walk starting from N_1. For f_2 we may write up to a constant factor:

$$f_2(N_1, N_2 - N_1) =$$
$$\#\{N_1 \leq i_4 < i_3 < i_2 < i_1 \leq N_2 : \alpha_{\underline{u}}^{Z_{i_1} - Z_{i_4}} - \alpha_{\underline{u}}^{Z_{i_2} - Z_{i_4}} + \alpha_{\underline{u}}^{Z_{i_3} - Z_{i_4}} = 1\}.$$

Let $\mathcal{S}$ be the set of triples $\underline{\ell}_1, \underline{\ell}_2, \underline{\ell}_3 \in \mathbb{Z}^2$ solving of the equation

$$\alpha_{\underline{u}}^{\underline{\ell}_1} - \alpha_{\underline{u}}^{\underline{\ell}_2} + \alpha_{\underline{u}}^{\underline{\ell}_3} = 1.$$

By the S-unit theorem, $\mathcal{S}$ is finite.

Using the local limit theorem for the random walk and a Borel-Cantelli argument, it can be deduced that for a.e. ω

$$f_2(1, n) \leq C(\omega) n (\log n)^5.$$

It means that f_1^2 is the dominant part and we obtain

$$\begin{aligned} &\| \textstyle\sum_{i=N_1}^{N_2} A^{Z_i(\omega)} \chi \|_{L^4(\mu)}^4 \leq \\ &f_1^2(N_1, N_2 - N_1) + f_2(N_1, N_2 - N_1) \\ &\leq C(\theta^{N_1}\omega) \, [f_1(N_1, N_2 - N_1)]^2 := \varphi(N_1, N_2 - N_1) \end{aligned} \tag{5}$$

and $\varphi(N_1, N_2 - N_1)$ is a sub-additive function.

Observe that the bounds do not depend on the character, but only on the eigenvalues of A_1, A_2. Now, we will use Móricz's maximal inequality.

Móricz's maximal inequality: A non-negative function $\psi(a,r)$ is called sub-additive if

$\psi(a,s)+\psi(a+s,r-s) \leq \psi(a,r)$, for every $a \geq 0$, $r \geq 1$and $0 \leq s \leq r$.

Theorem. Móricz (1977). *Let (ζ_k) be a sequence of random variables. Assume that for some sub-additive non-negative function $\psi(a,s)$ we have*

$$\Big\| \sum_{k=a}^{a+s} \zeta_k \Big\|_4^4 \leq \psi^2(a,s), \forall a \geq 0, s \geq 1.$$

Then, there is an absolute constant C such that

$$\Big\| \max_{1\leq s\leq r} \Big| \sum_{k=a}^{a+s} \zeta_k \Big| \Big\|_4^4 \leq C\psi^2(a,r). \tag{6}$$

Using (5) and (6) we conclude

$$\begin{aligned} &\Big\| \max_{N_1\leq k\leq N_2} \big| \textstyle\sum_{i=N_1}^{k} A^{Z_i(\omega)}\chi \big| \Big\|_{L^4(\mu)}^4 \\ &\leq C(\theta^{N_1}\omega)\,(N_2-N_1)^2\,(\ln(N_2-N_1))^2, \end{aligned} \tag{7}$$

for a constant $C(\omega) > 0$ which is a.e. finite and does not depend on the character χ.

Concluding the tightness:

For $M > 0$ big enough, $\Omega_M := \{\omega : C(\omega) \leq M\}$ has a probability $\mathbf{P}(\Omega_M) \geq \frac{1}{2}$. Let us fix $\omega \in \Omega$. The properties below will hold for $\mathbf{P}$-a.e. The previous bound (7) yields

$$\left\| \max_{N_1 \leq k \leq N_2} |\sum_{i=N_1}^{k} A^{Z_i(\omega)} \chi| \right\|_{L^4(\mu)}^4 \tag{8}$$
$$\leq M\,(N_1 - N_2)^2\,(\ln(N_1 - N_2))^2, \text{ if } \theta^{N_1}\omega \in \Omega_M.$$

Let δ be a positive number. If n is big enough, we can find $0 \leq \rho_1 < \rho_2 < ... < \rho_v \leq n$, visit times of ω to Ω_M under the iteration of the shift θ, such that $\rho_{i+1} - \rho_i$ is of order δn and v is of order $1/\delta$. This follows from Birkhoff ergodic theorem and the elementary Lemma.

Now, we consider the normalised sums

$$M_n(\chi, t) := \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\lfloor nt \rfloor} A^{Z_k} \chi.$$

For $\delta > 0$, let $v := \lfloor 1/\delta \rfloor$ and $t_i = \rho_i / n$. By our bound (8), we have

$$\Big\| \sup_{t_{i-1} \le s \le t_i} |M_n(\chi, s) - M_n(\chi, t_{i-1})| \Big\|_4^4 \le$$

$$M \frac{(t_i - t_{i-1})^2 n^2 \log^2[(t_i - t_{i-1}) n]}{n^2 \log^2 n} =$$

$$M \delta^2 [1 + \frac{\log \delta}{\log n}]^2;$$

hence

$$\sum_{i=1}^{v} \| \sup_{t_{i-1} \le s \le t_i} |M_n(\chi, s) - M_n(\chi, t_{i-1})| \|_4^4 \le$$

$$M \sum_{i=1}^{\lfloor 1/\delta \rfloor} \delta^2 [1 + \frac{\log \delta}{\log n}]^2 \le M \delta [1 + \frac{\log \delta}{\log n}]^2.$$

Taking the $\limsup_n$ and then $\delta \to 0$ yields the tightness by Billingsley's bound (2). As we mentioned we may pass to $f \in AC_0(\mathbb{T}^\rho)$.

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Thank You For Your Attention!