The group of disjoint 2-spheres in 4-space

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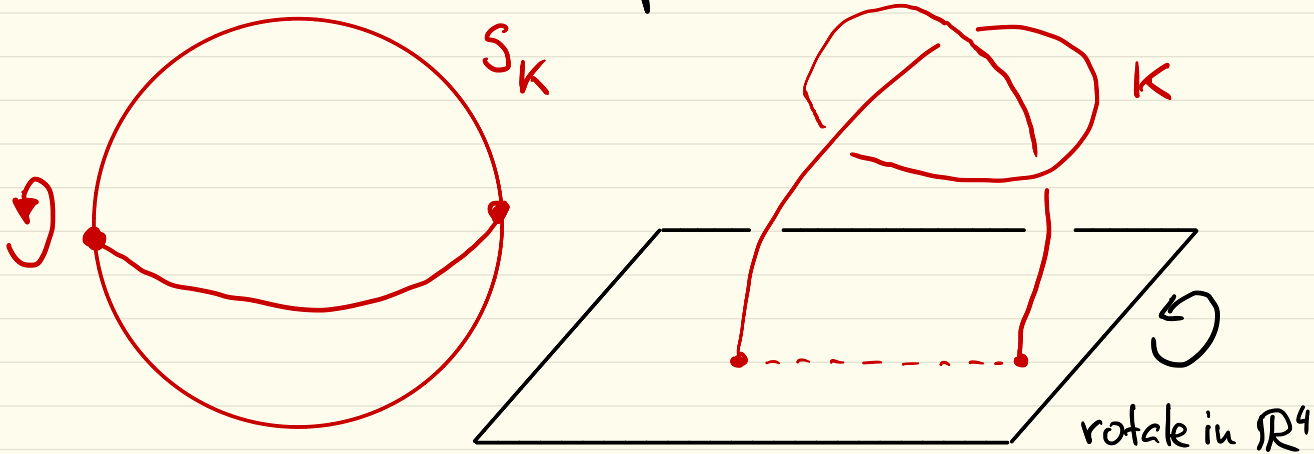
Brief history of 2-knots & -links

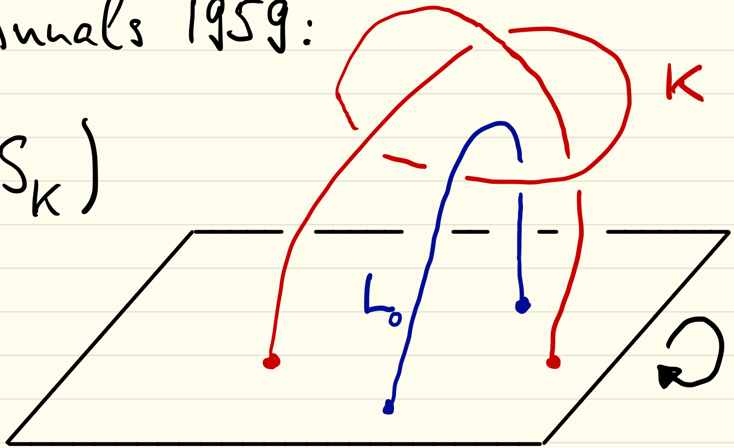
E. Artin, 1925: Spun 2-knots $\subseteq \mathbb{R}^4 \subseteq S^4$



$\pi_1(\mathbb{R}^4 \setminus S^2_K) \cong \pi_1(\mathbb{R}^3 \setminus K)$ so 2-knots are more complicated than 1-knots!

Andrews-Curtis, Annals 1959:

$$0 \neq S_{L_0} \in \pi_2(S^4 \setminus S_K)$$



Note:

S_{L_0} is a **2-unknot**, so $\pi_2(S^4 \setminus S_{L_0}) = 0$
$\cong$
S^1

But:

Tying trefoil into L_0 symmetrically gives L with $S_K \neq 0 \in \pi_2(S^4 \setminus S_L)$.

It's not hard to see that (S_K, S_L) is link homotopic to the unlink in the following sense (c.f. J. Milnor 1950's):
There is a homotopy through link maps $f: X \to Y$, $\pi_0(X) \hookrightarrow \pi_0(\mathrm{im}(f))$

Massey-Rolfsen, 1982, asked whether any 2-link is homotopically trivial?

Thm.: [A. Bartels & P.T., 1999] For $n \geq 2$, any link $S^n \sqcup \ldots \sqcup S^n \hookrightarrow S^{n+2}$ is link hom. trivial.

Proof via stratified Morse & surgery theory, homology of nilpotent groups...

$$LM^n_{p,q} := \left\{ \frac{\text{link maps } S^p \sqcup S^q \to S^n}{\text{link homotopy}} \right\}.$$

[Koschorke]:

For $1 \leq p,q \leq n-2$, $\#$ makes these into abelian monoids.

For example, $LM^n_{1,n-2} \cong \mathbb{Z}$ via linking number.

First unknown group is $LM^4_{2,2}$!

Theorem 1: [Schneiderman - T., 2016]

(i) $LM^4_{2,2} \cong \mathbb{Z}^{\mathbb{N}}$, i.p. reflection of S^4 gives inverses for $\#$.

(ii) A link map $(f_1, f_2) \in LM^4_{2,2}$ is trivial if f_1 is a (topological) embedding. Question of [Kirk].

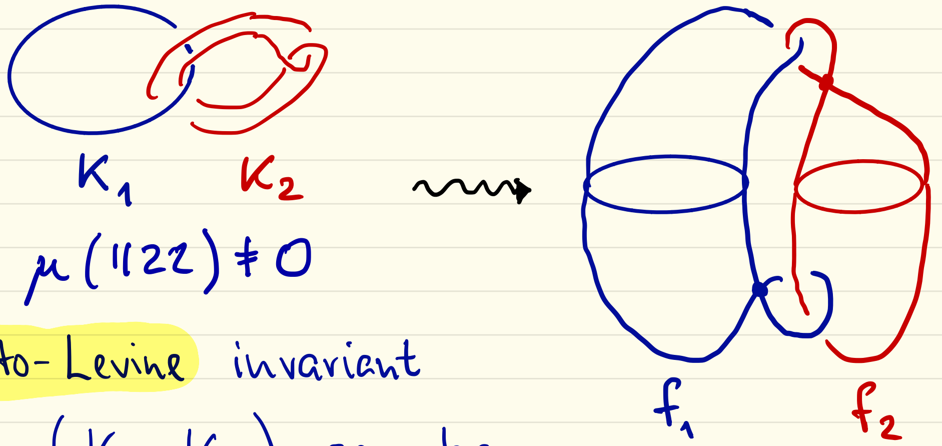
Focus today on "direct" proof of Corollary (ii).

Fenn-Rolfsen, 1986 : $LM^4_{2,2} \neq 0$, i.e.

there is an essential link map

$$(f_1, f_2) : S^2 \amalg S^2 \longrightarrow S^4,$$

a "suspension" of the Whitehead link in S^3:



$\mu(1122) \neq 0$

Sato-Levine invariant

of (K_1, K_2) can be

read off from self-intersections $\lambda(f_1, f_1) \in \mathbb{Z}[\mathbb{Z}]$

$H_1(S^4 \setminus f_2)$

Linked 'embeddings' in dimensions 3 and 4:

[1986]

$$JK : \left\{ \begin{array}{l} (K_1, K_2) : S^1 \sqcup S^1 \hookrightarrow S^3 \\ \bullet\ K_i \text{ are unknots} \\ \bullet\ lk(K_1, K_2) = 0 \end{array} \right\} \xrightarrow{\text{Jin-Kirk}} LM^4_{2,2}$$

Cochran's $\beta := \beta_1^i, \beta_2^i$

Kirk's $\sigma := \lambda(f_1, f_1), \lambda(f_2, f_2)$

lifts of $\mu(111\cdots122)$, $\mu(222\cdots211)$

$x = (1-t)(1-t^{-1})$

$$\mathbb{Z}^{\mathbb{N}} \oplus \mathbb{Z}^{\mathbb{N}} \cong x\cdot\mathbb{Z}[x] \oplus x\cdot\mathbb{Z}[x]$$

Theorem 2: [Schneiderman - T., 2016]

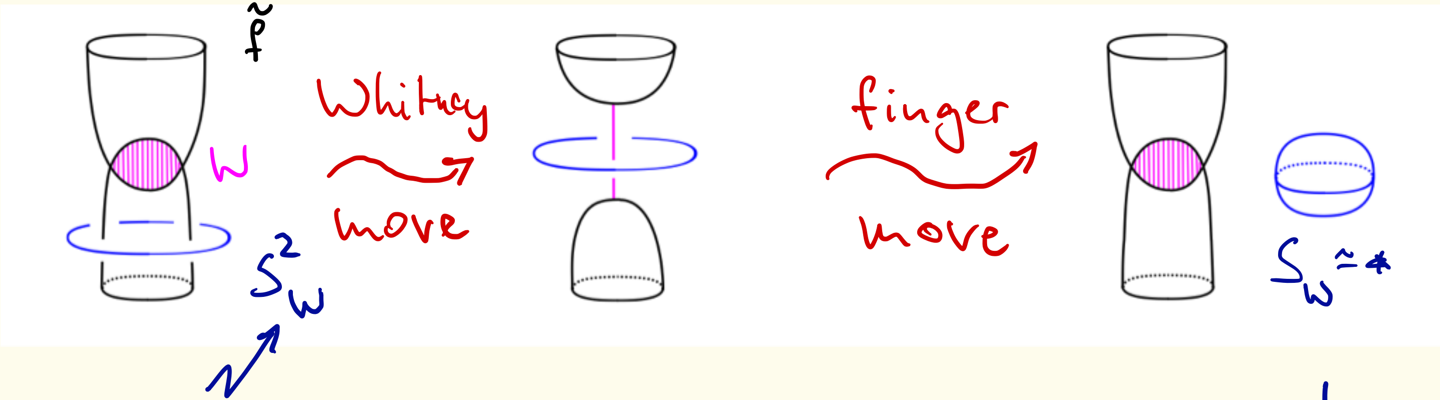
(i) JK is onto, (also from T. 1996 - stratified Morse)

(ii) σ is a monomorphism,

(iii) $\operatorname{cok}(\sigma) = \mathbb{Z}$, determined by $\mu(1122) = \mu(2211)$

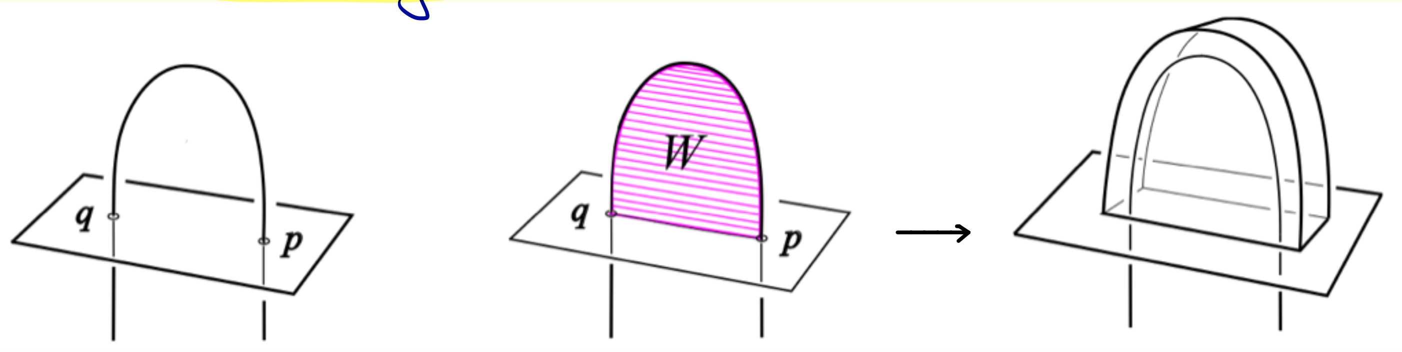
Let's revisit some basic tools in dimension 4:

- Any $f: \Sigma^2 \to M^4$ is homotopic to $\tilde{f}: \Sigma^2 \looparrowright M$, a generic immersion with algebraically 0 self-intersections.
- A homotopy between such immersions is homotopic (rel ∂) to a sequence of finger moves & Whitney moves:

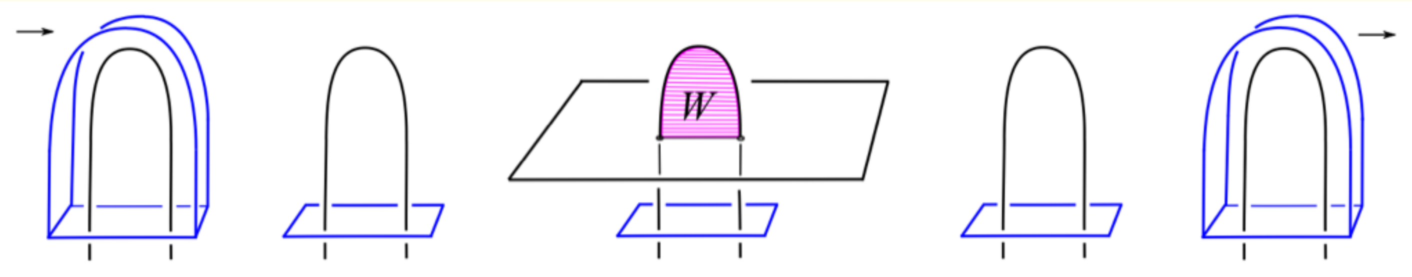


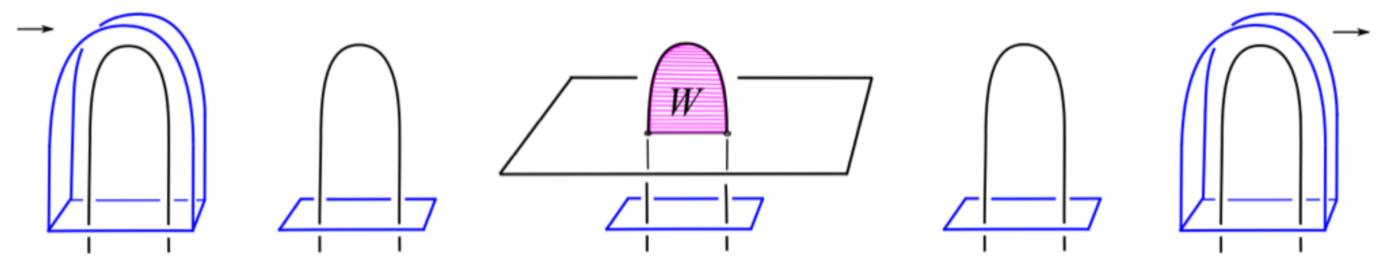
Whitney sphere shrinks after Whitney move!

Whitney move, dual version:



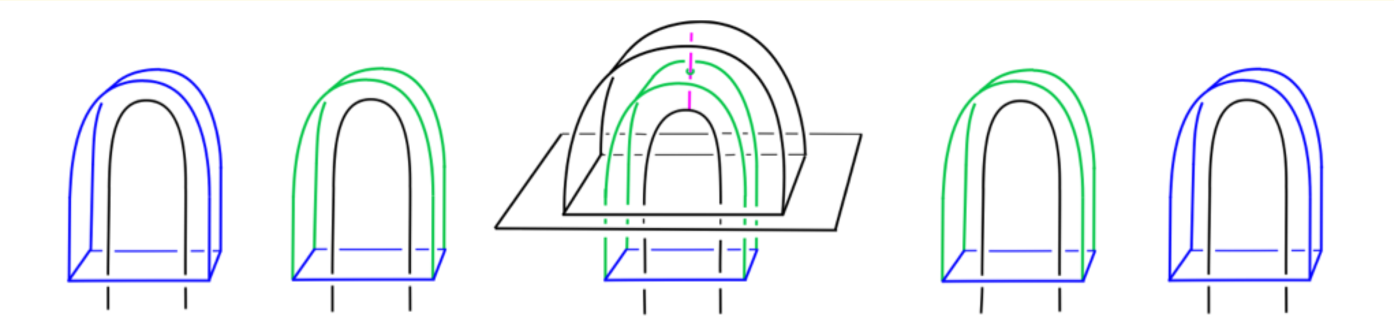
Precise picture of Whitney sphere S_W:





After the Whitney move we see that

Whitney sphere bounds 3-ball:



$S_W \simeq *$ is called a Whitney homotopy.

Metabolic Unlinking Theorem [S.-T. ,2016] TFAE:

(1) $(f_1, f_2) \in LM^4_{2,2}$ is trivial

(2) $\exists$ homotopy $f_1 \simeq f$ in $S^4 \smallsetminus f_2$ such that $f : S^2 \looparrowright S^4$ admits Whitney discs W_i leading back to unknot &

(*) $f_2 \in \langle S_{W_i} \rangle \subseteq \pi_2(S^4 \smallsetminus f)$.

(3) ....

$\Downarrow$ S_{W_i} are disjointly embedded : W_i are framed, disjointly embedded.

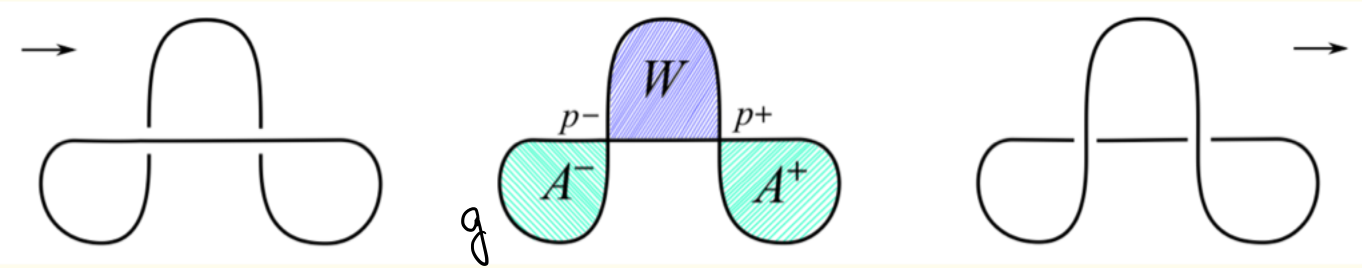
Remarks: (i) $\lambda(f_2, f_2) = 0$

(ii) The W_i have geometric duals spheres S_j in $S^4 \smallsetminus f$, in particular $\mathbb{Z}[\mathbb{Z}]$-intersection form is metabolic:

$$\begin{array}{cc} W_i & S_j \end{array}$$
$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & * \end{pmatrix}$$

Note: $\langle t \rangle = \mathbb{Z} \cong \pi_1(S^4 \smallsetminus \text{unknot}) \cong \pi_1(S^4 \smallsetminus f)$.

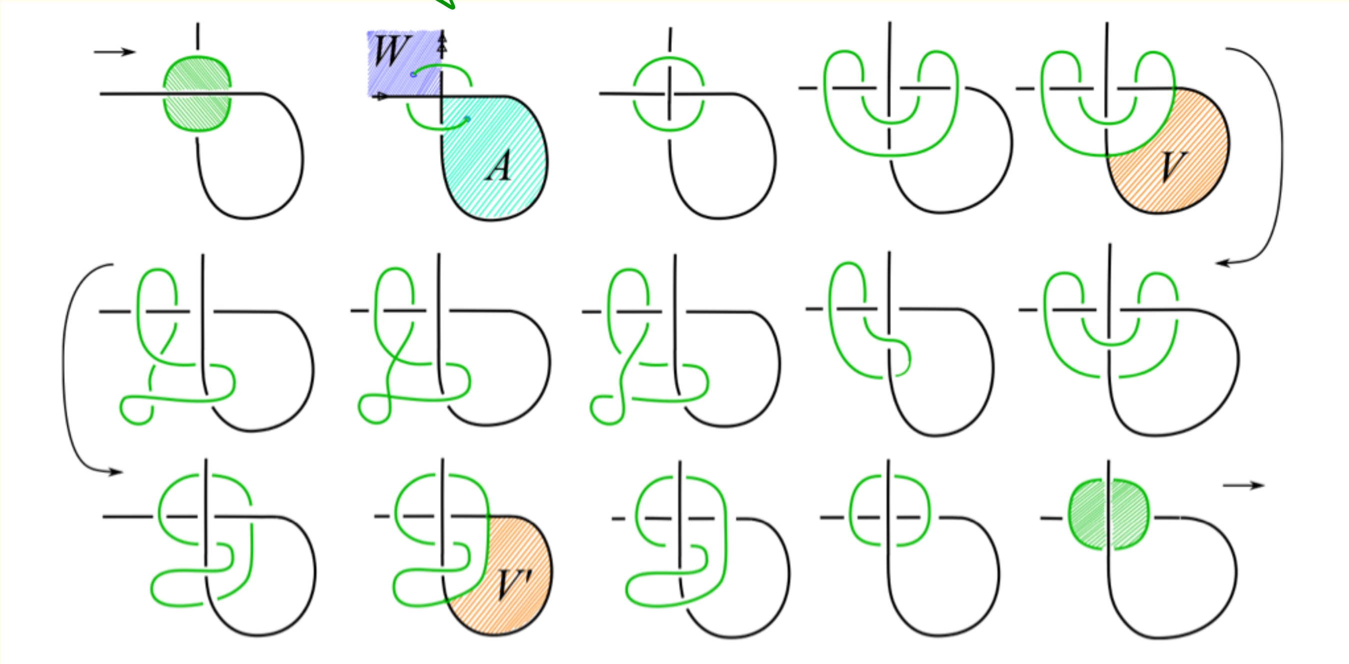
Dual spheres from accessory disks:



g

$A^{\pm}$ exist iff double point loop g is null homotopic. In the standard model $A^{\pm}$ are embedded, just like W.

Accessory sphere S_A:



$S_A \pitchfork A = pt$ and $S_A \pitchfork W = pt \Rightarrow$

W has a dual sphere!

Metabolic Unlinking Theorem [S.–T., 2016] TFAE:

(1) $(f_1, f_2) \in LM^4_{2,2}$ is trivial

(2) $\exists$ homotopy $f_1 \simeq f$ in $S^4 \setminus f_2$ such that .... &

$$(*) \quad f_2 \in \langle S_{w_i} \rangle \subseteq \pi_2(S^4 \setminus f).$$

(3) $\exists$ homotopy $f_1 \simeq f$ in $S^4 \setminus f_2$ such that $f: S^2 \hookrightarrow S^4$ admits framed, disjointly embedded Whitney discs W_i which in $S^4 \setminus f$ admit **algebraic dual spheres** S_j such that $\lambda(f_2, f_2) = 0$ and $(*)$ holds modulo $I^2 \cdot \pi_2(S^4 \setminus f)$, where $I^2 = (t-1)^2 \cdot \mathbb{Z}[t^{\pm 1}] \subseteq \mathbb{Z}[t^{\pm 1}]$.

$$\begin{array}{cc} W_i & S_j \end{array}$$
$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & * \end{pmatrix}$$

Lemma: For a link map $(f_1, f_2): S^2 \sqcup S^2 \longrightarrow S^4$ with f_1 an embedding, conditions (3) of our unlinking theorem hold.

Proof: By Alexander duality, know that

$$H_1(S^4 \setminus f_1) \cong \mathbb{Z}$$

$\Longrightarrow \exists\ \mathbb{Z}$-cover $X \twoheadrightarrow S^4 \setminus f_1$ s.t. for gen. $t \in \mathbb{Z}$

$$\longrightarrow H_2 X \xrightarrow{\cdot (t-1)} H_2 X \longrightarrow H_2(S^4 \setminus f_1) = 0$$

$$\underset{\omega}{}$$

$\forall N\ \exists a_N : f_2 = (t-1)^N \cdot a_N \Longrightarrow \lambda(f_2, f_2) = 0$

Also get a link homotopy $f_1 \simeq f =$ generic immersion with W_i for f s.t. $(t-1)^2$ divides $\lambda(f_2, W_i)$. QED

Sketch of proof for Metabolic Unlinking Theorem:

(1) $\Rightarrow$ (2) follows from our basic tools and the fact that f = unknot + k finger moves.

(2) $\Rightarrow$ (3) We saw

$$0 \longrightarrow \langle S_{w_i} \rangle \longrightarrow \pi_2(S^4 \setminus f) \xrightarrow{\lambda(w_{i},-)} \bigoplus_{i=1}^{k} \mathbb{Z}[\mathbb{Z}] \longrightarrow 0$$

$$f_2 \longmapsto 0$$

accessory spheres $\quad S_{A_j} \longmapsto \delta_{ij}$

$$\left(\pi_2(S^4 \setminus f), \lambda\right) \text{ is metabolic} : \quad \lambda = \begin{array}{cc} w_i & S_{A_j} \\ \end{array} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & * \end{pmatrix}$$

(3) $\Rightarrow$ (1) Use $\lambda(f_2, f_2) = 0$ to embed f_2 into a Lagrangian of λ. $(*)$ mod $I^2 \Longrightarrow$

$\exists\ T_i \in \pi_2(S^4 \setminus f)$ such that $W_i' := W_i \# T_i$ and

$$\lambda_{W_i'} = \lambda_{W_i} + (t-1)^2 \cdot T_i$$

freely generate this new Lagrangian containing f_2.

Note: W_i' have algebraic dual spheres because we are controlling λ!

Since $\pi_1(S^4 \setminus f) \cong \mathbb{Z}$ is a good group, Freedman's disk embedding theorem gives disjointly embedded Whitney disks W_i'' homotopic to W_i'.

Using our Whitney homotopies on these W_i'' get a null homotopy of f_2 in $S^4 \setminus f$. QED