Coherent double covering diagrams of virtual links

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Knotted embeddings in dimensions 3 and 4
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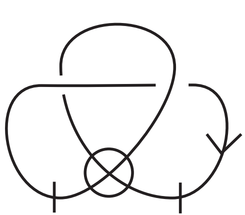
Twisted links

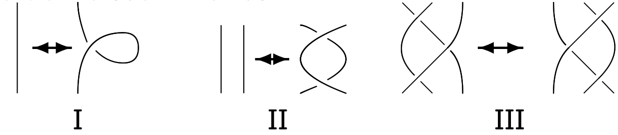
A twisted link diagram is a virtual link diagram possibly with some bars on its arcs.



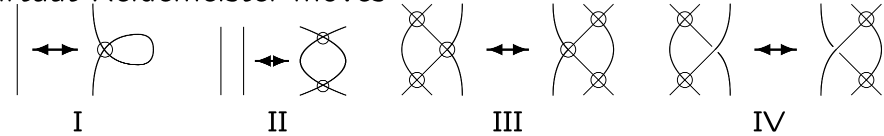
A twisted link is an equivalence class of twisted link diagrams under Reidemeister moves I, II, III, virtual Reidemeister moves I, II, III, IV and twisted moves I, II, III.

Generalized Reidemeister moves

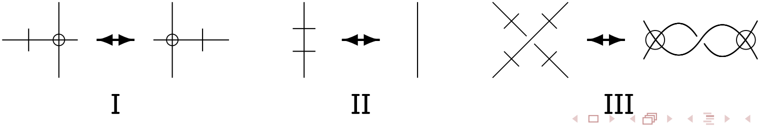
Reidemeister moves



Virtual Reidemeister moves

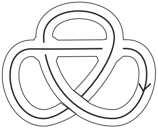


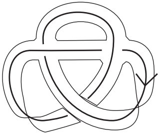
Twisted moves



Abstract link diagrams

An abstract link diagram (ALD) is a pair (Σ, D) of a compact surface Σ and a link diagram D in Σ such that $|D|$ is a deformation retract of Σ, where $|D|$ is a 4-valent graph obtained from D by replacing all crossings with vertices.

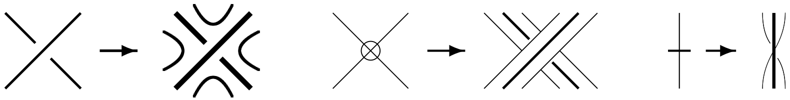


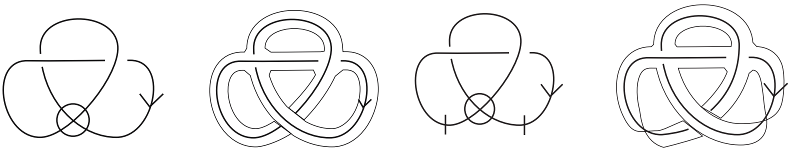


We say that two ALDs (Σ, D) and (Σ', D') are related by an abstract R-move if $\exists$ a closed surface F and embeddings $g : \Sigma \to F$ and $g' : \Sigma' \to F$ such that $g(D)$ and $g'(D')$ are related by an Reidemeister move in F.
An abstract link is an equivalence class of abstract link diagrams under the equivalence relation generated by abstract Reidemeister moves.

Twisted links VS abstract links

a twisted link diagram $\mapsto$ an abstract link diagram





Twisted links VS abstract links

Theorem [S. Kamada, N. K], [J, S. Carter, S. Kamada, M. Saito]

$$\{\text{virtual links}\} \Leftrightarrow \{\text{abstract links on orientable surfaces}\}$$

$$\Leftrightarrow \left\{\begin{array}{l}\text{stable equivalence classes of links}\\ \text{in thickened orientable surfaces}\end{array}\right\}$$

Theorem [M. Bourgoin]

$$\{\text{twisted links}\} \Leftrightarrow \{\text{abstract links}\}$$

$$\Leftrightarrow \left\{\begin{array}{l}\text{stable equivalence classes of links}\\ \text{in thickened surfaces}\end{array}\right\}$$

For a virtual link diagram D, Kauffman defined two knot groups $G_+(D)$ and $G_-(D)$ by "upper" and "lower" Wirtinger presentations, and proved that if $D \sim D'$ then $G_+(D) \cong G_+(D')$ and $G_-(D) \cong G_-(D')$.
Thus $G_+(L)$ and $G_-(L)$ are defined for a virtual link L.

Remark. For a classical link L, $G_+(L) \cong G_-(L) \cong \pi(\mathbb{R}^3 - L)$.

The following theorem gives a geometric interpretation of $G_+(L)$ and $G_-(L)$.

Theorem [S. Kamada, N. K]

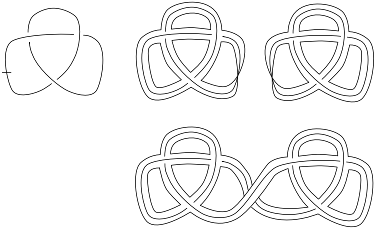
Let D be a virtual link diagram and let (Σ, D') be the associated abstract link diagram. Let L' be a link in $\Sigma \times [-1, 1]$ presented by the diagram D'. Then
$G_+(D) \cong \pi(\Sigma \times [-1, 1] - L'/\Sigma \times \{1\})$
$G_-(D) \cong \pi(\Sigma \times [-1, 1] - L'/\Sigma \times \{-1\})$

For a twisted link diagram D, Bourgoin defined a twisted knot group $\tilde{G}(D)$ and proved that if $D \sim D'$ then $\tilde{G}(D) \sim \tilde{G}(D')$.
Thus $\tilde{G}(L)$ is defined for a twisted link L.

A geometric interpretation of $\tilde{G}(D)$ is obtained by using the double covering of a twisted link, which we explain in this section.

Double covering of a twisted link

D: a twisted link diagram
$\Longrightarrow$ (Σ, D'): an abstract link diagram associated with D
$\Longrightarrow$ $(\tilde{\Sigma}, \widetilde{D'})$: the double covering of (Σ, D')
associated with the orientation double covering $\tilde{\Sigma} \to \Sigma$
$\Longrightarrow$ $\widetilde{D}$:a virtual link diagram



Double covering of a twisted link

Theorem [S. Kamada, N. K]

The double covering of a twisted link is well defined. Namely, if $D \sim D'$ (as a twisted link) then $\widetilde{D} \sim \widetilde{D}'$ (as a virtual link)

$$\{\text{twisted links}\} \longrightarrow \{\text{virtual links}\}, [D] \mapsto [\widetilde{D}]$$

Corollary

For a given virtual link invariant X, we can obtain a twisted link invariant $\widetilde{X}$ by $\widetilde{X}(D) := X(\widetilde{D})$.

Double covering of a twisted link

Corollary

For a given virtual link invariant X, we can obtain a twisted link invariant $\widetilde{X}$ by $\widetilde{X}(D) := X(\widetilde{D})$.

- Twisted knot group (X: knot group G_+ or G_-)

M. O. Bourgoin, *Twisted link theory*, Algebr. Geom. Topol. 8 (2008), 1249–1279.

- Twisted knot quandle (X: knot quandle)

N. K., *Polynomial invariants and quandles of twisted links*, Topology Appl. 159 (2012), 999–1006.

- Doubled JKSS invariant (X: JKSS invariant)

N. K., *A twisted link invariant derived from a virtual link invariant*, to appear.

Double covering of a twisted link

- Twisted knot group (X: knot group G_+ or G_-)

M. O. Bourgoin, *Twisted link theory*, Algebr. Geom. Topol. 8 (2008), 1249–1279.

Theorem [S. Kamada, N,K,]

Let D be a twisted link diagram and $\widetilde{D}$ the double covering. Then $\tilde{G}(D) \cong G_+(\widetilde{D}) \cong G_-(\widetilde{D})$.

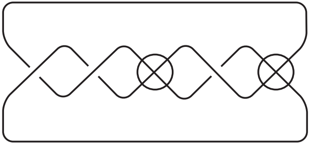
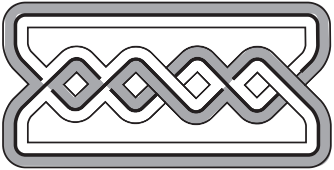
Since we know a geometric interpretation of knot group G_+ and G_- for a virtual link diagram, we obtain a geometric interpretation of the twisted knot group $\tilde{G}$ for a twisted link diagram as follows.

$$\begin{aligned} \tilde{G}(D) &\cong G_+(\widetilde{D}) = \pi_1(\Sigma \times [-1,1] - \widetilde{L}/\Sigma \times \{1\}) \\ &\cong G_-(\widetilde{D}) = \pi_1(\Sigma \times [-1,1] - \widetilde{L}/\Sigma \times \{-1\}) \end{aligned}$$

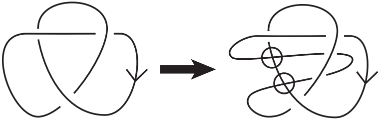
where $\widetilde{L}$ is a link in $\Sigma \times [-1,1]$ presented by $\widetilde{D}$.

Checkerboard colorable diagrams

D : virtual link diagram
(Σ, D_Σ) : an abstract link diagram associated with D
D: checkerboard colorable $\Leftrightarrow$ (Σ, D_Σ) is checkerboard colorable

Note that checkerboard colorability is not necessary to be preserved under generalized Reidemeister moves.



A virtual link L is said to be checkerboard colorable, if there is a checkerboard colorable virtual diagram of L.

Remark on checkerboard colorable diagrams

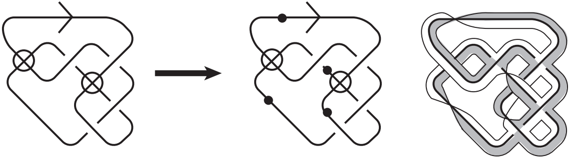
1. Classical link diagrams are checkerboard colorable.

$$\left\{\begin{array}{l}\text{classical link}\\\text{diagrams}\end{array}\right\} \subset \left\{\begin{array}{l}\text{checkerboard}\\\text{colorable}\\\text{virtual link}\\\text{diagrams}\end{array}\right\} \subset \left\{\begin{array}{l}\text{virtual link}\\\text{diagrams}\end{array}\right\}$$

Therefore classical links are checkerboard colorable.
2. Some classical link invariants related to checkerboard colorability, as singnature, Khovanov homology, can be defined for checkerboard colorable virtual links.
3. Checkerboard colorability can be defined for twisted link.
4. (O. Viro, O. Manturov) If D and D' are checkerboard colorable virtual link diagrams and if $D \sim D'$ then $\exists D_0 = D, D_1, D_2, \ldots, D_n = D'$, a sequence of checkerboard colorable virtual link diagrams such that D_{i+1} is obtained from D_i by a Reidemeister move or a virtual Reidemeister move.

Cut point(Dye)

D: a virtual link diagram
P: a set of points on D
P : a cut system of D $\Leftrightarrow$ The abstract link diagram associated with (D, P) is checkerboard colorable, where the bands are twisted at points in P.



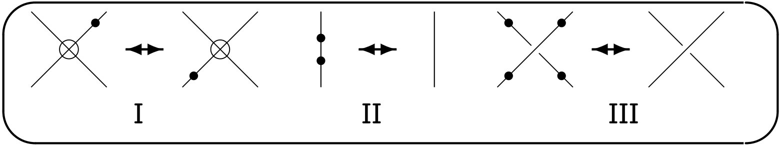
Each point of a cut system is called a cut point.

Cut point

Proposition

Two cut systems of a virtual link diagram are related by a sequence of cut point moves.

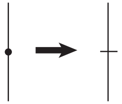
Cut point move



Square double covering diagrams of virtual link diagrams

In this section, we introduce a method of converting a virtual link diagram into a checkerboard colorable diagram, which we call a square double.

$$t : \left\{\begin{array}{l}\text{virtual link diagrams}\\ \text{with cut systems}\end{array}\right\} \to \{\text{twisted link diagrams}\}$$



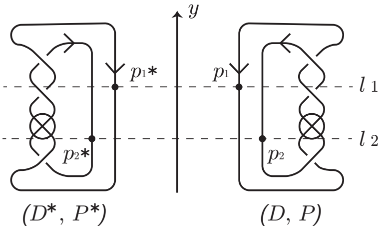
(D, P) $t(D, P)$

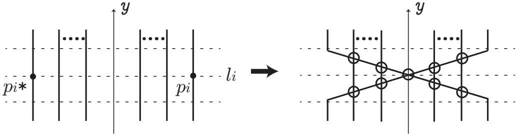
(D, P): a virtual link diagram with a cut system

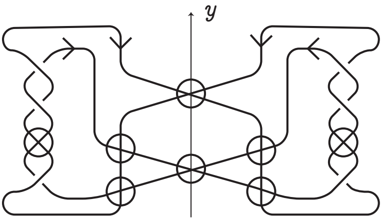
Take a double covering diagram of $t(D, P)$. The covering diagram $\widetilde{t(D, P)}$ is a virtual link diagram.

$$\phi : \left\{\begin{array}{l}\text{virtual link diagrams}\\ \text{with a cut system}\end{array}\right\} \to \{\text{virtual link diagrams}\}$$

Square double of a virtual link diagram







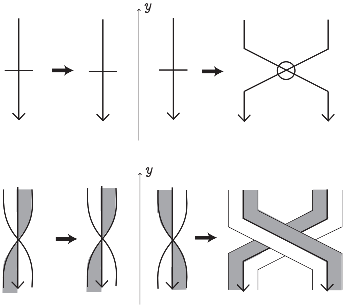
Square double of a virtual link diagram

Theorem (N.K.)

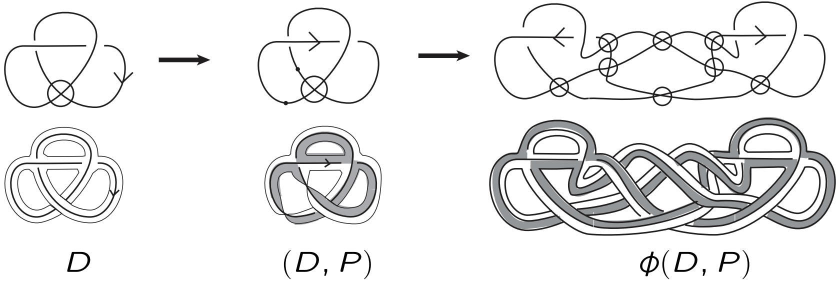
$$\phi : \left\{ \begin{array}{l} \text{virtual link diagrams} \\ \text{with cut systems} \end{array} \right\} \to \left\{ \begin{array}{l} \text{checkerboard colorable} \\ \text{virtual link diagrams} \end{array} \right\}$$

Lemma

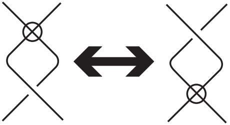
D : a twisted link diagram
If D is checkerboard colorable, then $\widetilde{D}$ is checkerboard colorable.



Square double of a virtual link diagram: Example



Square double of a virtual link diagram: K-equivalence



Two virtual link diagrams are said to be K-equivalent if they are related by a sequence of Reidemesiter moves, virtual Reidemeister moves and Kauffman flypes (K-flypes).

Lemma

Let P_1 and P_2 be cut systems for a virtual link diagram D. Then $\phi(D, P_1)$ is K-equivalent to $\phi(D, P_2)$.

Theorem

Let (D_1, P_1) and (D_2, P_2) be virtual knot diagrams with cut systems. If D_1 is equivalent (or K-equivalent) to D_2, then $\phi(D_1, P_1)$ is K-equivalent to $\phi(D_2, P_2)$.

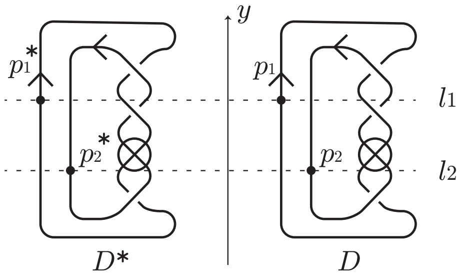
Coherent double of a virtual link diagram

In the previous section, we discuss the square double as a method of converting a virtual link diagram into a checkerboard colorable virtual link diagram.
In this section, we introduce another method which we call a coherent double.

Let D be a virtual link diagram.
Taking a cut system P, we consider a virtual link diagram with a cut system, (D, P).

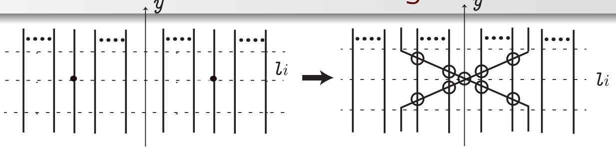
Coherent double of a virtual link diagram

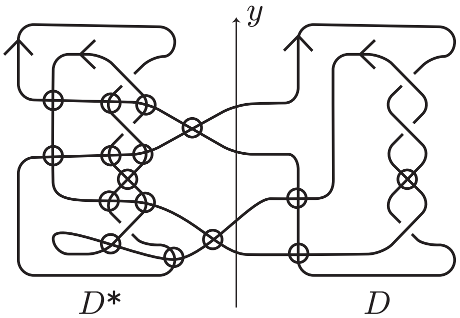
Let (D, P) be a virtual link diagram with a cut system.
Step1
Put (D, P) on the right of the y-axis such that cut points have distinct y-coordinates.
Make a copy of (D, P) on the left of the y-axis, which we denote by (D^*, P^*).



Coherent double of a virtual link diagram

Step2



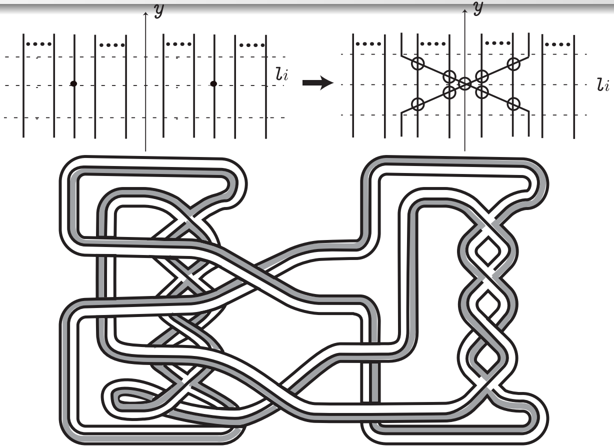


We denote the diagram by $\varphi(D, P)$ and call it a coherent double.

The coherent double is always checkerboard colorable.

Coherent double of a virtual link diagram

Step2



We denote the diagram by $\varphi(D, P)$ and call it a coherent double.

The coherent double is always checkerboard colorable.

Coherent double of a virtual link diagram

Theorem

For a virtual link diagram with a cut system (D, P), the coherent double $\varphi(D, P)$ is checkerboard colorable.

$$\varphi : \left\{\begin{array}{l}\text{virtual link diagrams} \\ \text{with cut systems}\end{array}\right\} \to \left\{\begin{array}{l}\text{checkerboard colorable virtual link} \\ \text{diagrams}\end{array}\right\}$$

Theorem

The virtual link type represented by the coherent double $\varphi(D, P)$ does not dependent on D and P.
Namely, if D_1 is equivalent to D_2 as a virtual link, then $\varphi(D_1, P_1)$ is equivalent to $\varphi(D_2, P_2)$ for any P_1 and P_2.

$$\{\text{virtual links}\} \to \left\{\begin{array}{l}\text{checkerboard colorable virtual} \\ \text{links}\end{array}\right\}$$

Application

- Odd writhe
- Linking invariant

etc.

Thank you for your attention.