Khovanov Homology and Virtual Knot Cobordism

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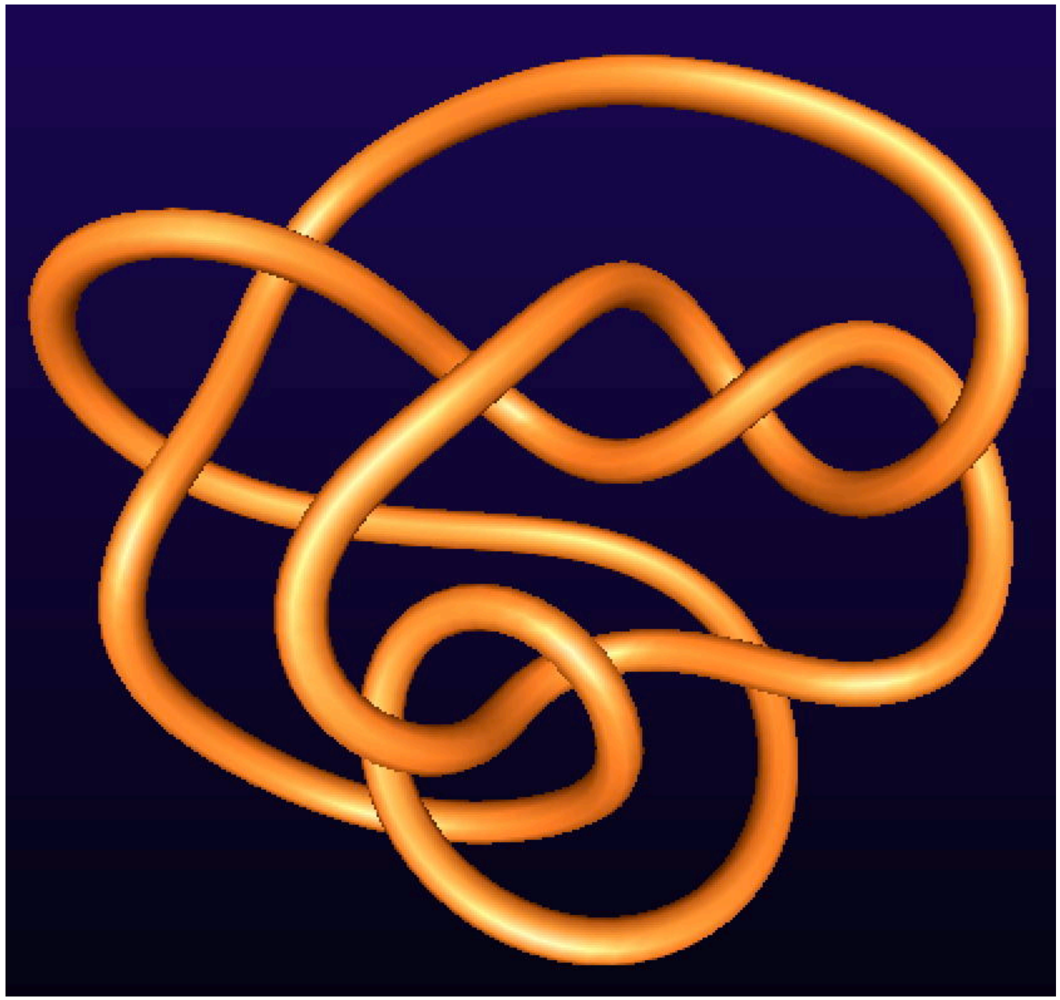
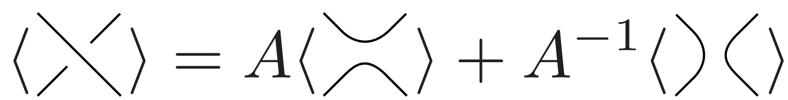
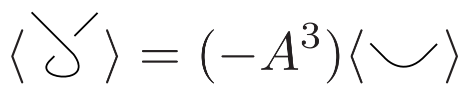


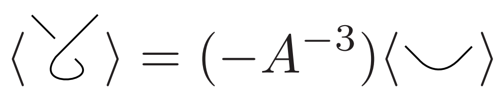
Khovanov Homology

Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as $\langle K\rangle$. There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic* $\langle K\rangle = \chi_q\langle H(K)\rangle$ for some homology theory associated with $\langle K\rangle$.

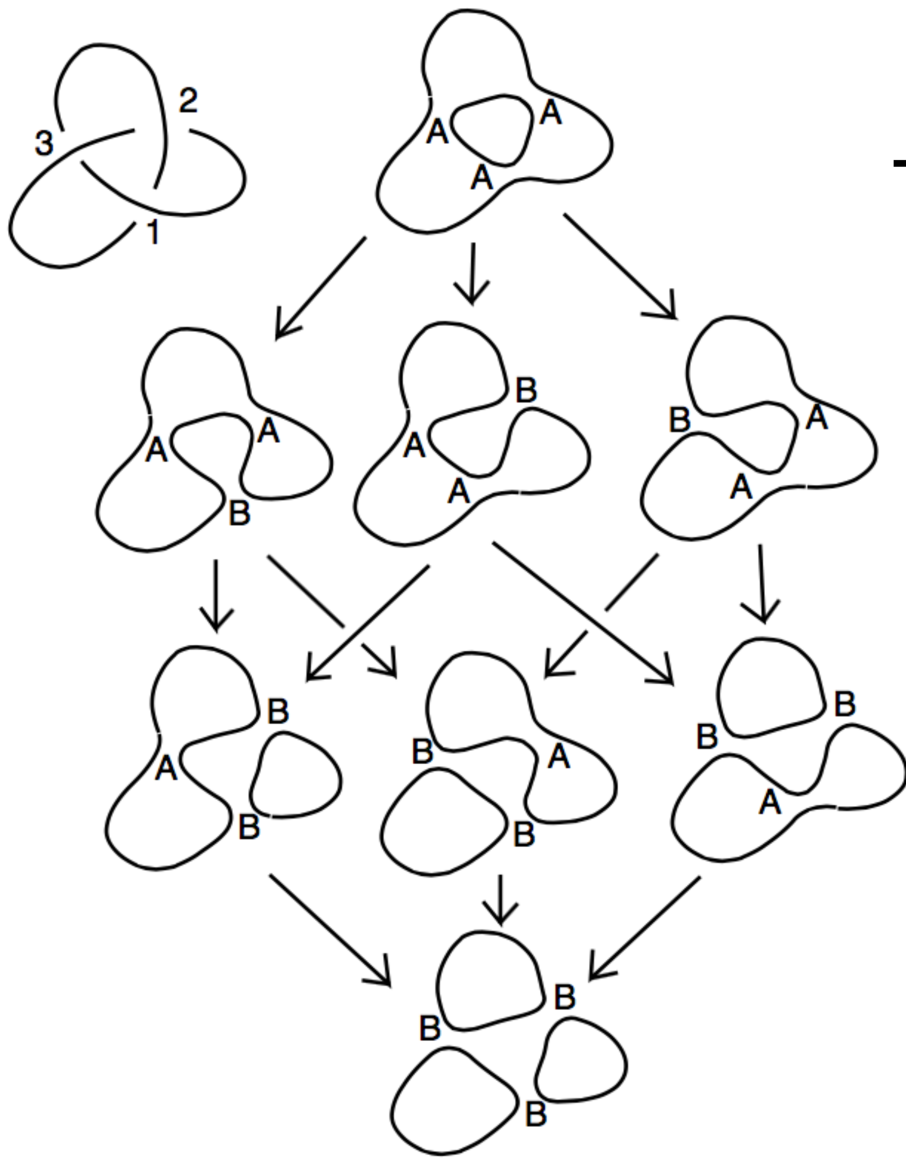


$$\langle K \bigcirc \rangle = (-A^2 - A^{-2})\langle K\rangle$$





Cubism



The bracket states form a category. How can we obtain topological information from this category?

Exploration: Examine the Bracket Polynomial for Clues.

Let c(K) = number of crossings on link K.

Form $A^{-c(K)}$ <K> and replace A by $-q^{-1}$.

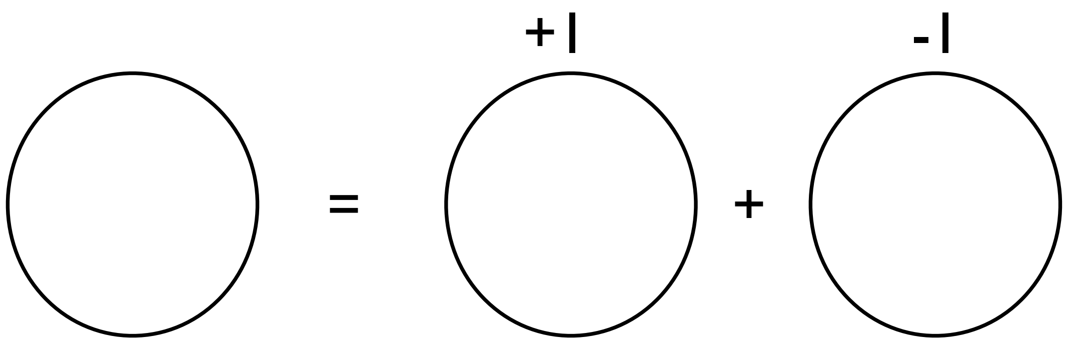
Then the skein relation for <K> will be replaced by:

$$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$$

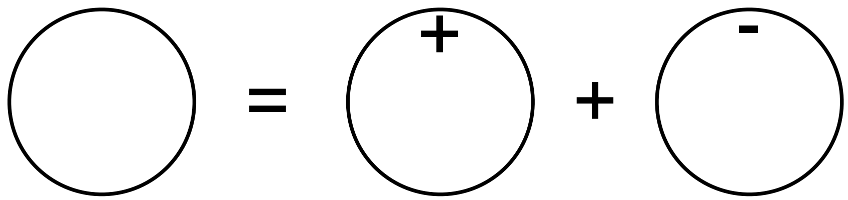
$$\langle \bigcirc \rangle = q + q^{-1}$$

$$\langle K \bigcirc \rangle = (q + q^{-1}) \langle K \rangle$$

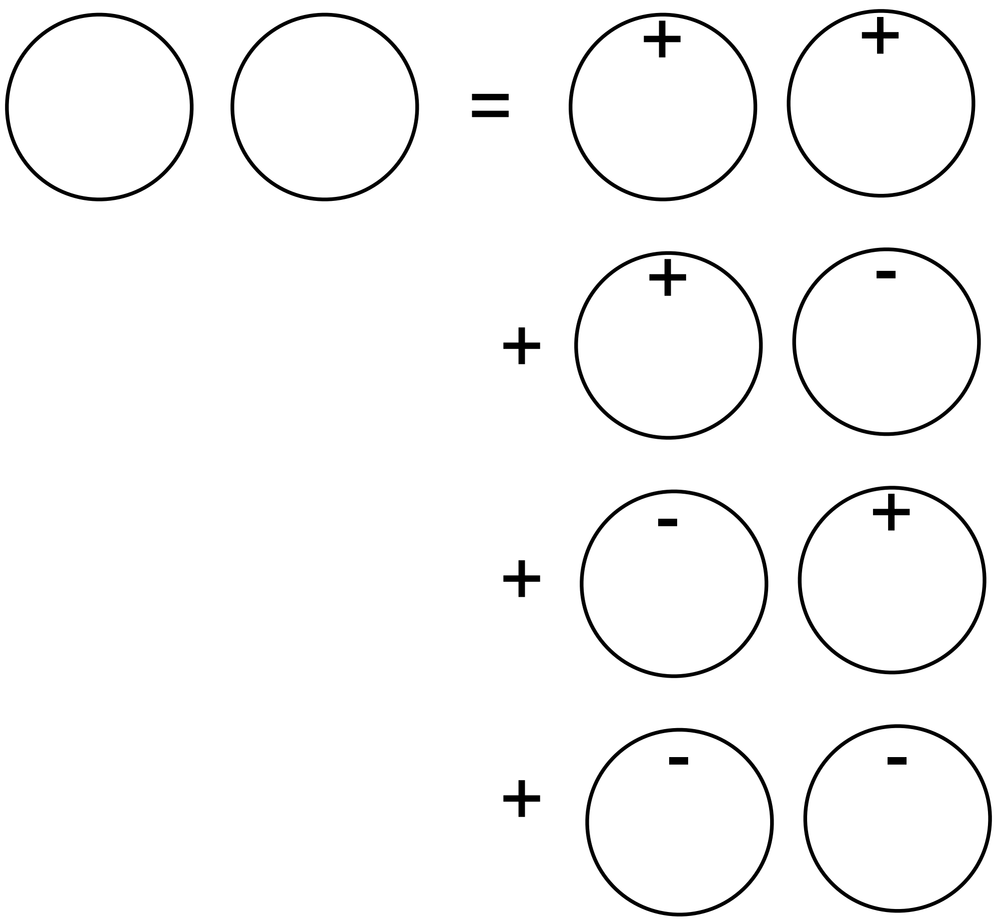
Use enhanced states by labeling each loop with +1 or -1.



$$\longleftrightarrow \quad q + q^{-1}$$



$= q + q^{-1}$



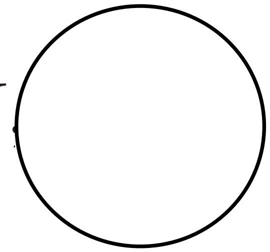
$= qq$
$+ qq^{-1} + q^{-1}q$
$+ q^{-1}q^{-1}$

$= qq + 2 + (qq)^{-1}$

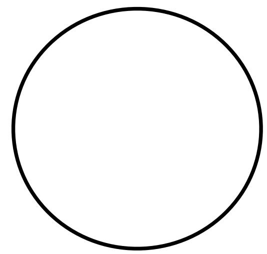
$= (q + q^{-1})^2$

Enhanced States

$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X$$



$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let -1 be denoted by X and +1 be denoted by 1.

$$\langle K \rangle = \sum_{s} (-1)^{n_B(s)} q^{j(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

$n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

C^{ij} = module generated by enhanced states with $i = n_B$ and j as above.

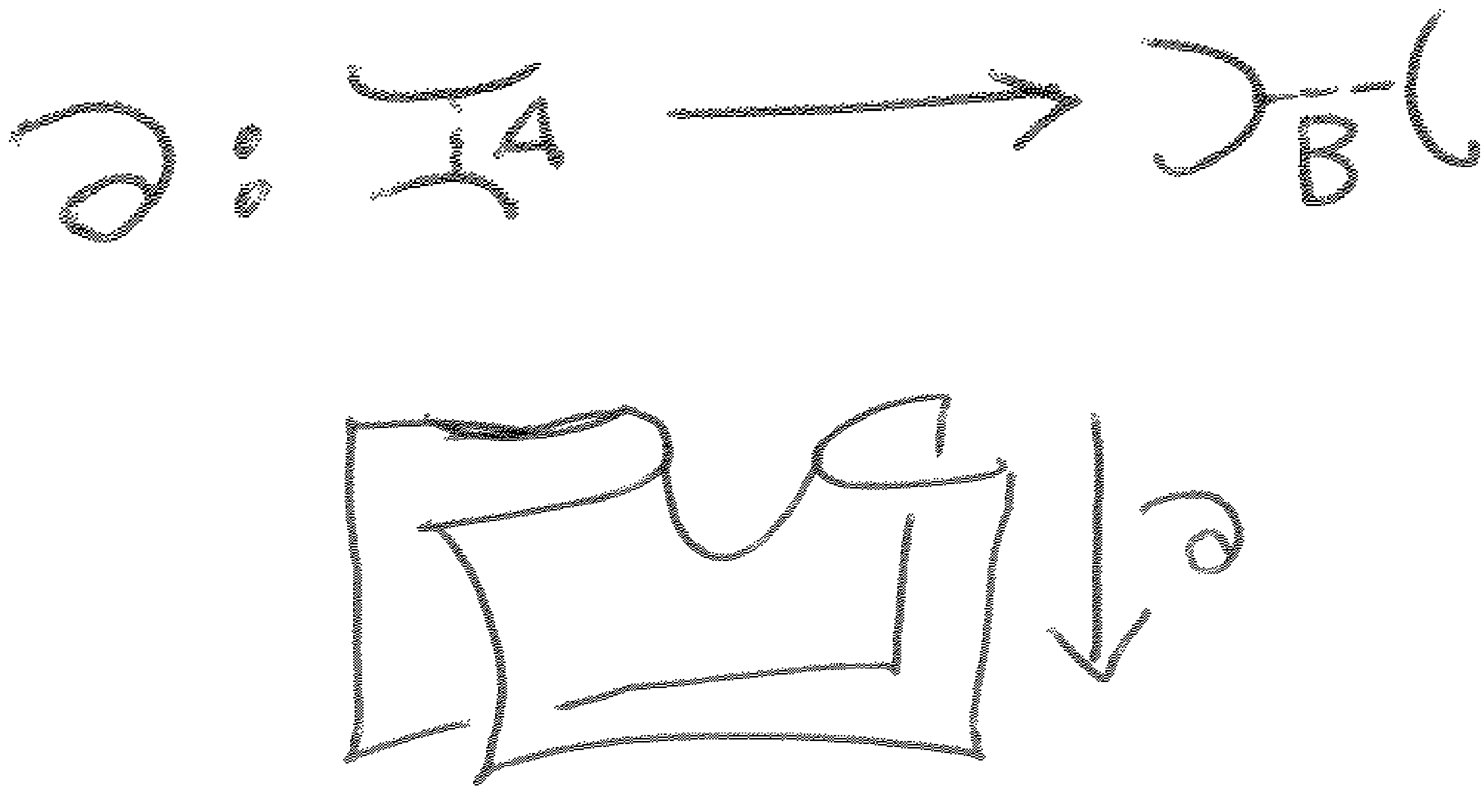
$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

Wanted: differential acting in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

For j to be constant as i increases by 1, we need

$\lambda(s)$ to decrease by 1.



∂ :
A
B
∂

The differential should increase the homological grading i by 1 and leave fixed the quantum grading j.

Then we would have

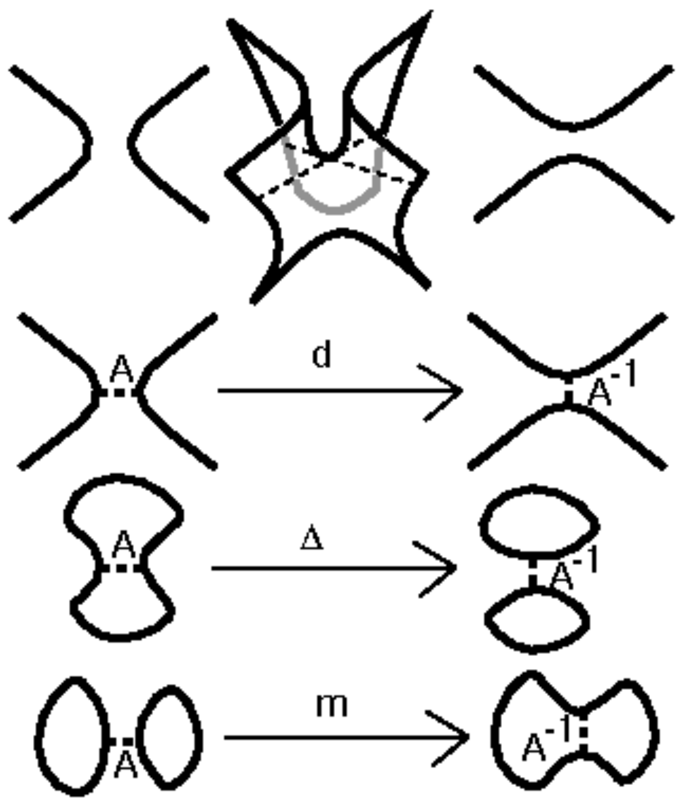
$$\langle K \rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet\, j})$$

$$\chi(H(\mathcal{C}^{\bullet\, j})) = \chi(\mathcal{C}^{\bullet\, j})$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet\, j}))$$

$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

The boundary is a sum of partial differentials corresponding to resmoothings on the states.

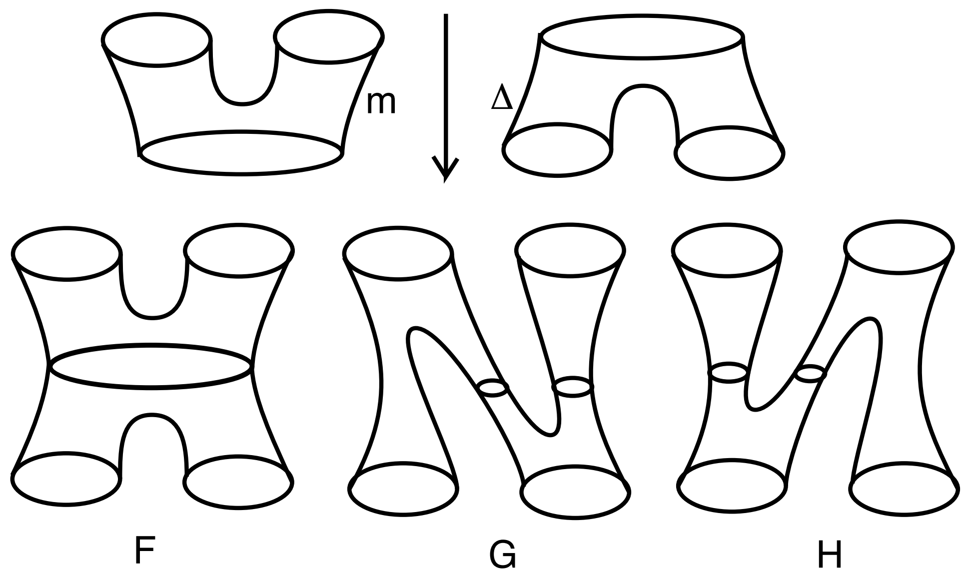


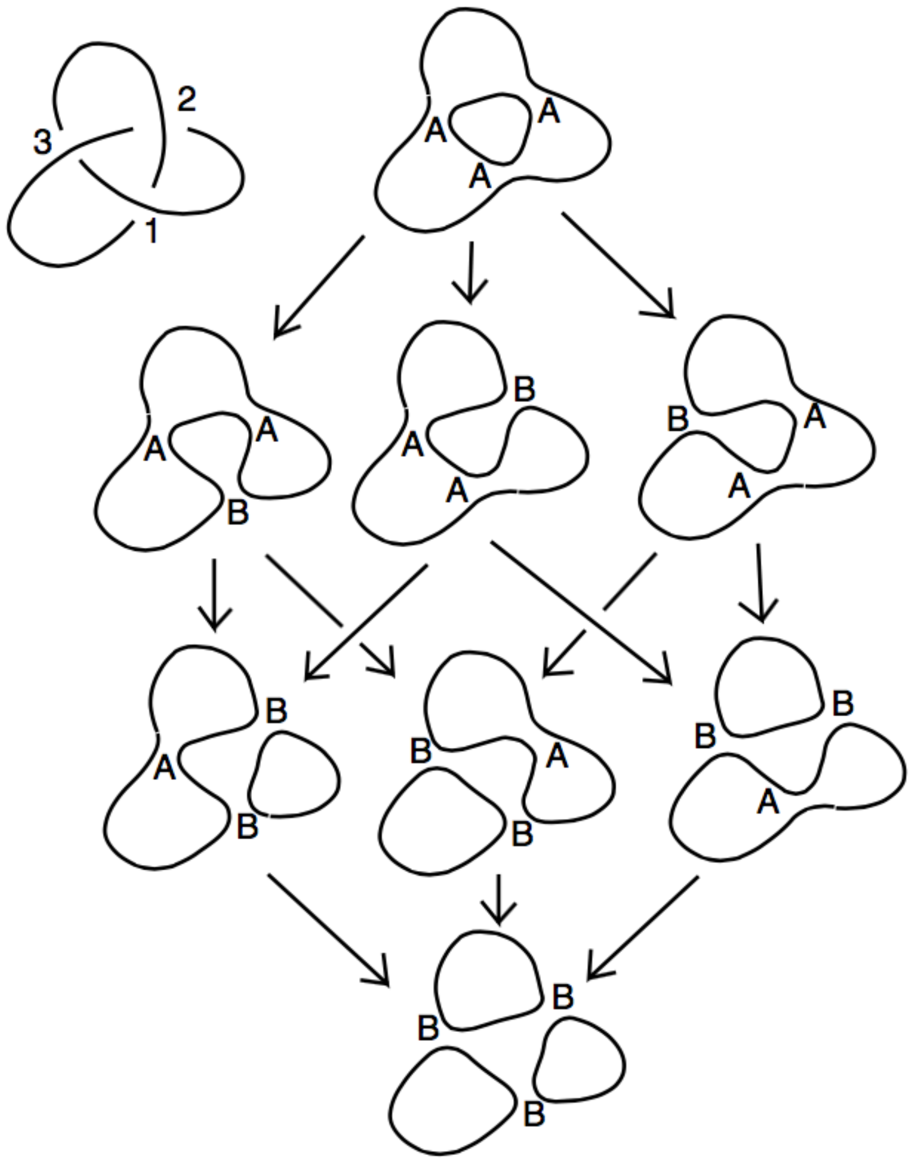
$$\Delta(X) = X \otimes X \text{ and } \Delta(1) = 1 \otimes X + X \otimes 1.$$

$$X^2 = 0$$

Proposition. The partial differentials $\partial_\tau(s)$ are uniquely determined by the condition that $j(s') = j(s)$ for all s' involved in the action of the partial differential on the enhanced state s. This unique form of the partial differential can be described by the following structures of multiplication and comultiplication on the algebra A = $k[X]/(X^2)$ where $k = Z/2Z$ for mod-2 coefficients, or $k = Z$ for integral coefficients.

1. The element 1 is a multiplicative unit and $X^2 = 0$.
2. $\Delta(1) = 1 \otimes X + X \otimes 1$ and $\Delta(X) = X \otimes X$.





$$
\begin{array}{c}
C^0 \\
\downarrow \\
C^1 \\
\downarrow \\
C^2 \\
\downarrow \\
C^3
\end{array}
$$

Bracket states form a category that assembles itself into a chain complex.

Levels in the chain complex are direct sums of modules corresponding to states with a constant number of B smoothings.

$\partial : C^{i,j} \longrightarrow C^{i+1,j}$

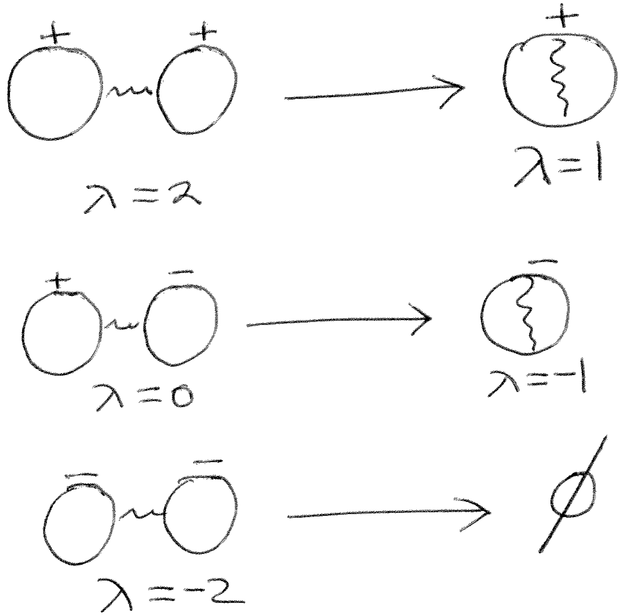
For j to remain fixed, we need $\boxed{\lambda \overset{\partial}{\longmapsto} \lambda - 1}$

where

$$\lambda(s) = \#(+1 \text{ loops}) - \#(-1 \text{ loops}).$$

<u>The ∂ is determined by this condition.</u>

③



$x \equiv -$

$1 \equiv +$

Then

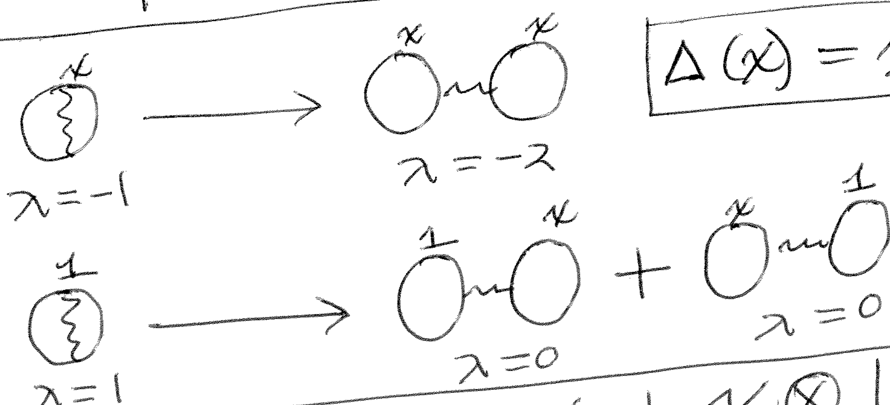
$x^2 = 0$

$1 \times 1 = 1$

$1 \times x = x \times 1 = x$

So far $\underline{V = \mathbb{Z}[x]/(x^2)}$.

For product $m : V \otimes V \longrightarrow V$.

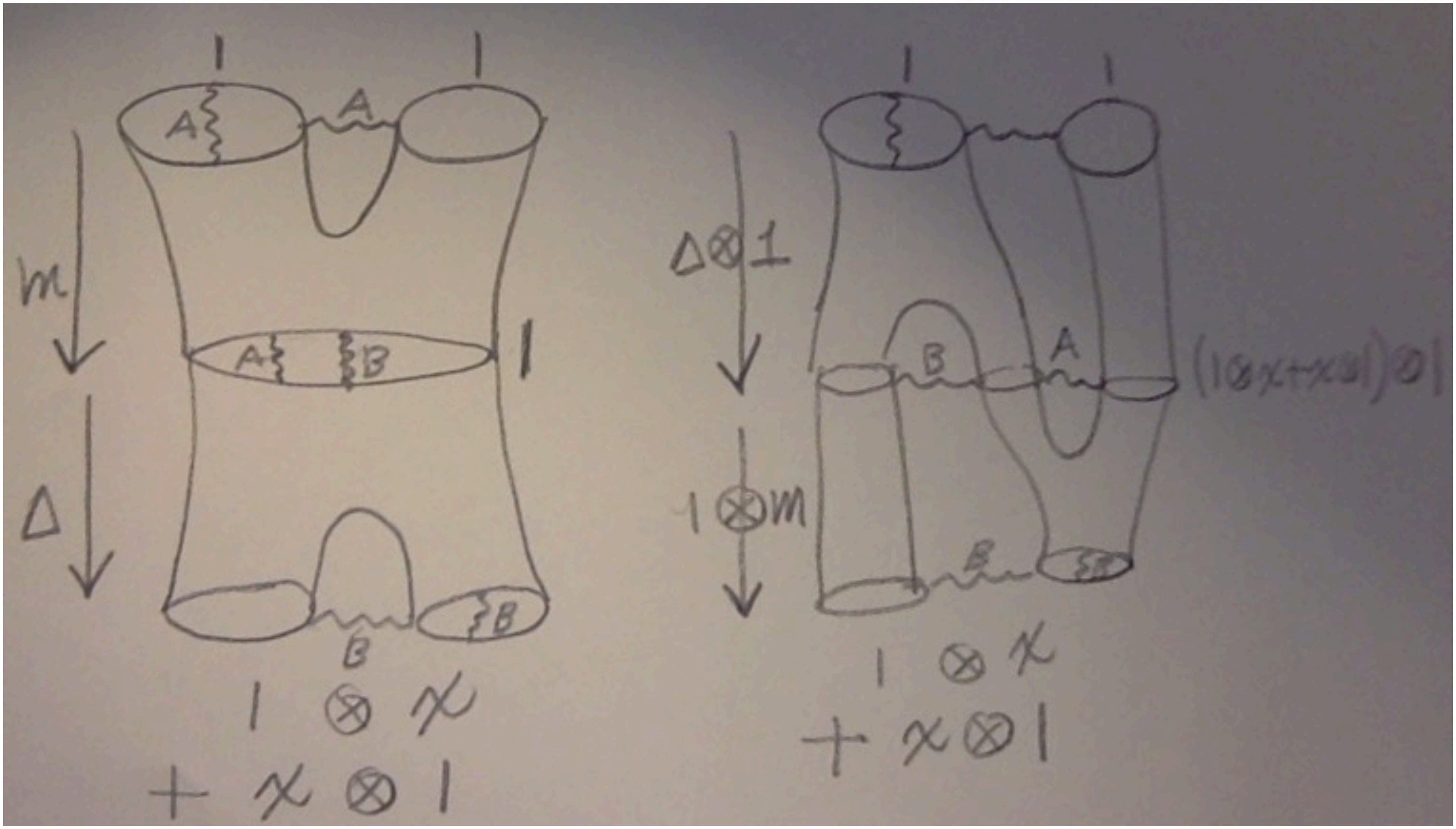


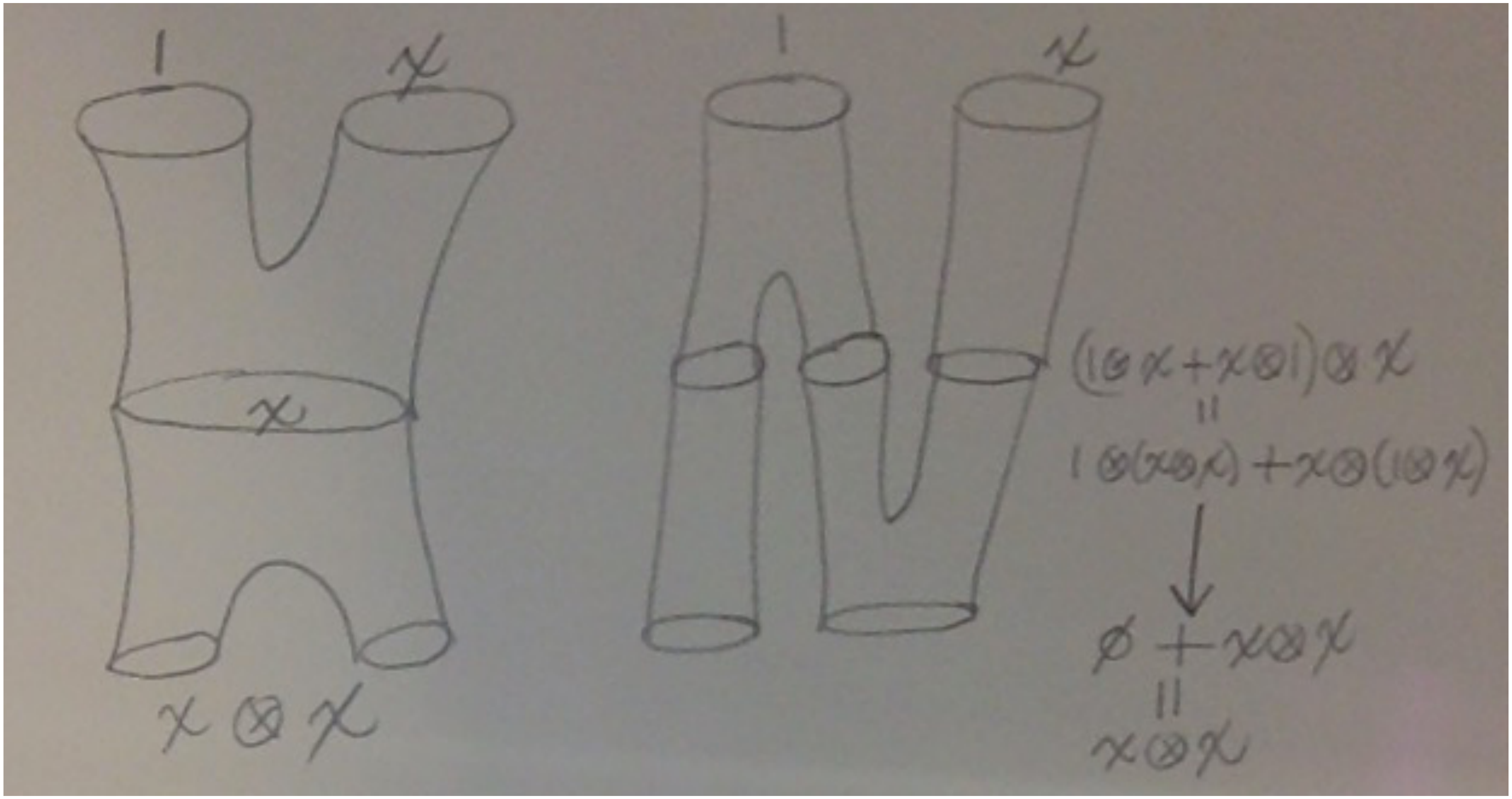
$\boxed{\Delta(x) = x \otimes x}$

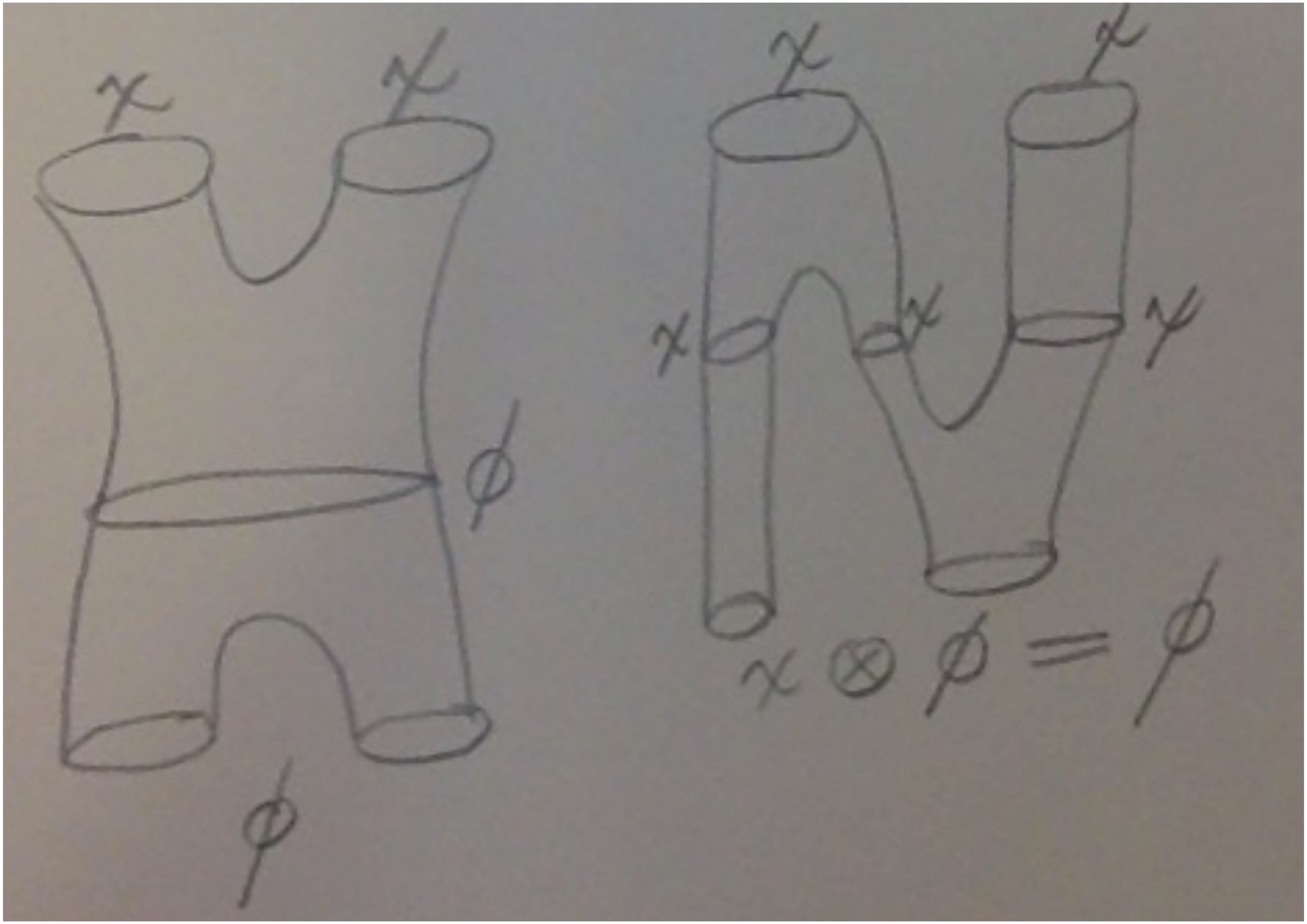
$\boxed{\Delta(1) = 1 \otimes x + x \otimes 1}$

Enhanced States
Plus
Boundary
Requirement
Yields
Frobenius Algebra.

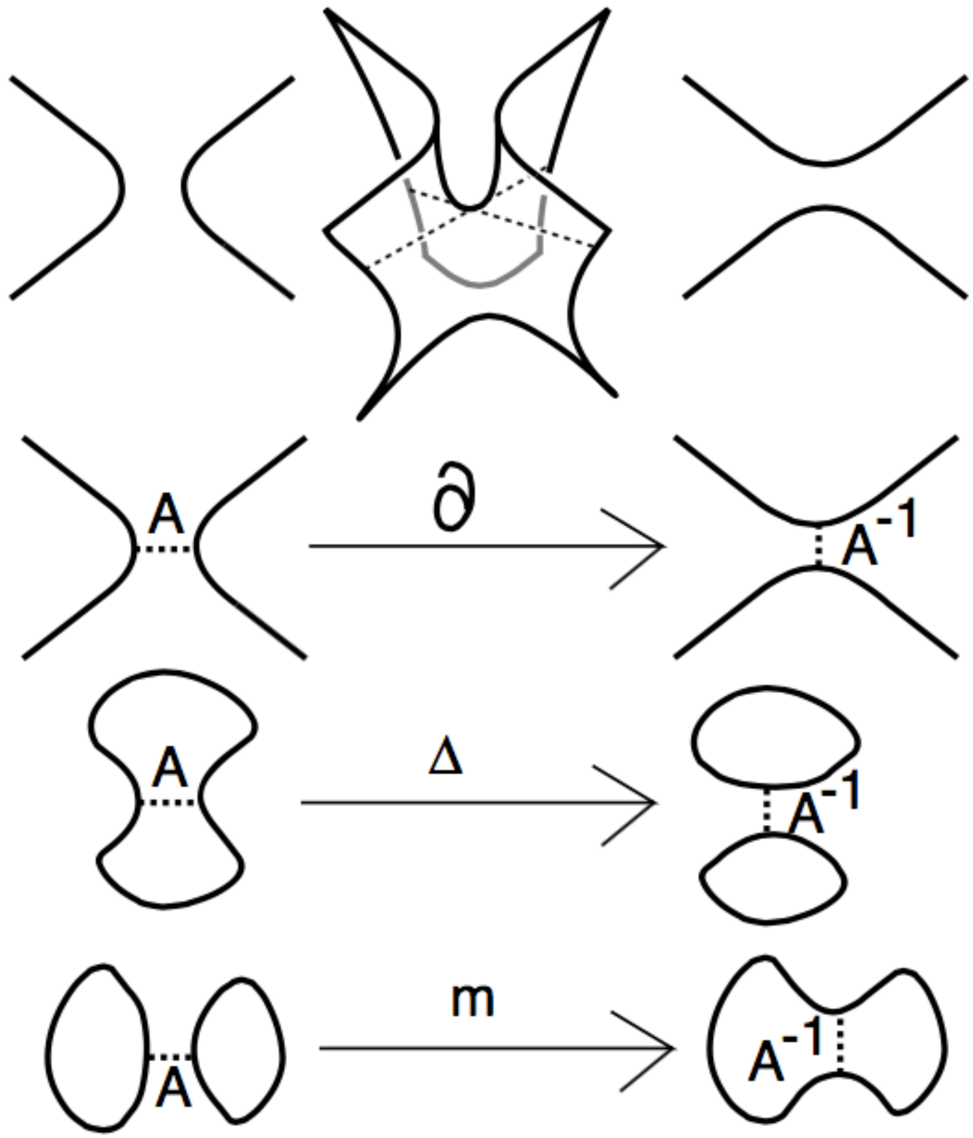
Checking Order Compatibility



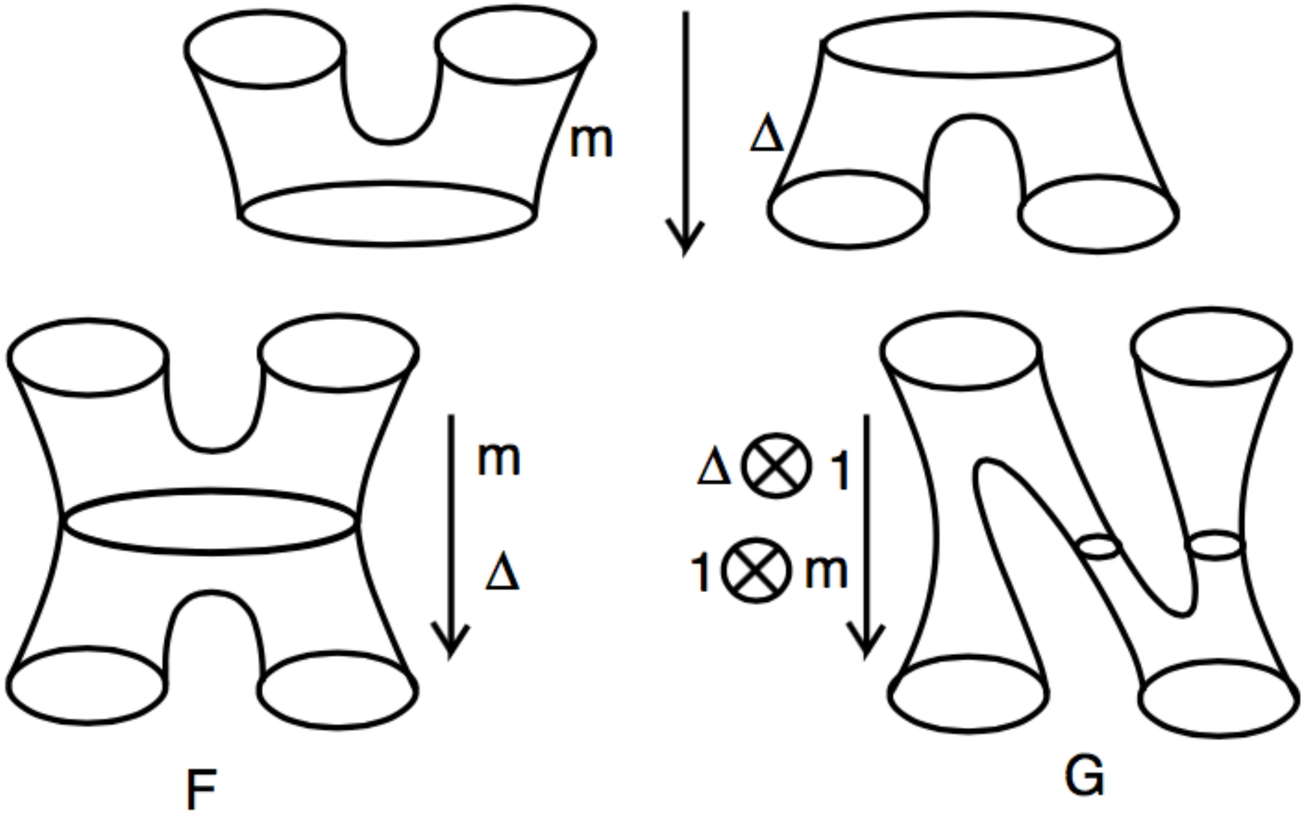

1
x
x
x ⊗ x
1
x
(1⊗x + x⊗1)⊗x
=
1⊗(x⊗x) + x⊗(1⊗x)
∅ + x⊗x
=
x⊗x



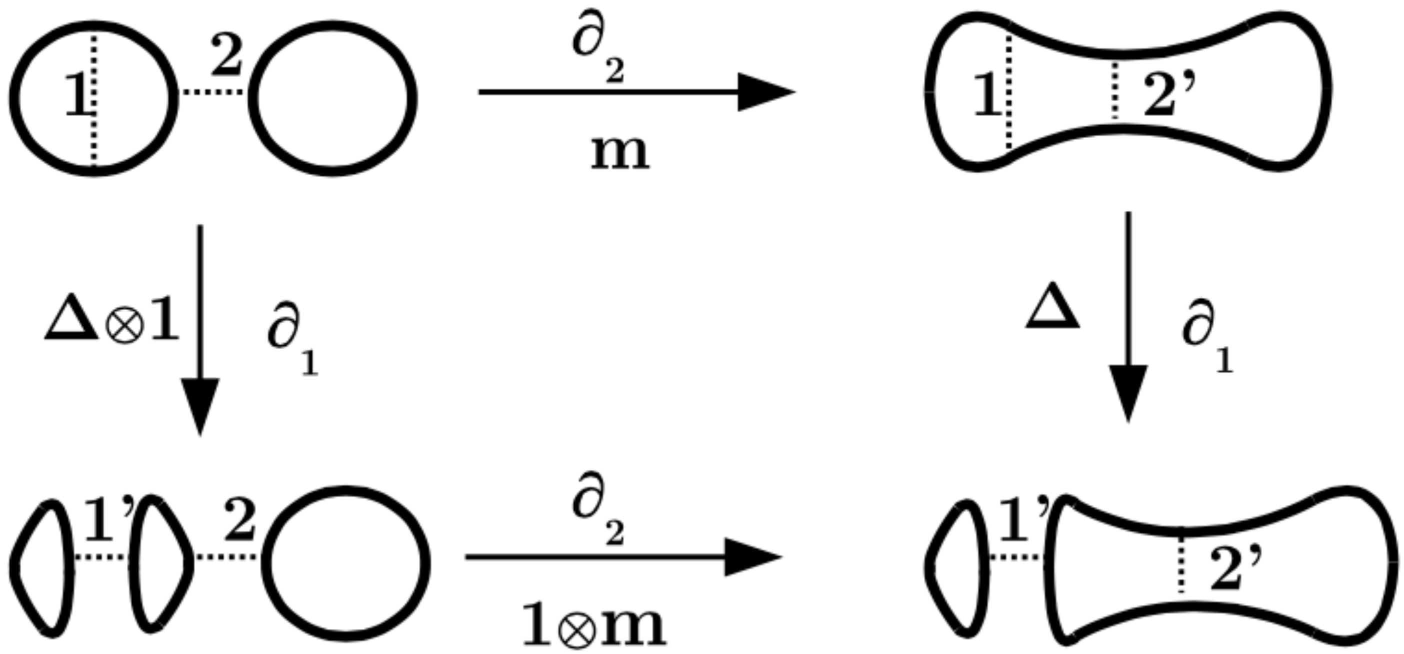
x
y
φ
φ
x
y
x
y
y
x ⊗ φ = φ



A
∂
A-1
A
Δ
A-1
A
m
A-1



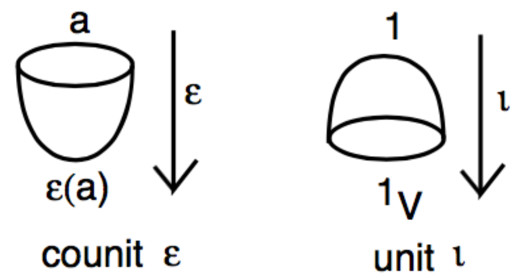
m
Δ
m
Δ
Δ⊗1
1⊗m
F
G



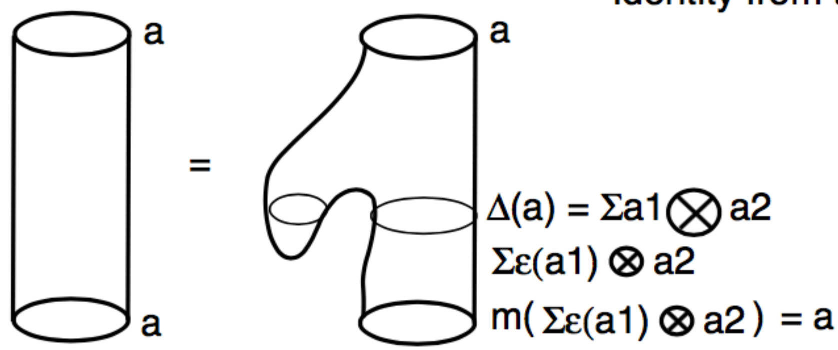
$$\partial_2\partial_1 = (1\otimes \mathbf{m})(\Delta\otimes 1)$$

$$\partial_1\partial_2 = (\Delta)(\mathbf{m})$$

$$\partial_1\partial_2 = \partial_2\partial_1$$



Evaluations at
successive levels.
Identity from topology.

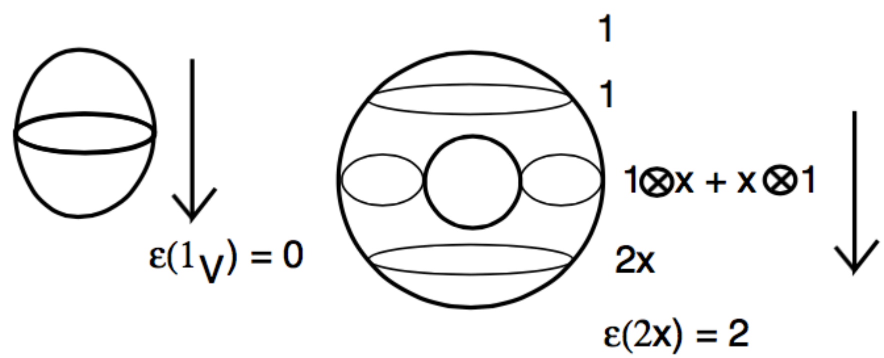


Using special case of a=1, we obtain:

$m(\varepsilon(1)\otimes x + \varepsilon(x)\otimes 1) = 1$

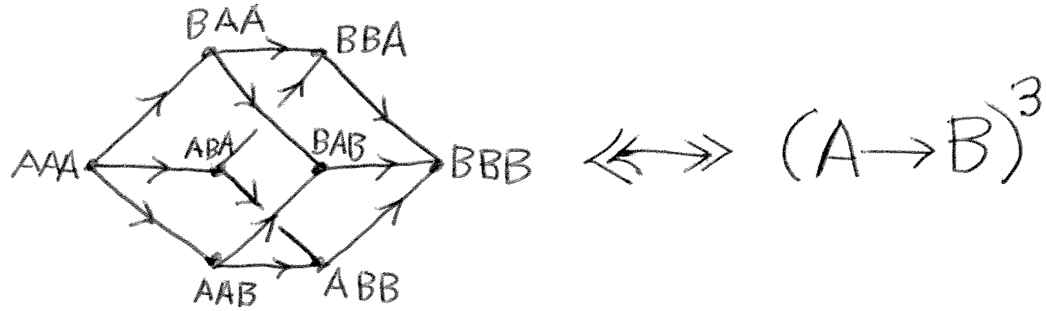
$\Longrightarrow \varepsilon(1)x + \varepsilon(x)1 = 1$

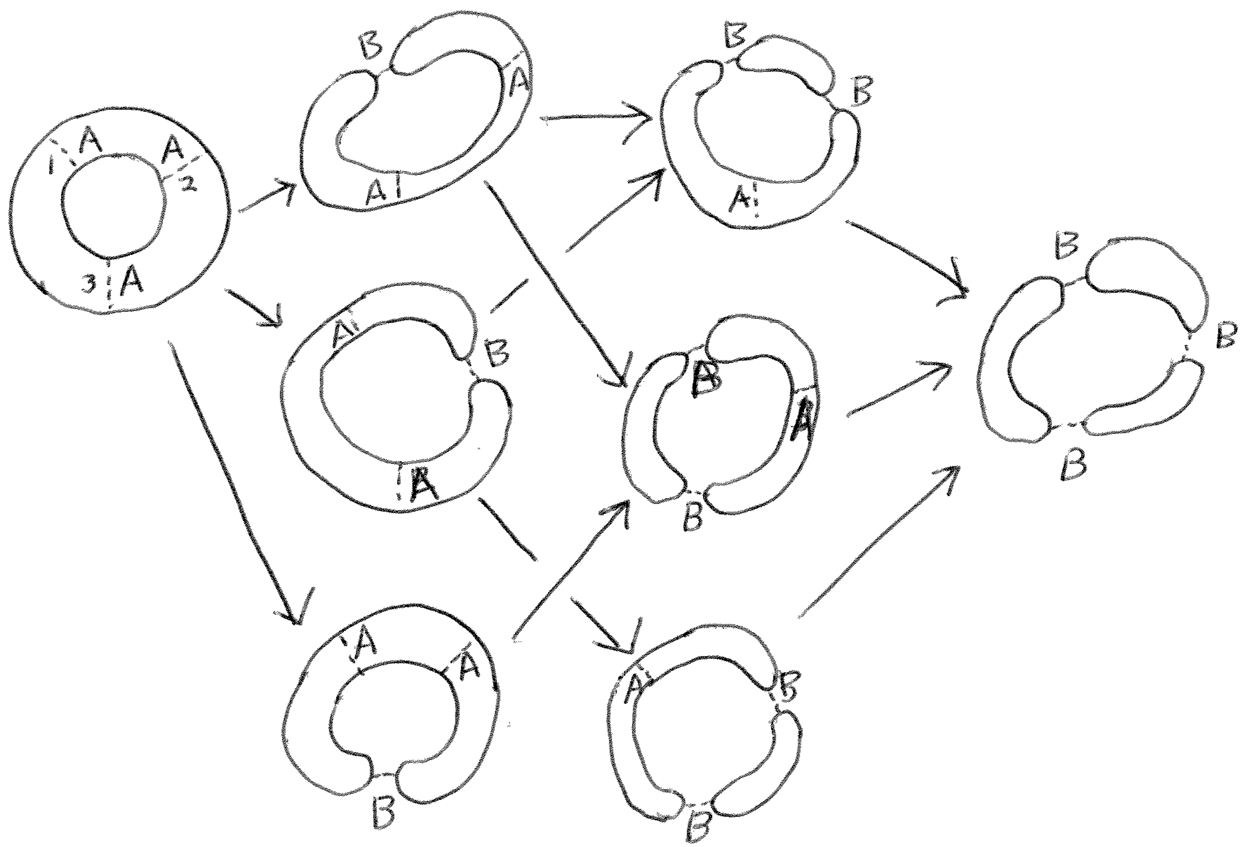
$\Longrightarrow \begin{array}{l} \varepsilon(1) = 0 \\ \varepsilon(x) = 1 \end{array}$



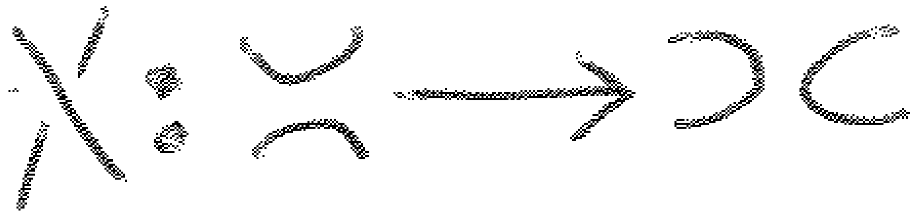
We have arrived at the Frobenius algebra, but there is still work to be done to see the invariance under ambient isotopy of knots and links.

Cubism Again





Categorification and the Morse Dream

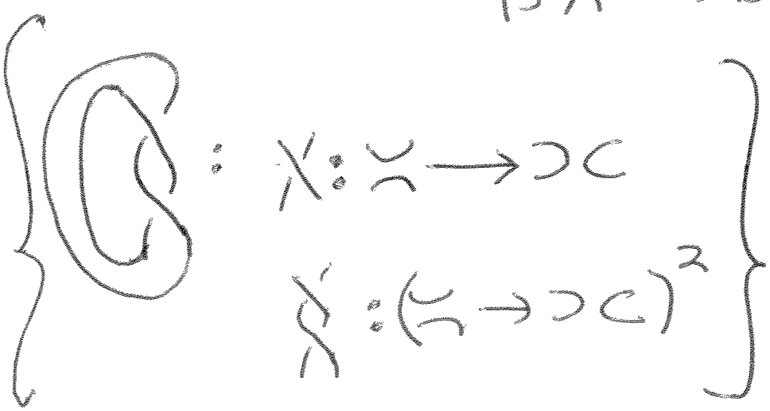


(flattening a higher category)

$$(A \to B)^2 = (A \to B)(A \longrightarrow B) = A(A \to B) \longrightarrow B(A \to B)$$

$$= (AA \to AB) \longrightarrow (BA \to BB)$$

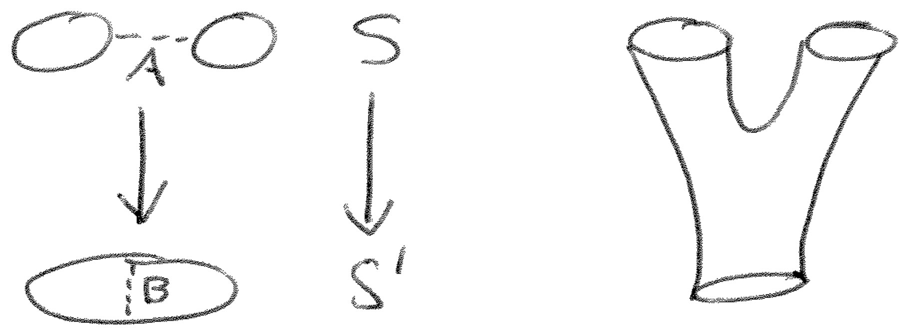
$$= \begin{array}{ccc} AA & \longrightarrow & AB \\ & \downarrow & \\ BA & \longrightarrow & BB \end{array} \quad = \quad \begin{tikzcd} AA \arrow[r] \arrow[d] & AB \arrow[d] \\ BA \arrow[r] & BB \end{tikzcd}$$



Morse Diagram $\longrightarrow$ <u>Cube Category</u>

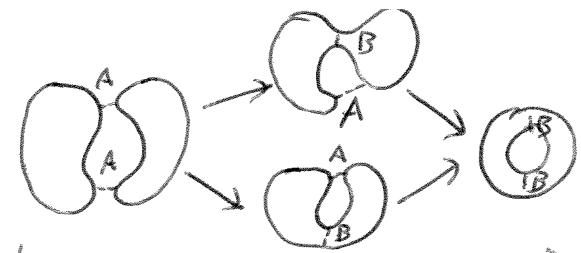
We have category $\mathcal{S}(K)$ whose objects are states S of K & morphisms gen by arrows $S \longrightarrow S'$, $b(S)+1 = b(S')$.

Regard the arrow as a surface cobordism.



Two surface cobs are = as morphisms iff the corresponding surfaces are homeomorphic.

$$\underbrace{\begin{tikzcd}[column sep=small] & AB \arrow[dr, "\partial_1"] & \\ AA \arrow[ur, "\partial_2"] \arrow[dr, "\partial_1"'] & & BB \\ & BA \arrow[ur, "\partial_2"'] & \end{tikzcd}}_{Cube[2]} \xrightarrow[\mathcal{F}_H]{}$$

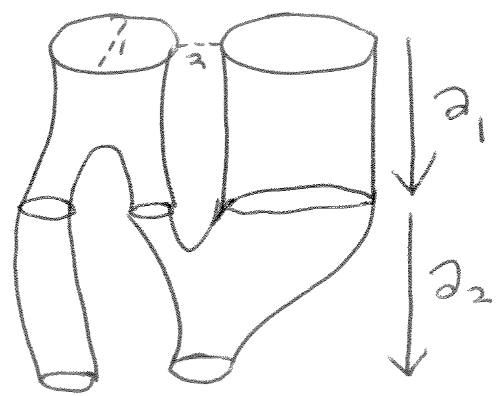


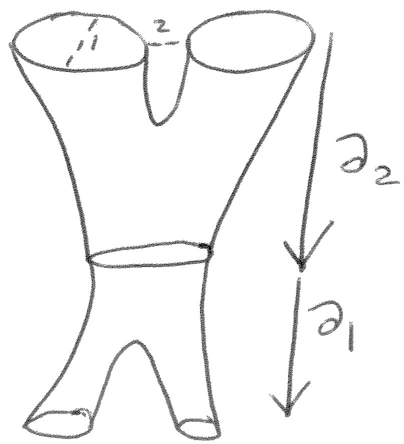
$\mathcal{S}$ (Hopf)  $= H$

$$\mathcal{S}(K) = \mathcal{F}_K\big(Cube[c(K)]\big)$$

and $\mathcal{F}$ extends to a <u>functor</u> from the cube category to the category $\mathcal{S}(K)$.

This means that all relevant squares commute. e.g.





Dror's Canopoly

An abstract categorical analog of a chain complex. That can be taken up to chain homotopy. The maps are additive combinations of surface cobordisms.

We make an abstract analogue of a chain complex from $\mathcal{S}(K)$ by extending to an additive category with dir sums.

$$A_1, \ldots, A_n \rightsquigarrow A = \bigoplus_{i=1}^{n} A_i$$

$$f : A \longrightarrow B, \quad B = \bigoplus_{j=1}^{m} B_j$$

$$f = (f_{ij}), \quad f_{ij} : A_i \longrightarrow B_j .$$

$$g : B \longrightarrow C$$

$$A_i \xrightarrow{f_{ik}} B_k \xrightarrow{g_{kj}} C_j$$

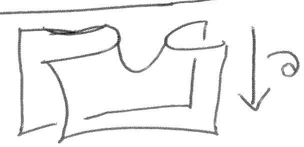
$$(g \circ f)_{ij} = \sum_k g_{kj} \circ f_{ik}$$

$$S \in \mathcal{O}bj(\mathcal{S}(K))$$

$$\mathcal{C}^i(K) = \bigoplus_{\substack{S \in \mathcal{O}bj(\mathcal{S}(K)) \\ b(S) = i}} S$$

$$\partial : \mathcal{C}^i(K) \longrightarrow \mathcal{C}^{i+1}(K)$$

$$\partial = \sum_{k=1}^{c(K)} \pm \partial_k$$



We say $f \sim g$ iff $\exists H: \mathcal{C} \to \mathcal{C}'$ s.t. $\partial H + H\partial = f - g$.

Work Mod 2 .

Categorical Chain Homotopy

Question: What is least equiv reln on $\mathcal{CS}(K)$ s.t. $[\mathcal{CS}(K)]$ (= chain homotopy equiv class of $\uparrow$) is invar under RM's ?

We examine this question as though we had not seen the Frobenius algebra.

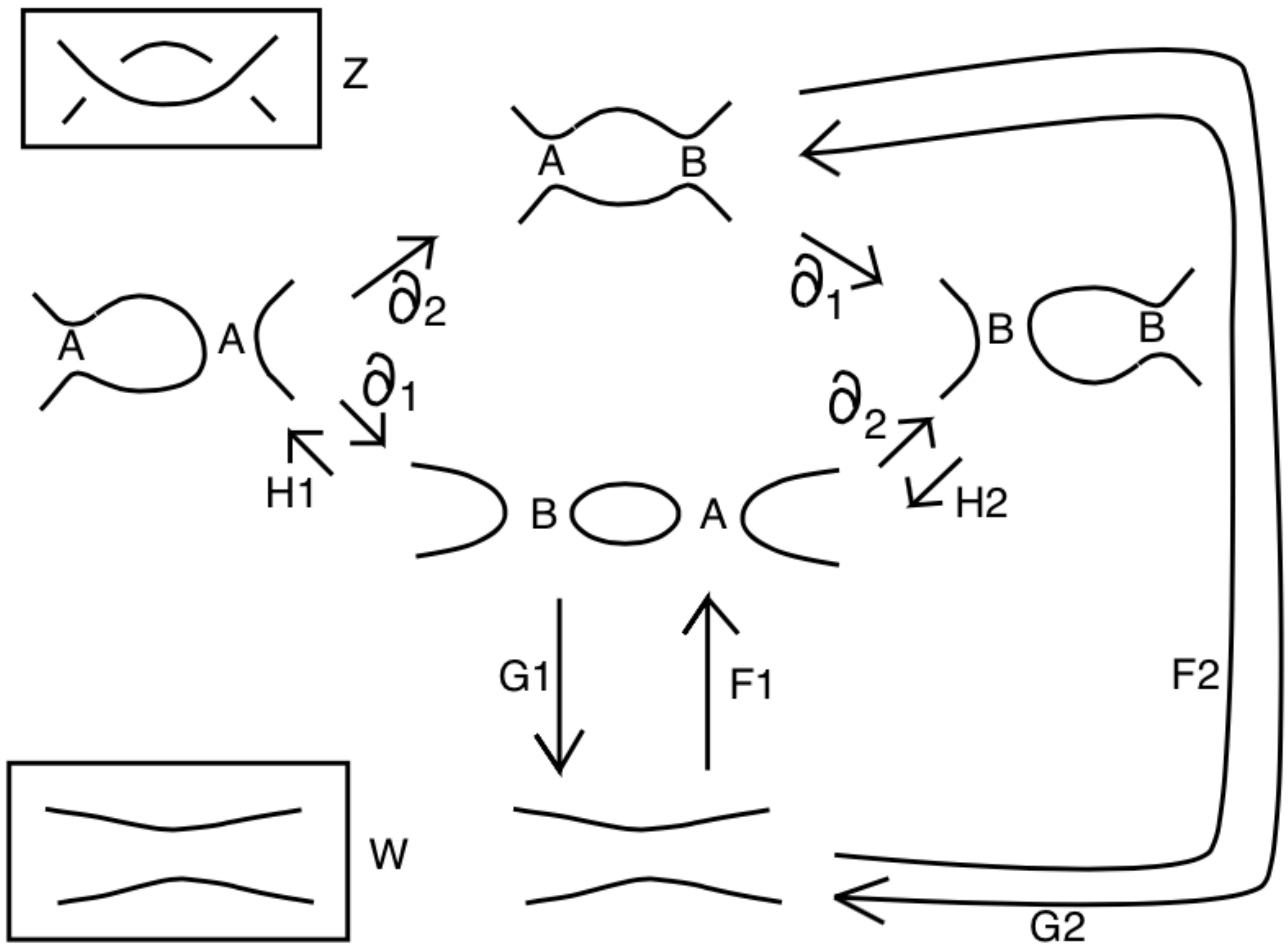


Figure 11: **Complexes for Second Reidemeister Move**

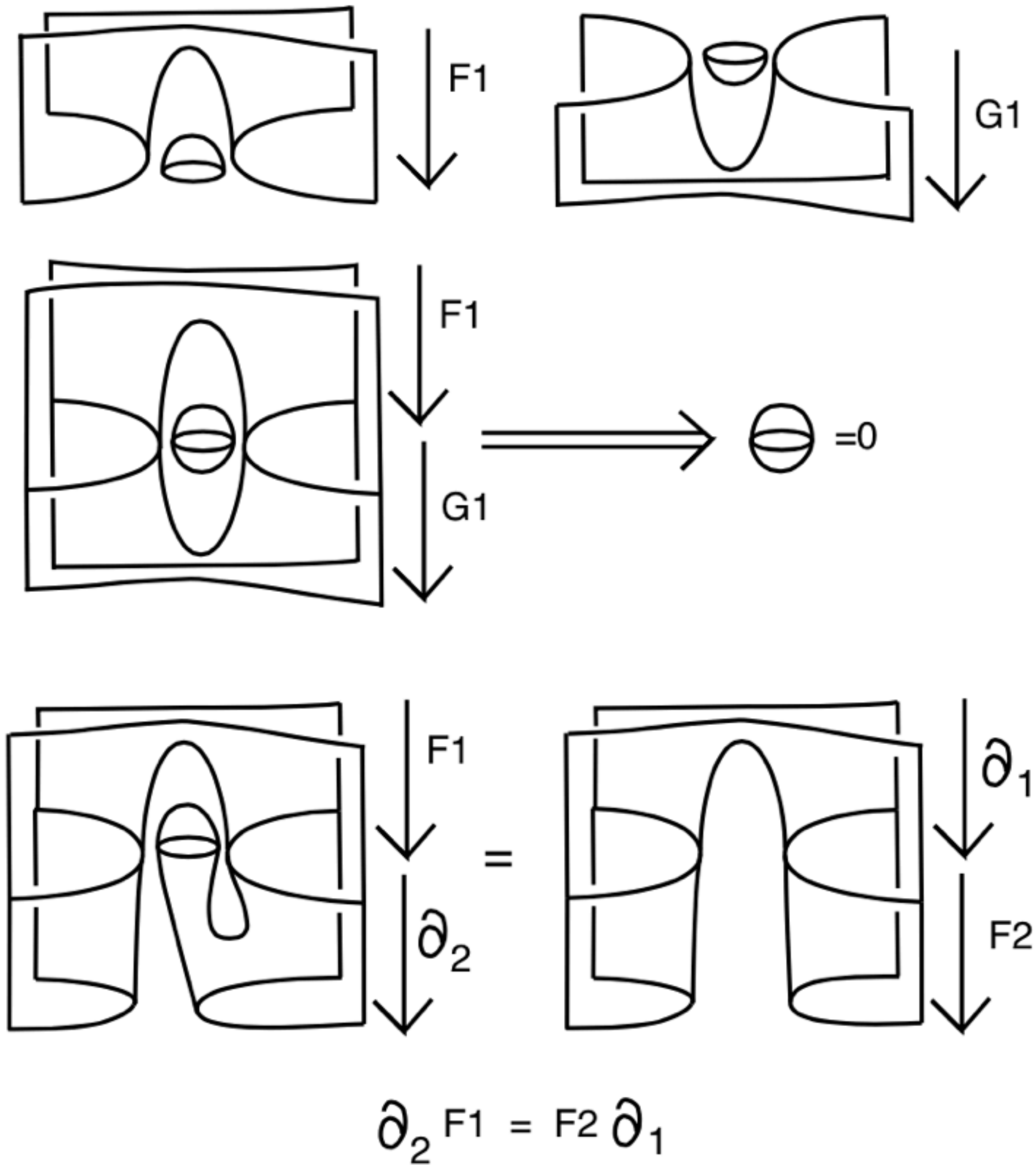


Figure 12: **Cobordism Compositions for Second Reidemeister Move**

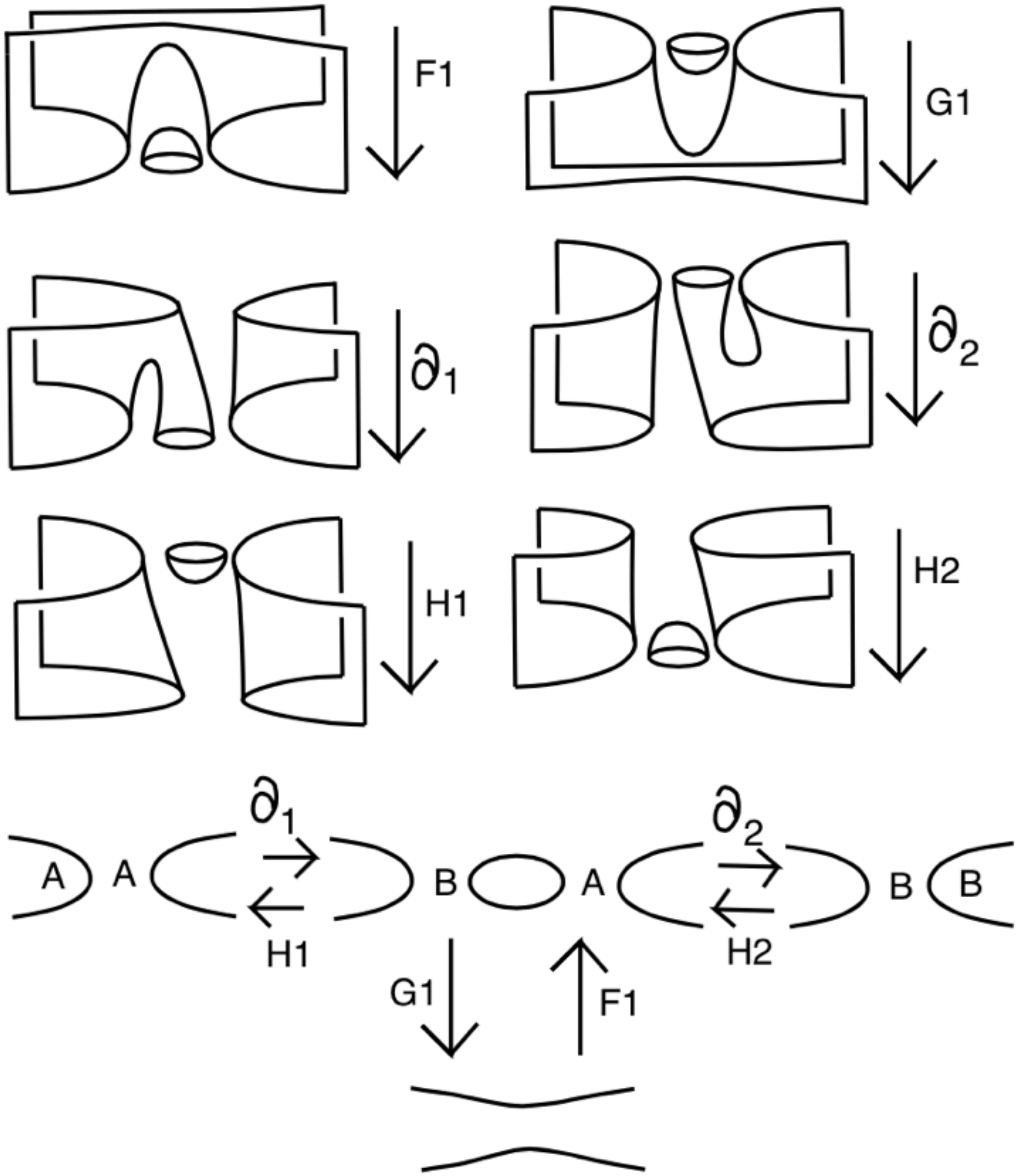


Figure 13: **Preparation for Homotopy for Second Reidemeister Move**

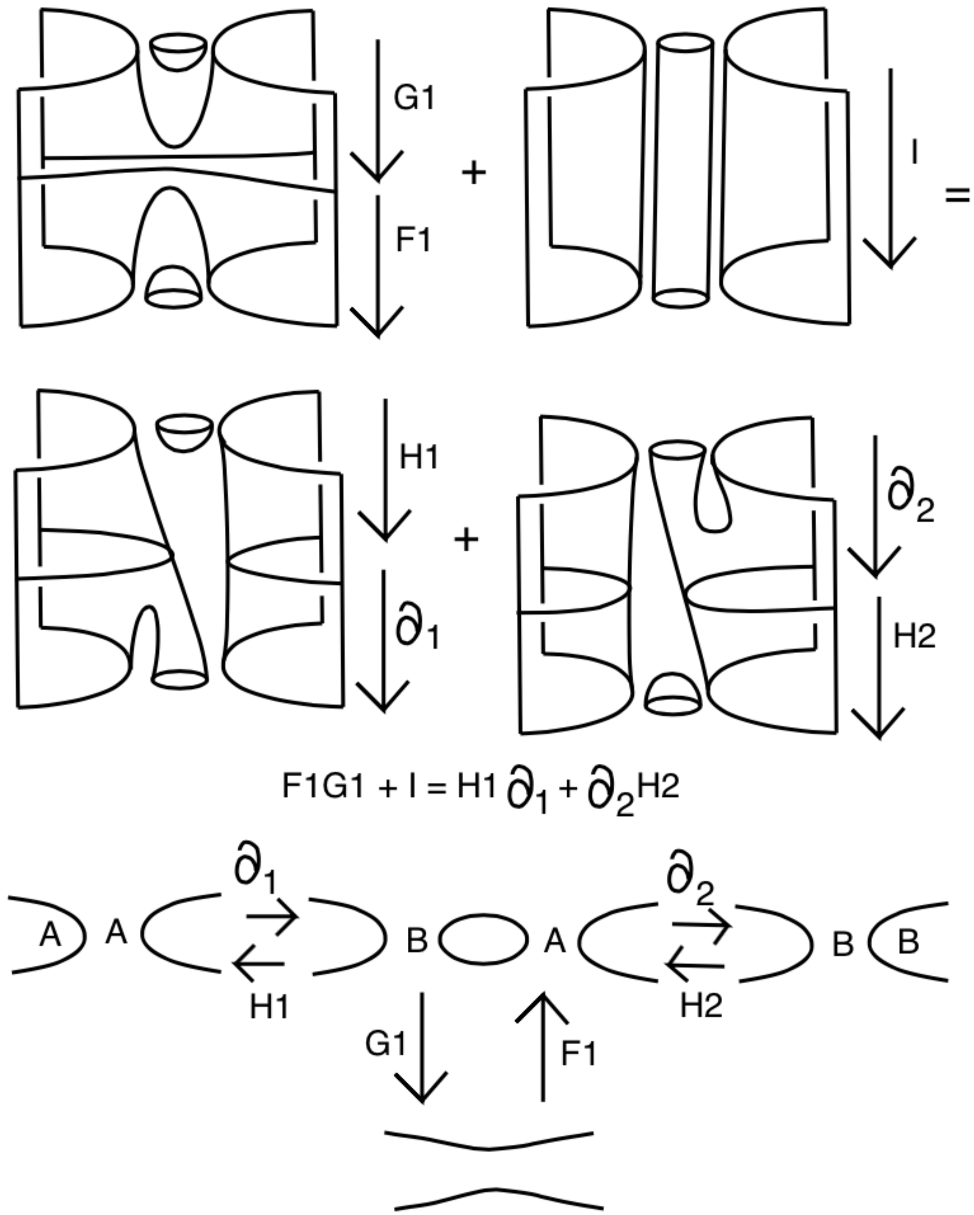


Figure 14: **Homotopy for Second Reidemeister Move**

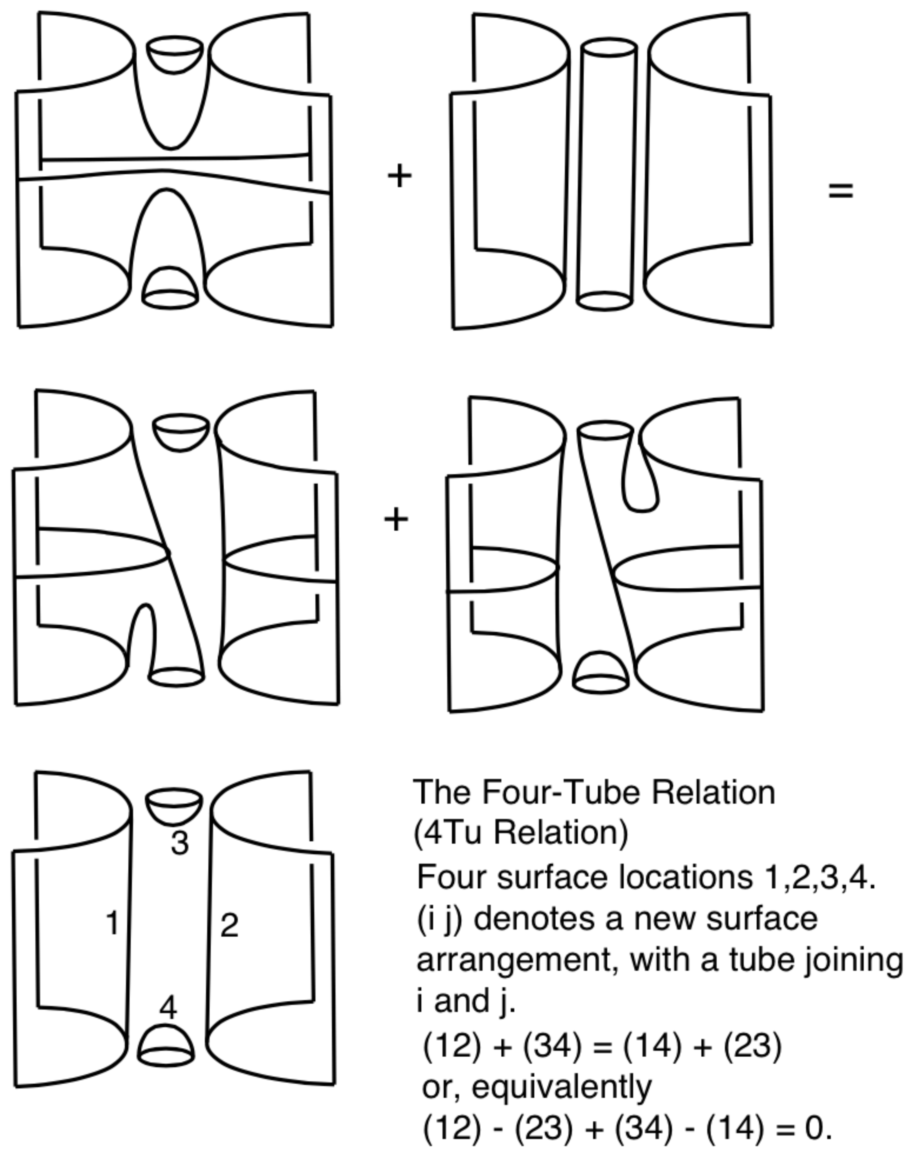
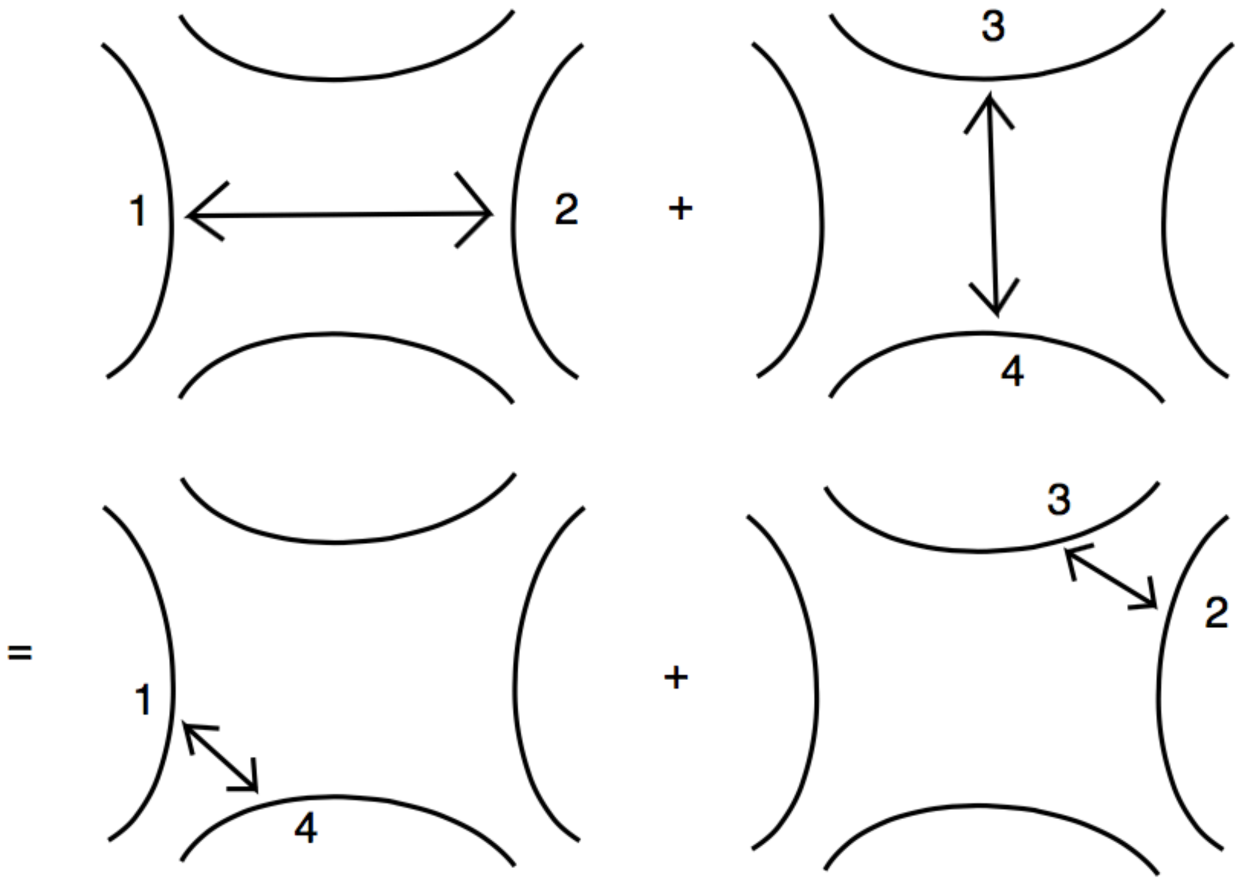


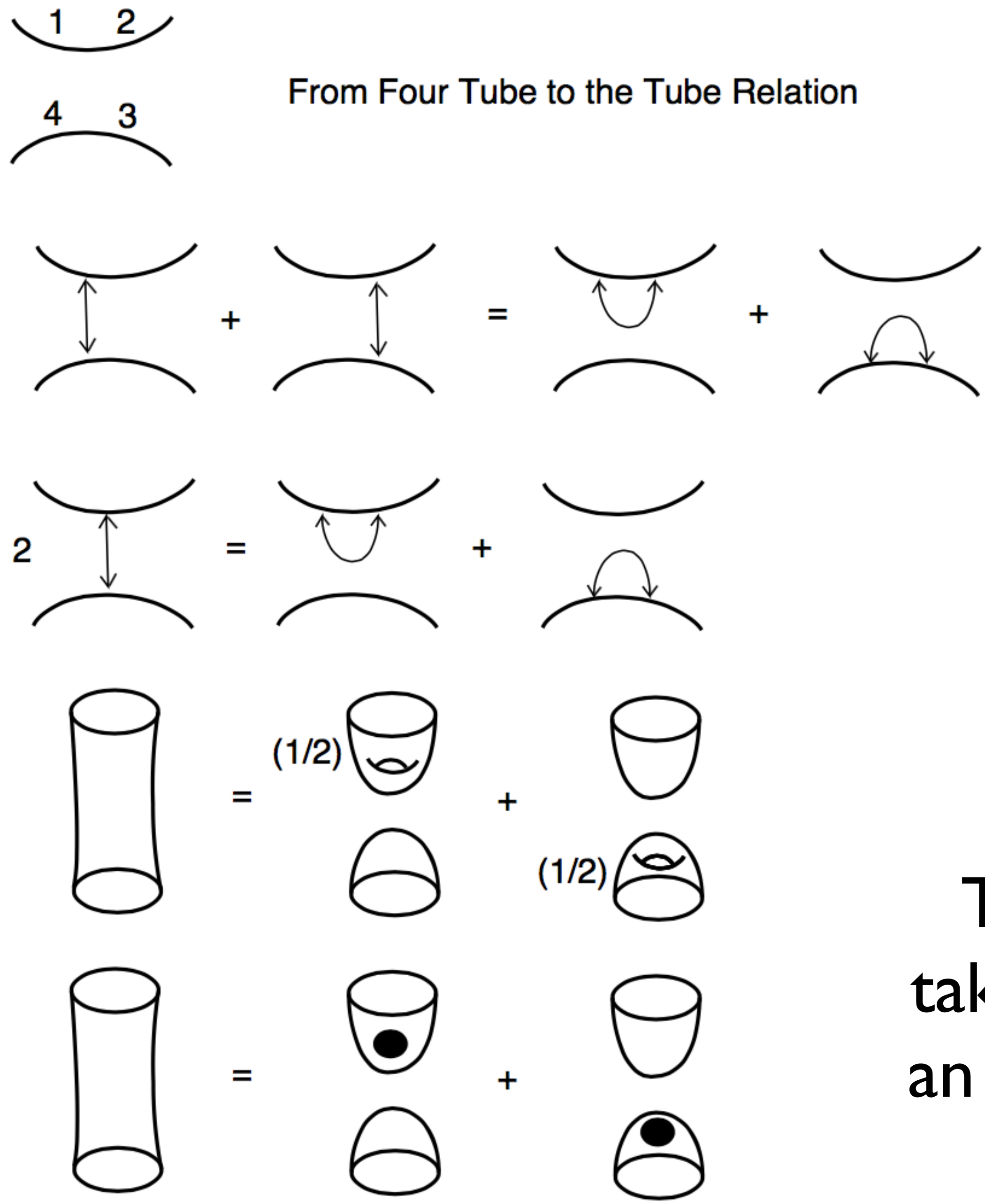
Figure 15: **Four-Tube Relation From Homotopy**



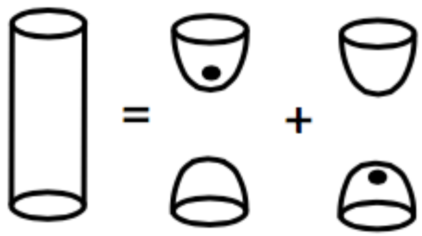
1234 - 1234 +1234 - 1234 = 0

Schematic Four-Tube Relation

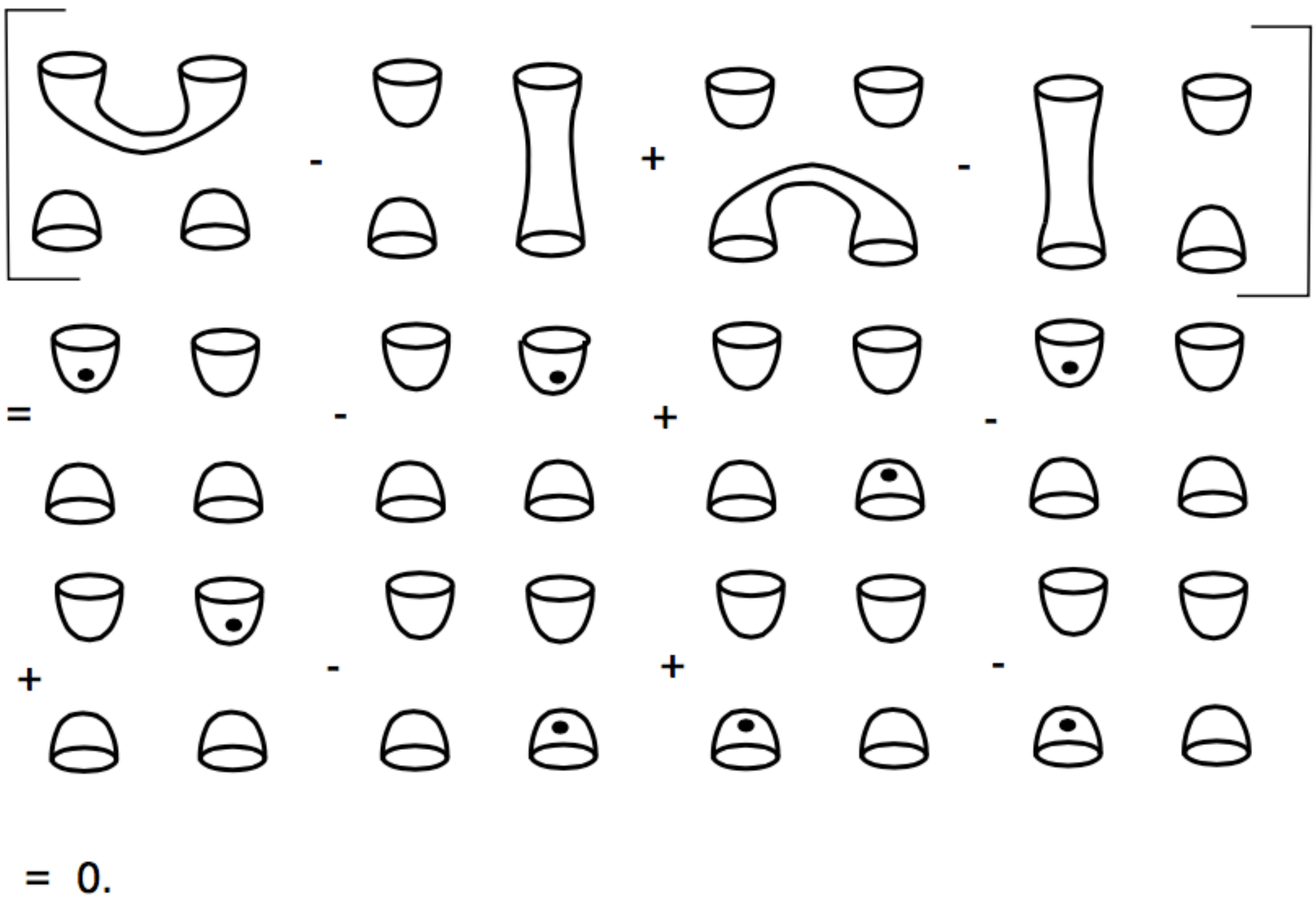
From Four Tube to the Tube Relation



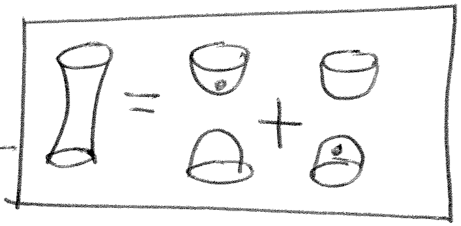
The dot can be taken to represent an algebra element x.



The Tube Relation implies the Four Tube Relation.

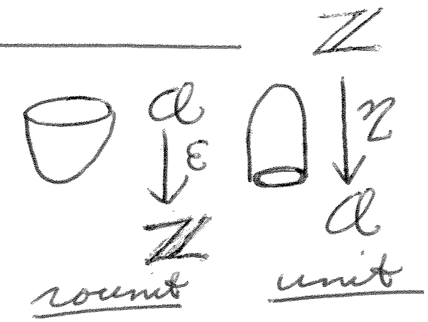


From 4Tu to Frobenius Algebra



Tube Relation

Now find algebra $\mathcal{A}$



$\bullet \equiv x \in \mathcal{A}$, $\alpha \in \mathcal{A}$

$$\Rightarrow \boxed{\alpha = \varepsilon(\alpha x)1 + \varepsilon(\alpha)x}$$

$$\Longrightarrow \left.\begin{aligned} x &= \varepsilon(x^2)1 + \varepsilon(x)x \\ 1 &= \varepsilon(x)1 + \varepsilon(1)x \end{aligned}\right\} \quad \boxed{\begin{aligned} \varepsilon(x) &= 1 \\ \varepsilon(x^2) &= 0 \\ \varepsilon(1) &= 0 \end{aligned}}$$

$$x^2 = \varepsilon(x^3)1 + \varepsilon(x^2)x$$

$$\Rightarrow x^2 = k1, \quad k \in \mathbb{Z}$$

$$\boxed{x^2 = k}$$

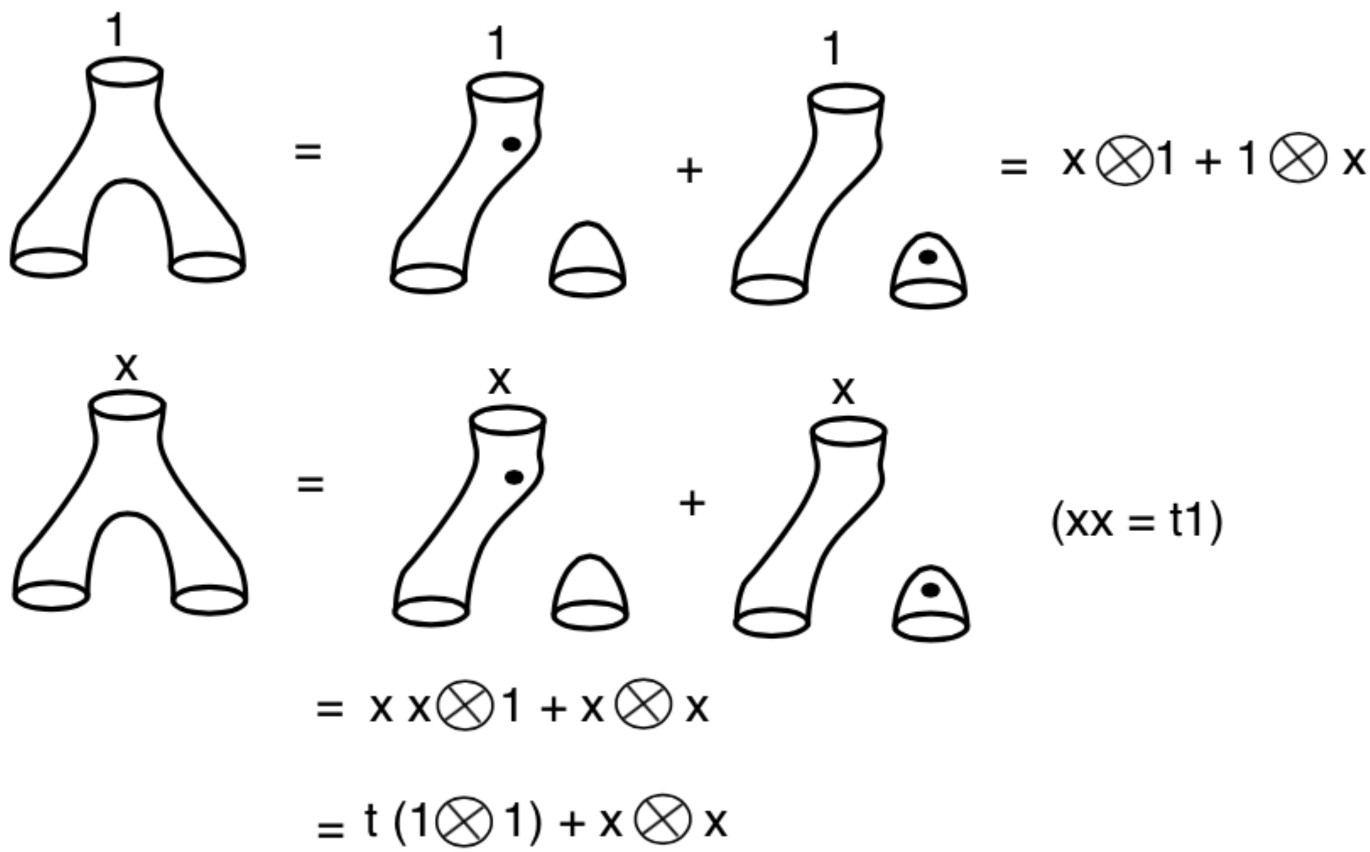


Figure 20: **Coproducts of 1 and x Via Tube-Cutting Relation**

Algebra from 4Tu - Guaranteed to Produce Link Homology

$$\mathcal{A} = \mathbb{Z}[x]/(x^2 - k)$$

$$\varepsilon(x) = 1, \quad \varepsilon(1) = 0$$

$$\Delta(1) = 1 \otimes x + x \otimes 1$$

$$\Delta(x) = k(1 \otimes 1) + x \otimes x$$

$k = 0$: Khovanov

$k = 1$: Lee

Lee's Algebra

$$
\begin{gathered}
x^2 = 1, \\
\Delta(1) = 1 \otimes x + x \otimes 1, \\
\Delta(x) = x \otimes x + 1 \otimes 1, \\
\epsilon(x) = 1, \\
\epsilon(1) = 0.
\end{gathered}
$$

This gives a link homology theory that is distinct from Khovanov homology. In this theory, the quantum grading j is not preseved, but we do have that

$$j(\partial(\alpha)) \geq j(\alpha)$$

for each chain α in the complex. This means that *one can use j to filter the chain complex for the Lee homology.* The result is a spectral sequence that starts from Khovanov homology and converges to Lee homology.

Lee's Algebra

$$\mathcal{A} = \mathbb{Q}[x]/(x^2-1)$$

$$\varepsilon(x) = 1, \ \varepsilon(1) = 0$$

$$\Delta(1) = 1 \otimes x + x \otimes 1$$

$$\Delta(x) = (1 \otimes 1) + x \otimes x$$

Let $r = \dfrac{1+x}{2}$, $g = \dfrac{1-x}{2}$

$$\varepsilon(r) = 1/2, \ \varepsilon(g) = -1/2$$

$$r + g = 1$$

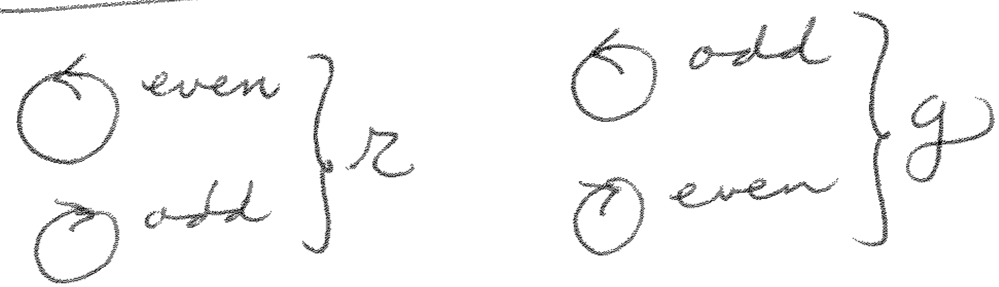
$$r^2 = r, \ g^2 = g$$

$$rg = 0$$

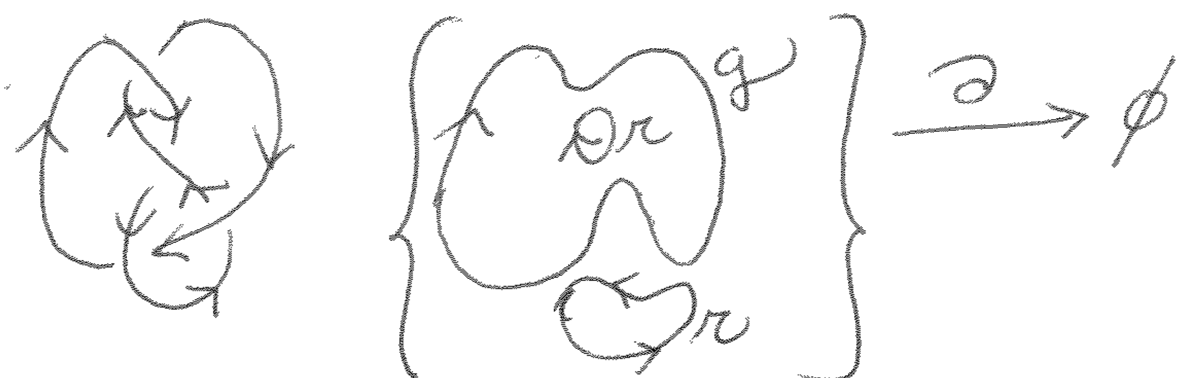
$$\Delta(r) = 2\, r \otimes r$$

$$\Delta(g) = -2\, g \otimes g$$

Lee Homology is gen by Seifert smoothing states for all choices of orientations of link.



e.g.



Lee homology is simple. One has that the dimension of the Lee homology is equal to $2^{comp(L)}$ where $comp(L)$ denotes the number of components of the link L. Up to homotopy, Lee's homology has a vanishing differential, and the complex behaves well under link concondance. In his paper [4] Dror BarNatan remarks "In a beautiful article Eun Soo Lee introduced a second differential on the Khovanov complex of a knot (or link) and showed that the resulting (double) complex has non-interesting homology. This is a very interesting result." Rasmussen [49] uses Lee's result to define invariants of links that give lower bounds for the four-ball genus, and determine it for torus knots. This gives an (elementary) proof of a conjecture of Milnor that had been previously shown using gauge theory by Kronheimer and Mrowka [29].

Rasmussen's result uses the Lee spectral sequence. We have the quantum (j) grading for a diagram K and the fact that for Lee's algebra $j(\partial(s)) \geq j(s)$. Rasmussen uses a normalized version of this grading denoted by $g(s)$. Then one makes a filtration $F^kC^*(K) = \{v \in C^*(K) | g(v) \geq k\}$ and given $\alpha \in Lee^*(K)$ define

$$S(\alpha) := max\{g(v) | [v] = \alpha\}$$

$$s_{min}(K) := min\{S(\alpha) | \alpha \in Lee^*(K), \alpha \neq 0\}$$
$$s_{max}(K) := max\{S(\alpha) | \alpha \in Lee^*(K), \alpha \neq 0\}$$

and

$$s(K) := (1/2)(s_{min}(K) + s_{max}(K)).$$

This last average of s_{min} and s_{max} is the Rasmussen invariant.

Grading

$$q(s) = j(s) + (n_+ - 2n_-)$$

$$\begin{cases} n_+ = \#\text{ of } + \text{ crossings in } K. \\ n_- = \#\text{ of } - \text{ crossings in } K. \end{cases}$$

$$\boxed{\begin{aligned} j(s) = \#(&B\text{ smoothings}) \\ &+ \#(1\text{'s}) - \#(X\text{'s}) \end{aligned}}$$

We now enter the following sequence of facts:

1. $s(K) \in Z$.
2. $s(K)$ is additive under connected sum.
3. If K^* denotes the mirror image of the diagram K, then

$$s(K^*) = -s(K).$$

4. If K is a positive knot diagram (all positive crossings), then

$$s(K) = -r + n + 1$$

   where r denotes the number of loops in the canonical oriented smoothing (this is the same as the number of Seifert circuits in the diagram K) and n denotes the number of crossings in K.

5. For a torus knot $K_{a,b}$ of type (a, b), $s(K_{a,b}) = (a-1)(b-1)$.
6. $|s(K)| \leq 2g^*(K)$ where $g^*(K)$ is the least genus spanning surface for K in the four ball.
7. $g^*(K_{a,b}) = (a-1)(b-1)/2$. This is Milnor's conjecture.

This completes a very skeletal sketch of the construction and use of Rasmussen's invariant.

Grading

$$q(s) = j(s) + (n_+ - 2n_-)$$

$$\begin{cases} n_+ = \#\text{ of } + \text{ crossings in } K. \\ n_- = \#\text{ of } - \text{ crossings in } K. \end{cases}$$

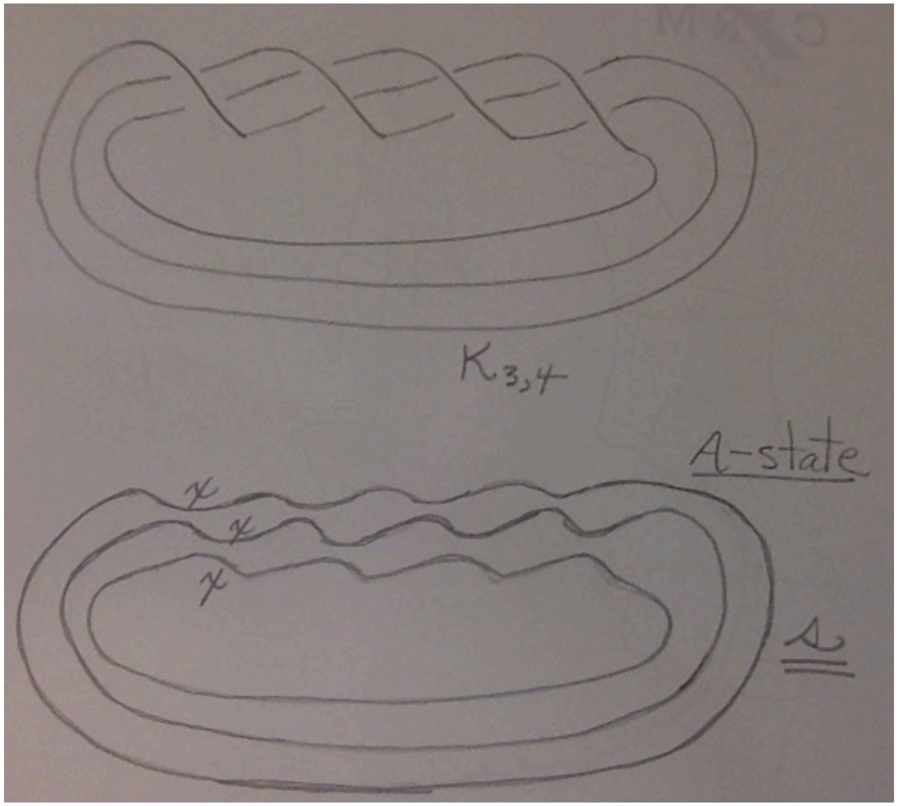
$$\boxed{j(s) = \#(B\text{ smoothings}) + \#(1\text{'s}) - \#(x\text{'s})}$$

Facts: $s_{max}(K) = s_{min}(K) + 2$

$s(K) = s_{min}(K) + 1$

A-State: $s(K) = 1 - (\#\text{loops}) + (\#\text{ crossings}) = 2\text{genus}(\text{Seifert}(K))$

For positive knot all loops labelled x.



For A-state of a $K_{p,q}$ torus knot have (with all x's) : • p loops

• $(p-1)q$ crossings

So $g(s) = (0-p) + (p-1)q$

$= pq - q - p$

$g(s) = (p-1)(q-1) - 1$

$$\Longrightarrow \boxed{s(K_{pq}) = (p-1)(q-1)}$$

Virtual Knot Theory
studies stabilized knots in thickened surfaces.

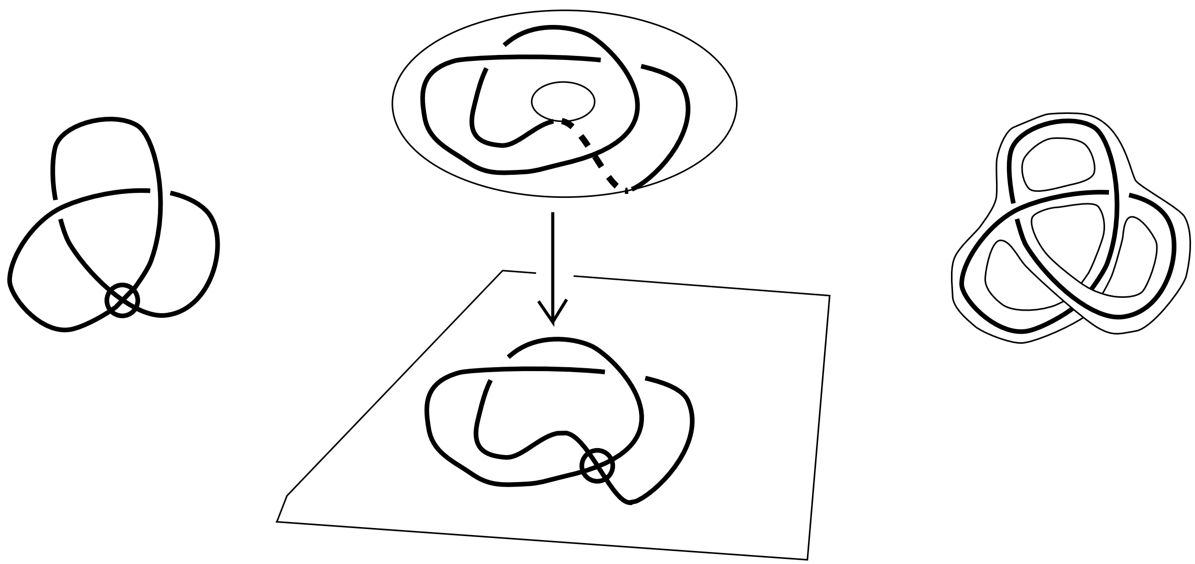
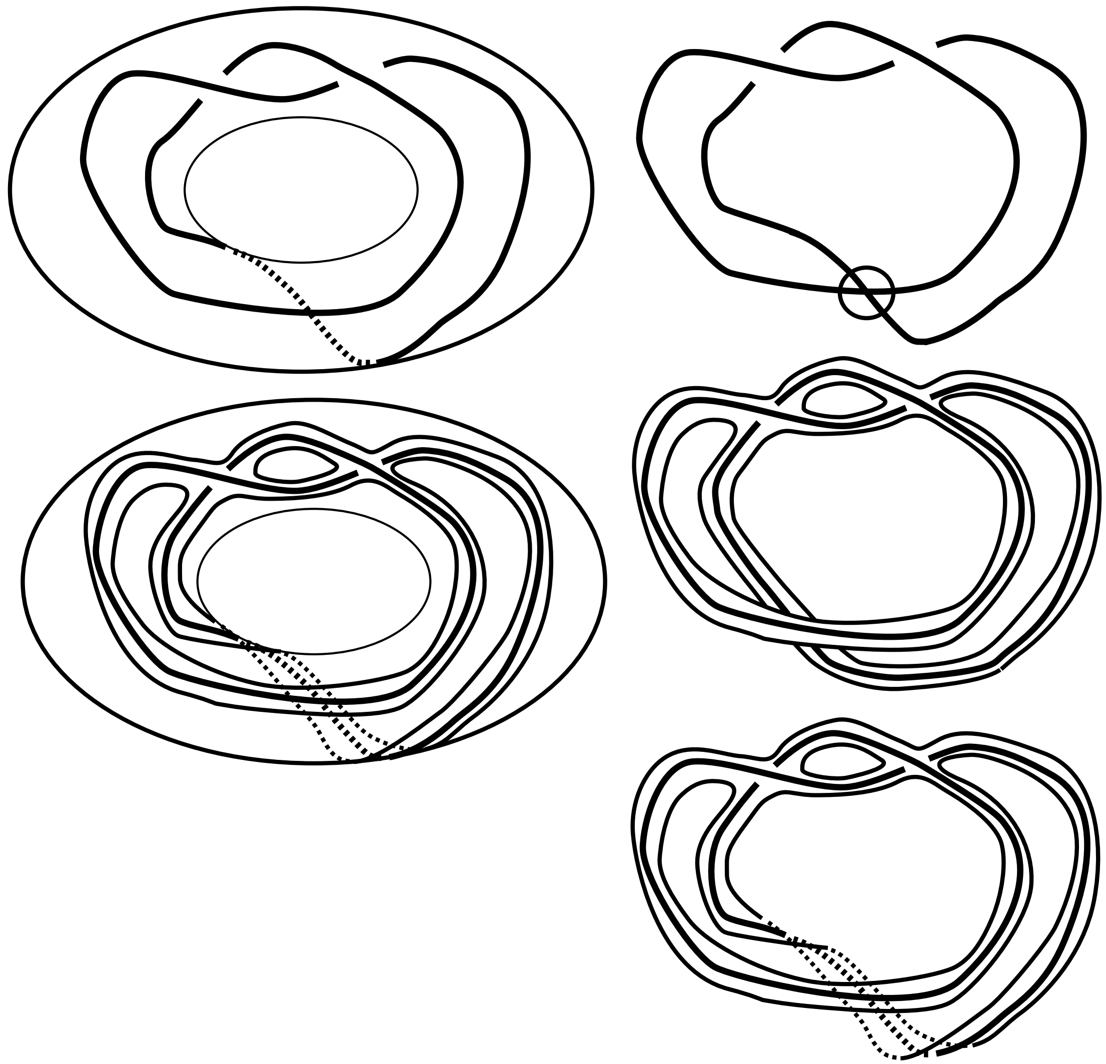
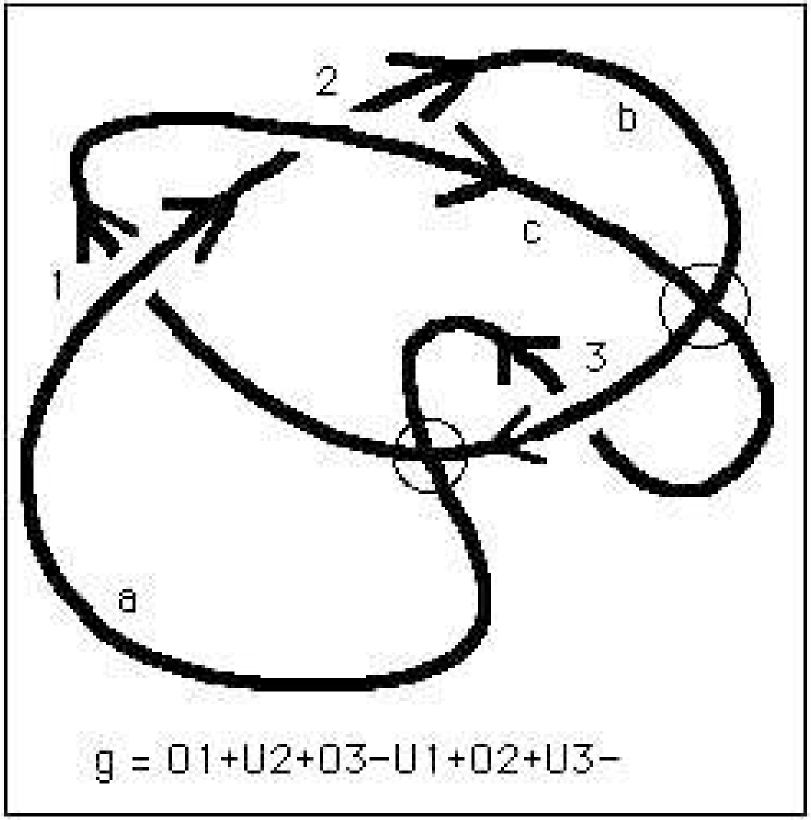


Figure 4: **Surfaces and Virtuals**





Virtual knots are all oriented (signed) Gauss codes taken up to Reidemeister moves on the codes.

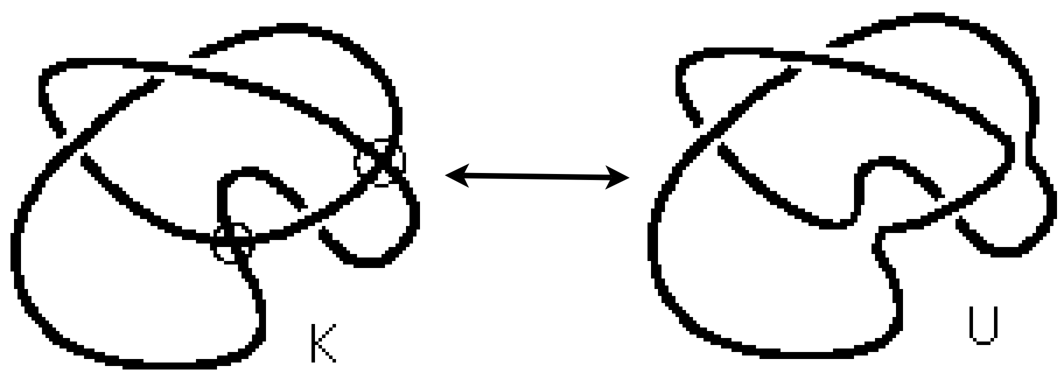
Virtual crossings are artifacts of the planar diagram.

$$g = O1 + U2 + O3 - U1 + O2 + U3 - .$$

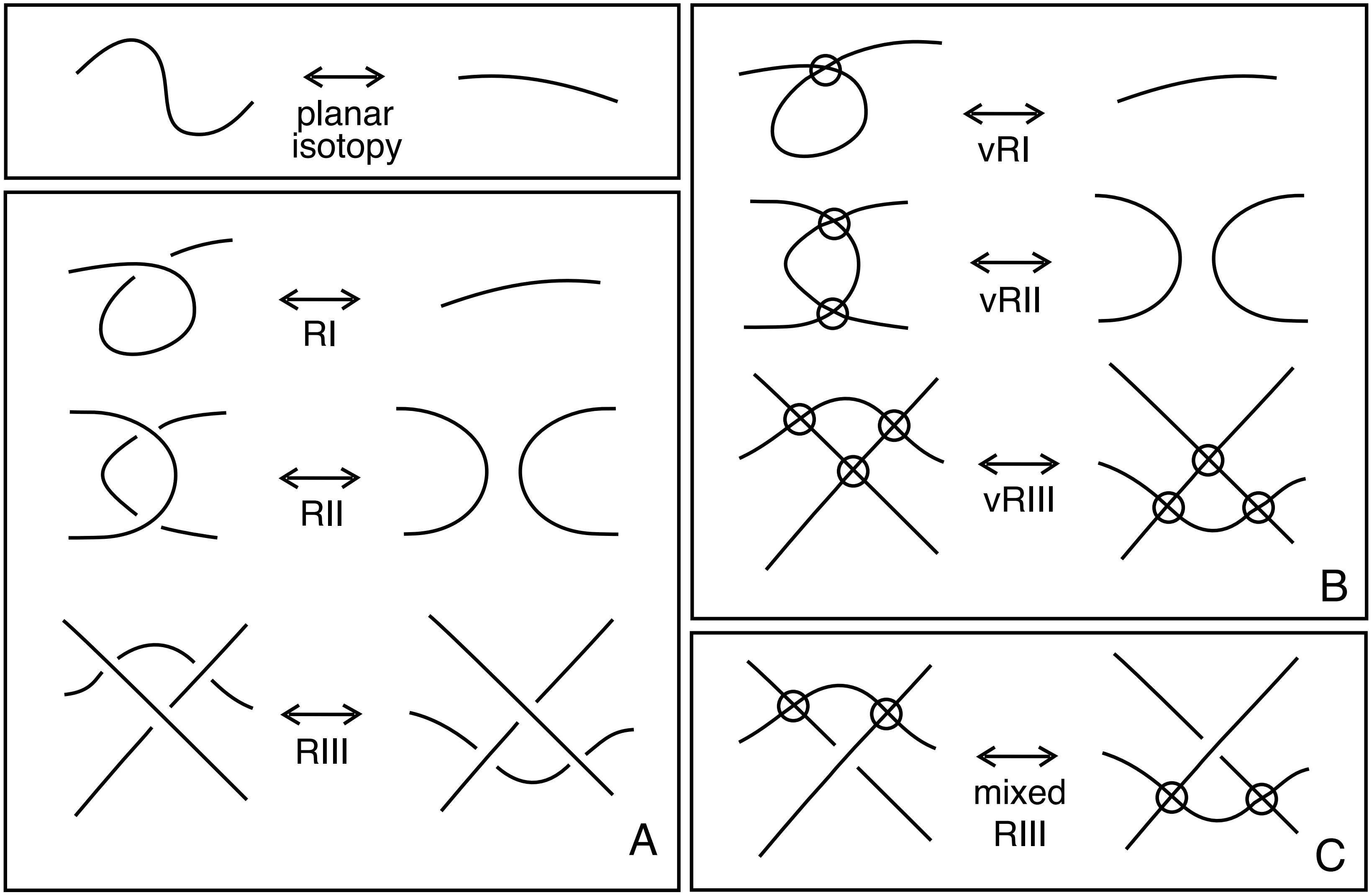
There exist infinitely many non-trivial K
with unit Jones polynomial.

Bracket Polynomial is Unchanged
when smoothing flanking virtuals.

Z-Equivalence



Generalized Reidemeister Moves for Virtual Knots and Links



Virtual Knot Cobordism

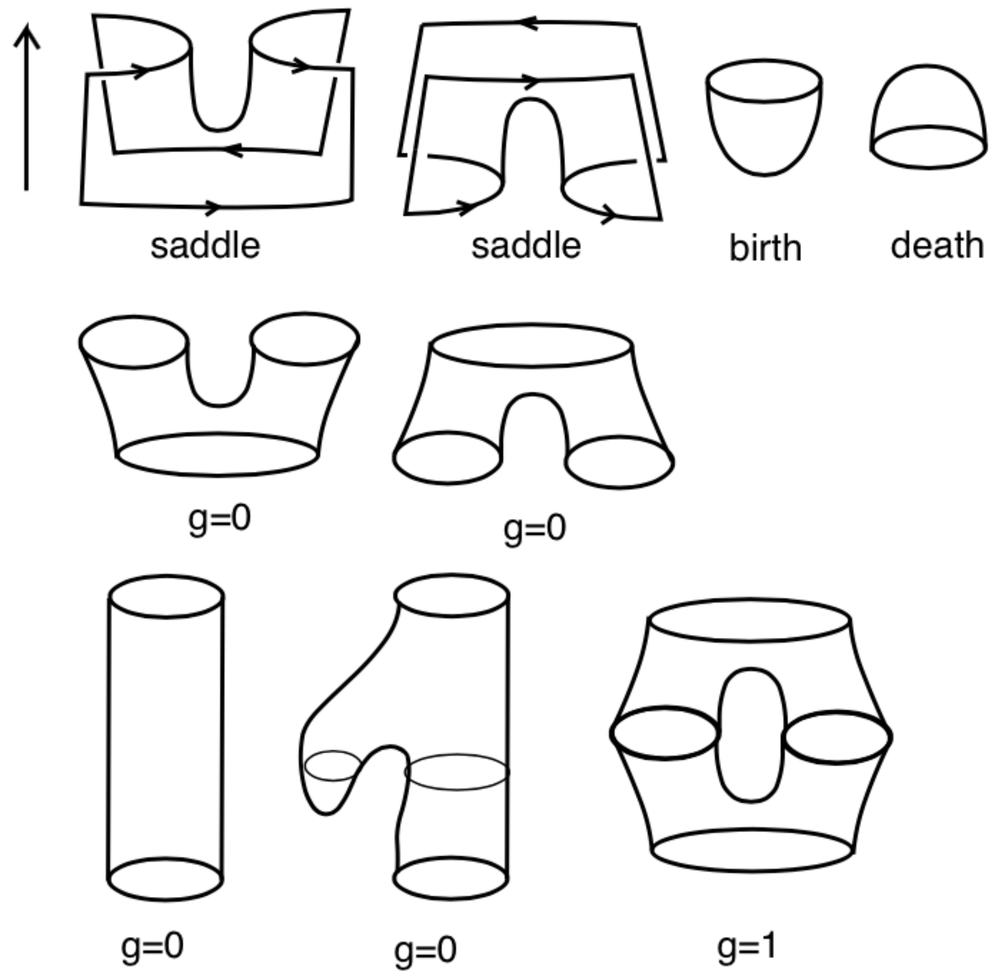


Figure 16: **Saddles, Births and Deaths**

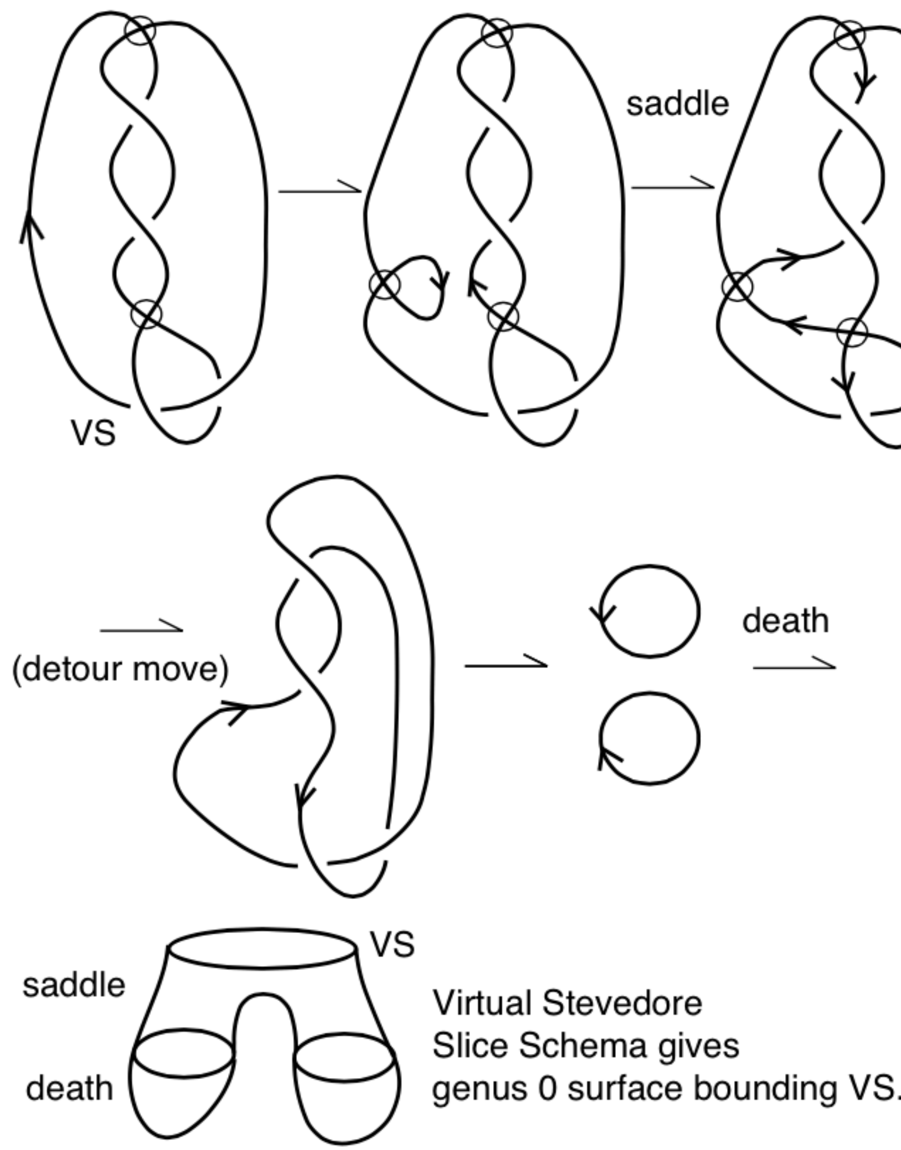


Figure 17: **Virtual Stevedore is Slice**

We say that K is concordant to K`

$$K \underset{c}{\sim} K`$$

if there exists a cobordism from K to K` of genus 0.

A virtual knot is said to be slice
if it is concordant to the unknot.

Spanning Surfaces for Knots and Virtual Knots.

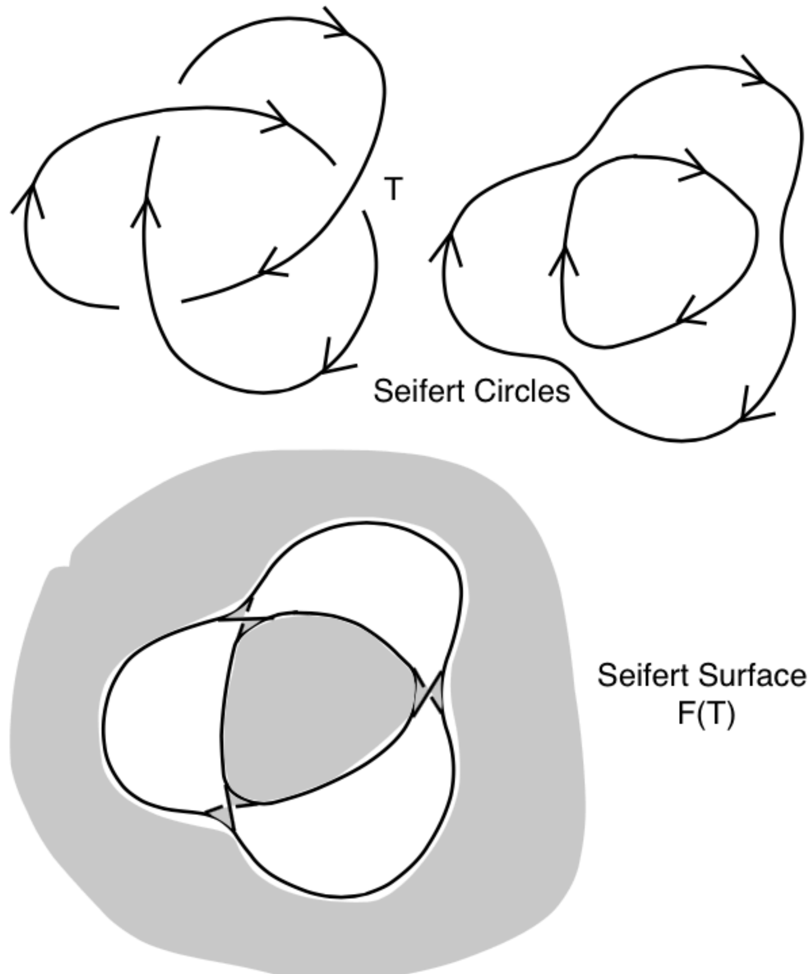
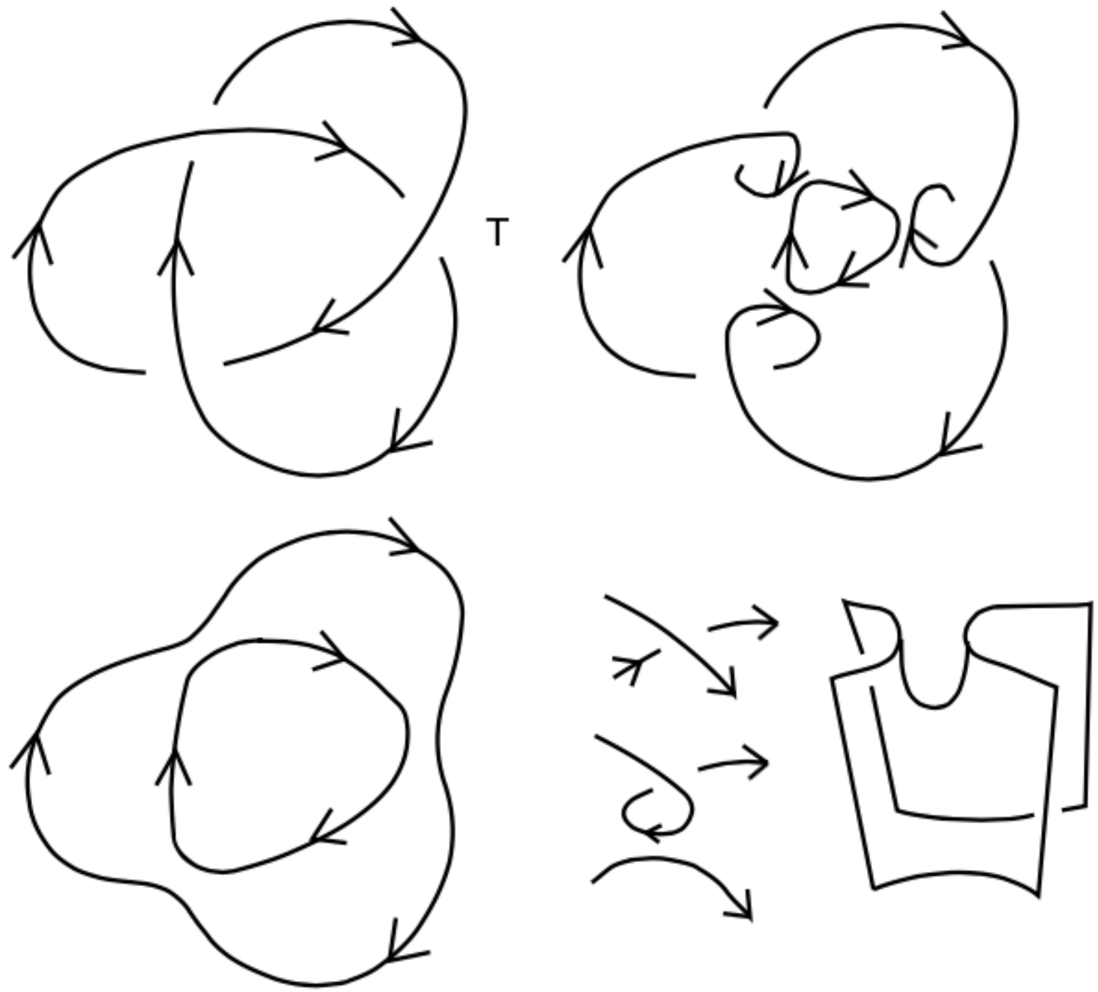
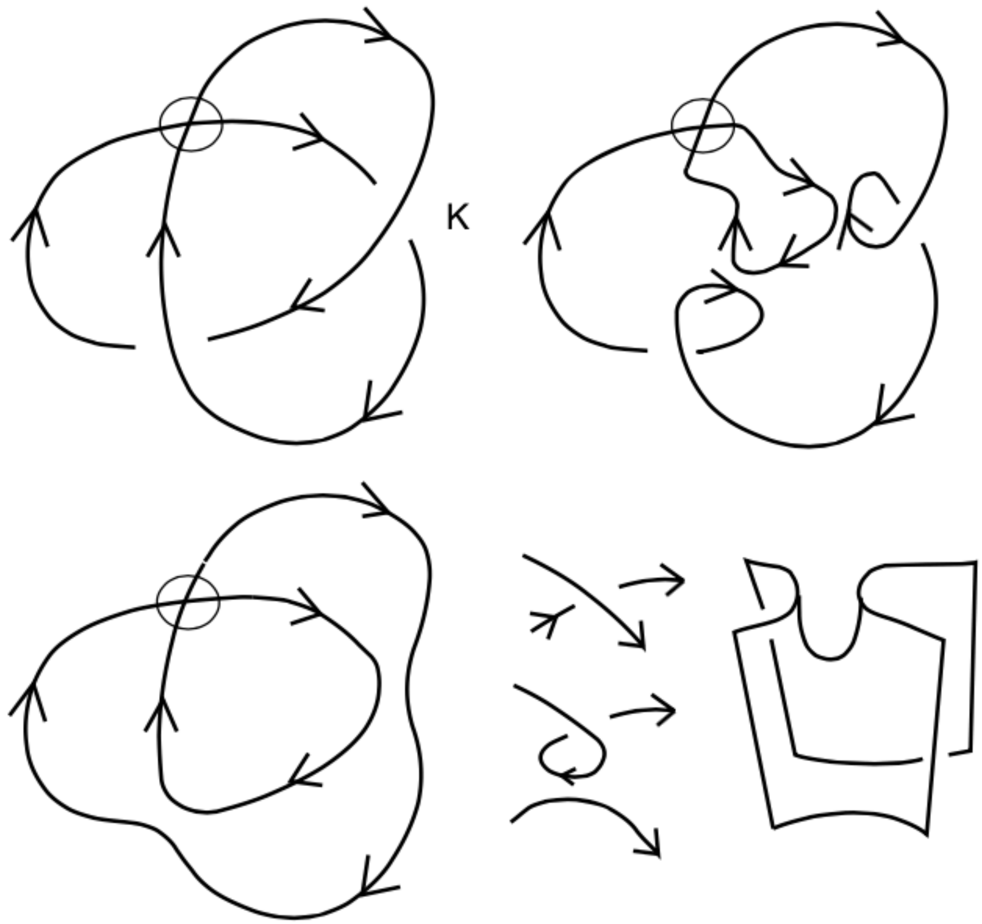


Figure 18: **Classical Seifert Surface**



Every classical knot diagram bounds a surface in the four-ball
whose genus is equal to the genus of its Seifert Surface.

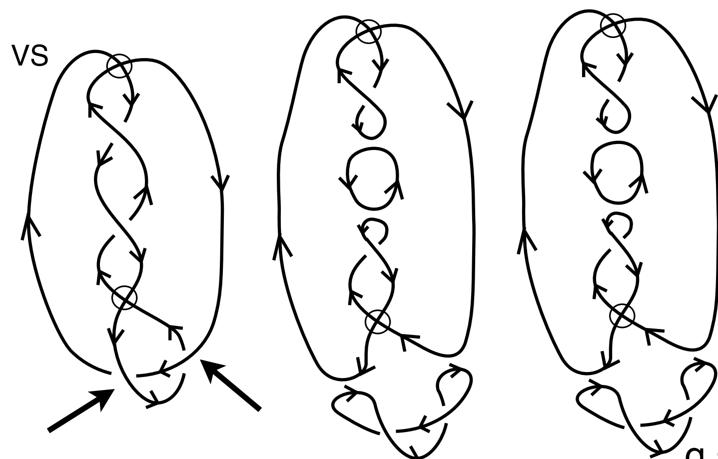
Figure 19: **Classical Cobordism Surface**



Every virtual diagram K bounds a virtual orientable surface of genus g = (1/2)(-r + n +1) where r is the number of Seifert circles, and n is the number of classical crossings in K.
This virtual surface is the cobordism Seifert surface when K is classical.

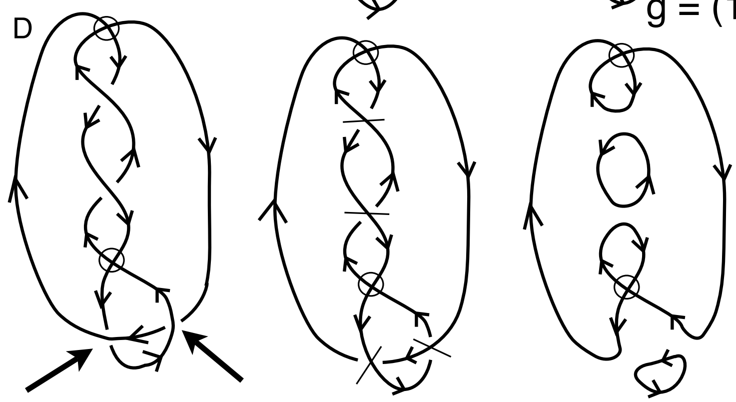
Figure 20: **Virtual Cobordism Seifert Surface**

Seifert Cobordism for the Virtual Stevedore and for a corresponding positive diagram D.



VS is the virtual stevedore
and bounds
another surface of
genus zero.

g = (1/2)(-r + n + 1) = (1/2)(-3 +4 + 1) = 1.



D is a positive
virtual diagram and
is NOT slice.

Heather Dye, Aaron Kaestner and LK, prove the following generalization of Rasmussen's Theorem, giving the four-ball genus of a positive virtual knot.

Theorem [2]. Let K be a positive virtual knot (all classcial crossings in K are positive), then the four-ball genus $g_4(K)$ is given by the formula

$$g_4(K) = (1/2)(-r + n + 1) = g(S(K))$$

where r is the number of virtual Seifert circles in the diagram K and n is the number of classcial crossings in this diagram. In other words, that virtual Seifert surface for K represents its minimal four-ball genus.

The virtual Seifert surface for positive virtual K represents the minimal four-ball genus of K.

The Theorem is proved by generalizing both Khovanov and Lee homology to virtual knots and generalizing the Rasmussen invariant to virtual knots.

Remarks on Generalizing Khovanov Homology, Lee Homology and Rasmussen Invariant

Extending Khovanov homology to virtual knots for arbitrary coefficients is complicated by the single cycle smoothing as depicted in Figure 8.

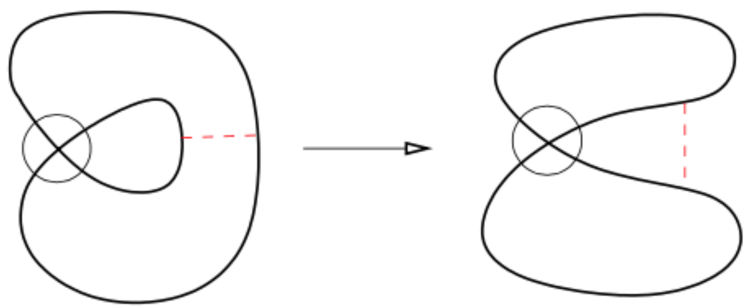
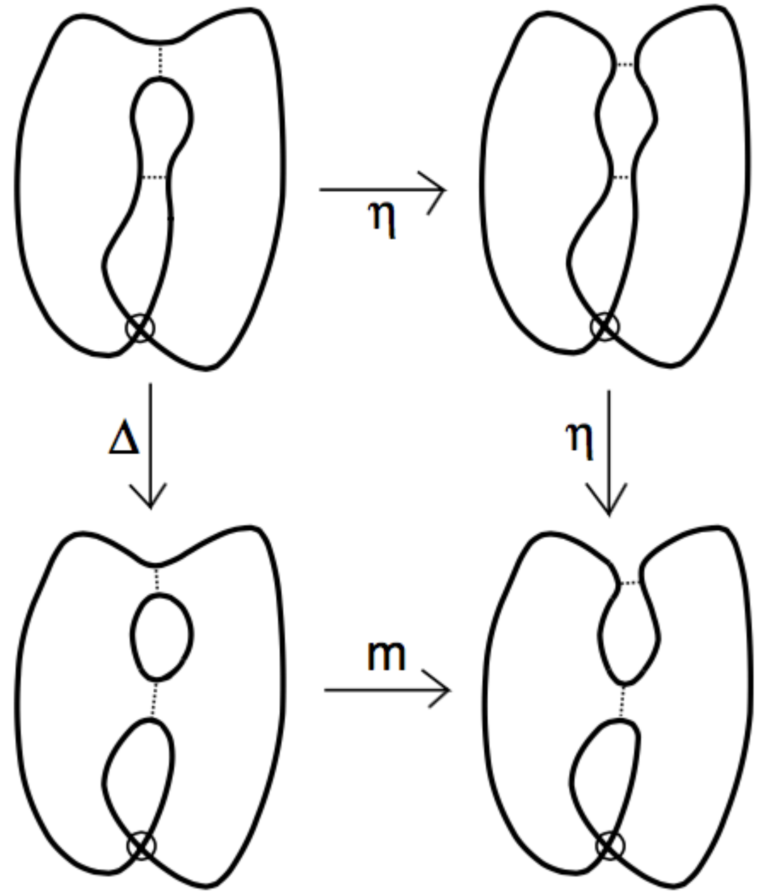


Figure 8: Single cycle smoothing



Composing along the top and right we have

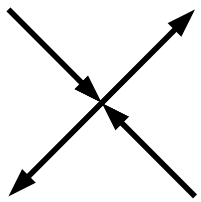
$$\eta \circ \eta = 0.$$

But composing along the opposite sides we see

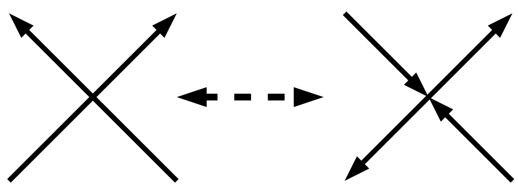
$$m \circ \Delta(1) = m(1 \otimes X + X \otimes 1) = X + X = 2X.$$

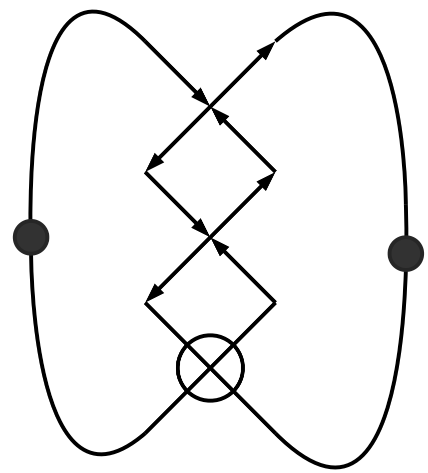
Mod 2 Khovanov Homology for Virtuals is OK.

Remarks on Generalizing Khovanov Homology, Lee Homology and Rasmussen Invariant



: Source-sink orientation

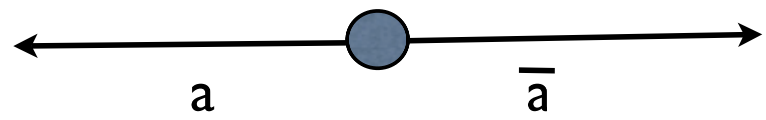




Cut loci for a two-crossing virtual knot

Canonical Source-Sink Orientation

Cut Locus Involution



The Frobenius algebra controlling the Khovanov homology differentials has
an order two function

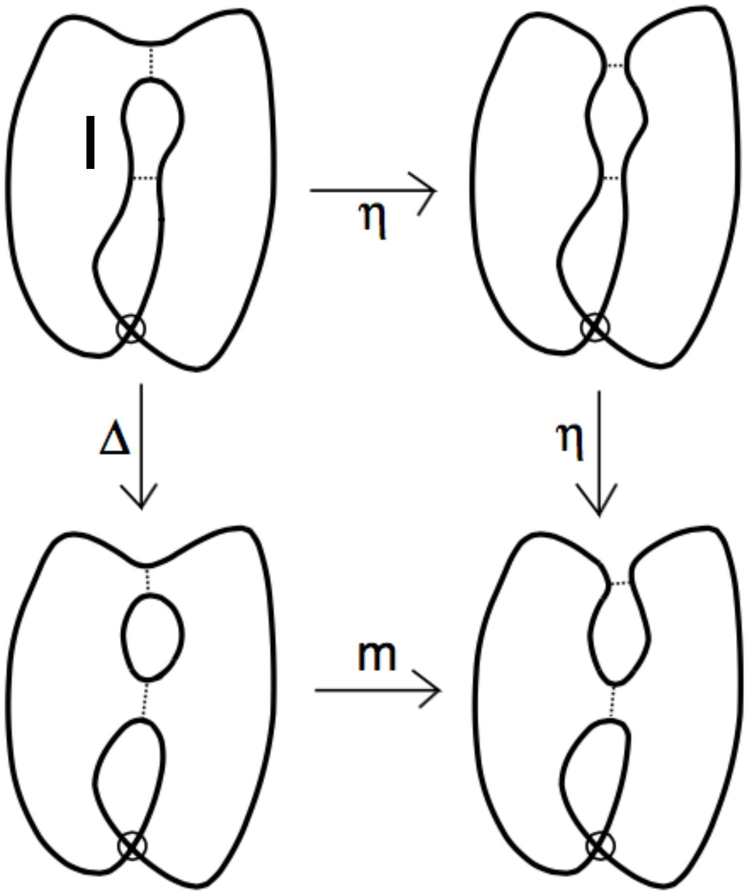
$$a \longrightarrow \bar{a}$$

that is applied whenever an algebra element is moved across a cut locus.

$$\bar{x} = -x$$

$$\bar{1} = 1$$

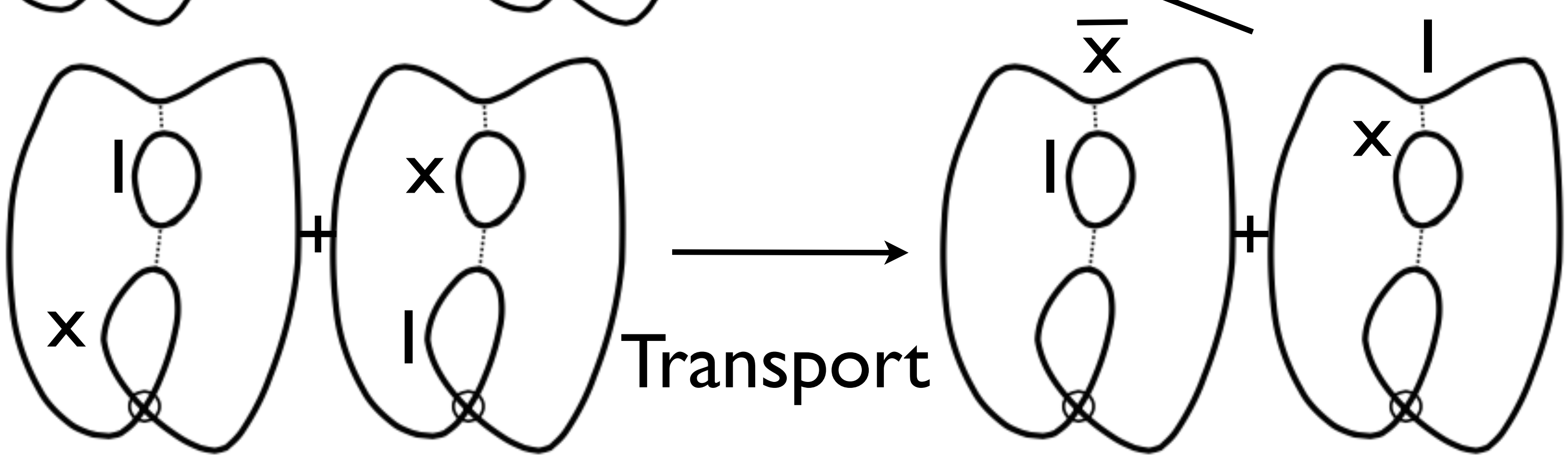
Transportation past cut loci bars the corrsponding elements and makes the square commute integrally.



$$m \circ \Delta(1) = m(1 \otimes \overline{X} + X \otimes \overline{1}) =$$

$\overline{x}$ + x = -x + x = 0.

multiply and add



Along with the bar operation and local coeffient transport there are other issues that demand extra care. We leave these for the reader to find in our paper.

Khovanov Homology, Lee Homology and a Rasmussen Invariant for Virtual Knots
Heather A. Dye, Aaron Kaestner, Louis H. Kauffman **arXiv:1409.5088**

Lee Algebra

$rr = r$

$gg = g$

$rg = gr = 0$

$\Delta(r) = 2r$

$\Delta(g) = 2g$

$r + g = 1$

$\overline{r} = g$

$\overline{g} = r$

$\overline{1} = 1$

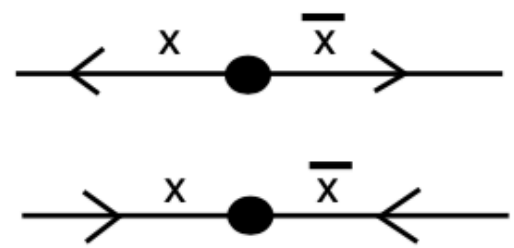
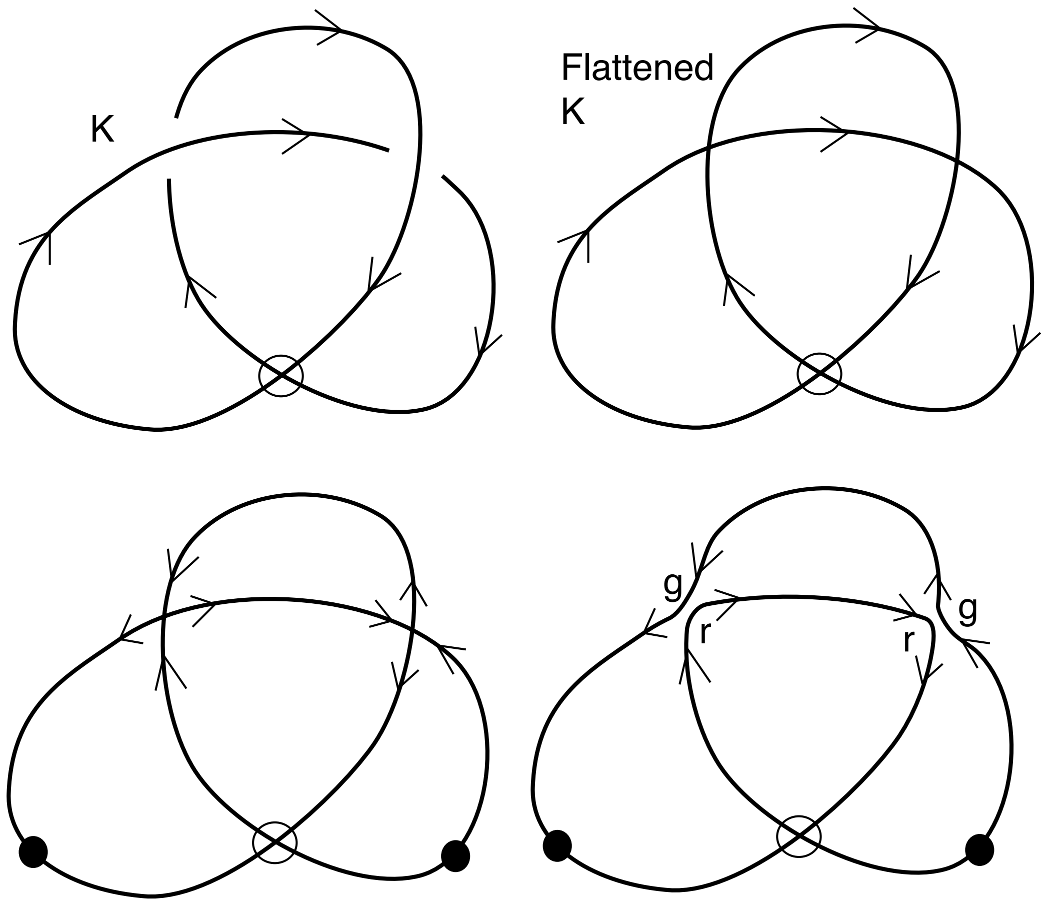


Figure 22: **Lee Algebra Undergoes Involution at a Cut Locus**

Barring Operations for the Lee Algebra



K with canonical source sink orientations and cut loci

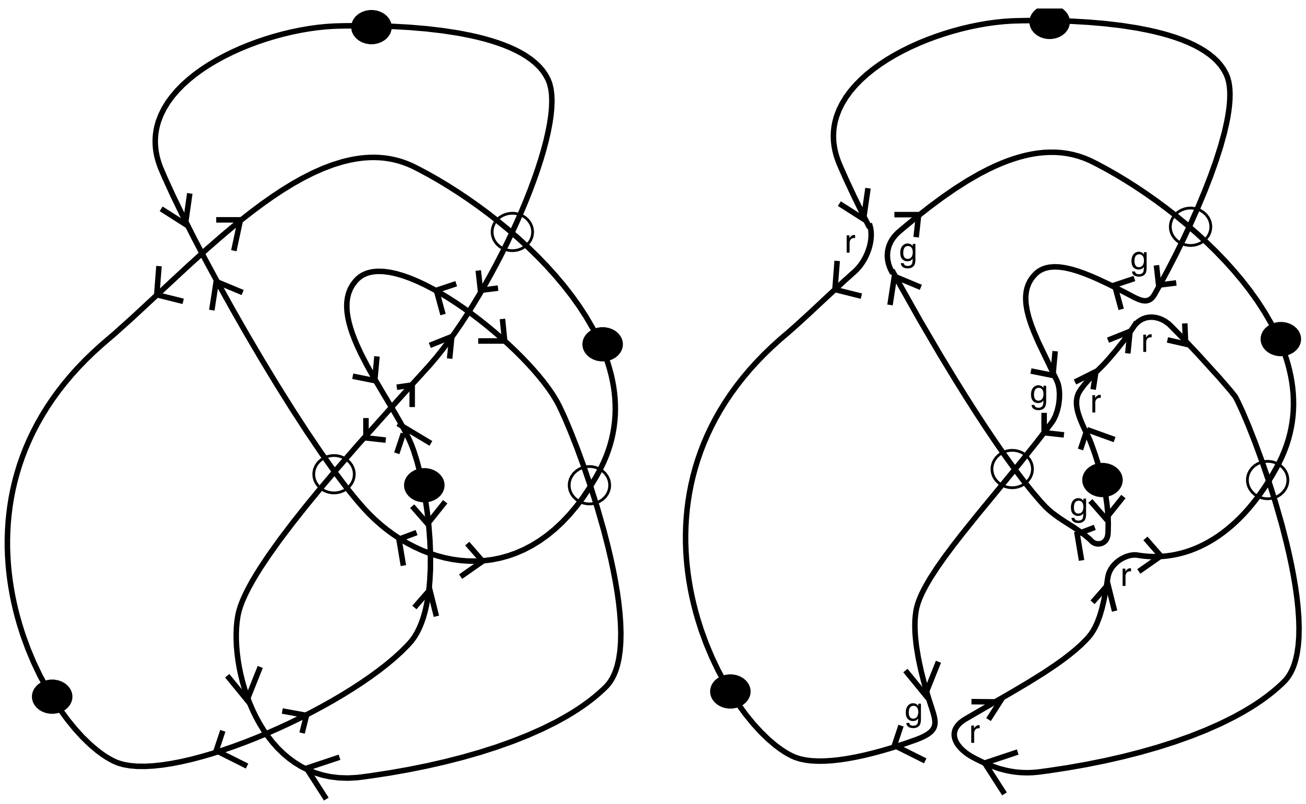
Seifert state labelled with Lee algebra is a non-trivial cycle.

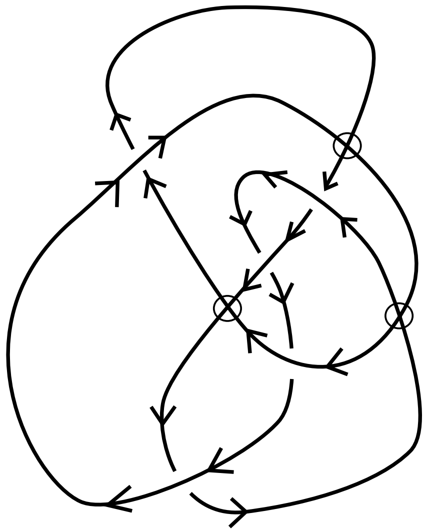
Lee Algebra

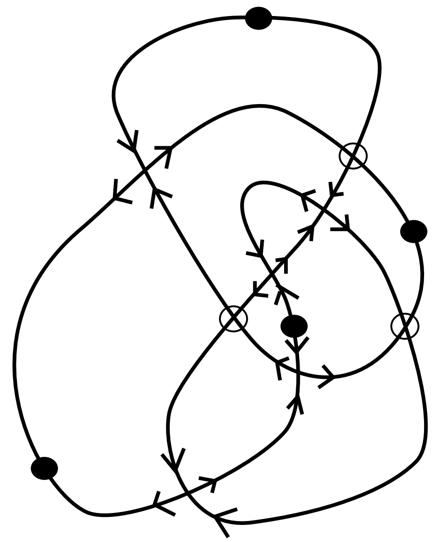
$$rg=gr = 0$$
$$rr = r$$
$$gg= g$$
$$r+g= 1$$
$$D(r) = 2r$$
$$D(g) = 2g$$

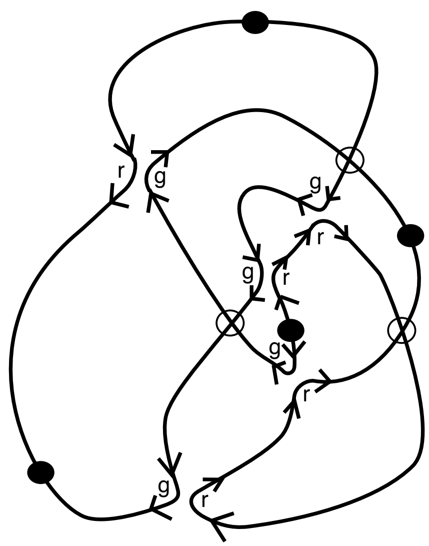
$$\overline{r} = g$$
$$\overline{g} = r$$

Another Example of a Virtual Lee Cycle









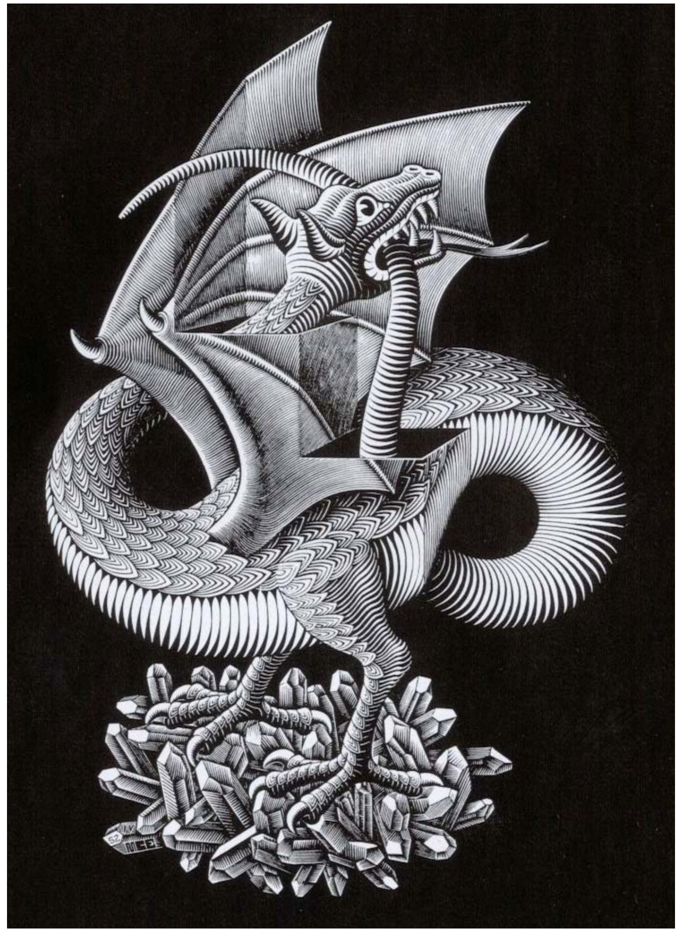
$$
\begin{aligned}
\text{genus} &= (1/2)(-r + n + 1) \\
&= (1/2)(-2 + 5 + 1) \\
&= 2.
\end{aligned}
$$

There are many questions about virtual knot cobordism and its relations with Khovanov homology and with variations on Khovanov homology. (See particularly the papers of William Rushworth.)

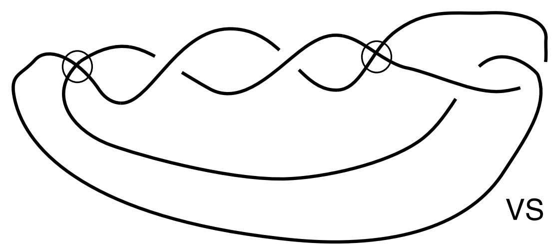
We are presently working on cobordism and KhoHomology of

knotoids, where a knotoid is a diagram with endpoints, not necessarily in the same region. Knotoids are taken up to Reidemeister moves that do not involve the endpoints.

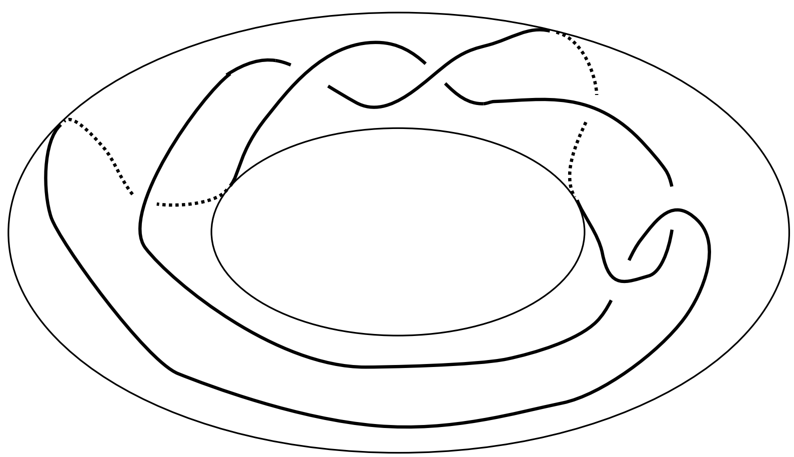
Thank you for your attention!



More about the Virtual Stevedore's Knot

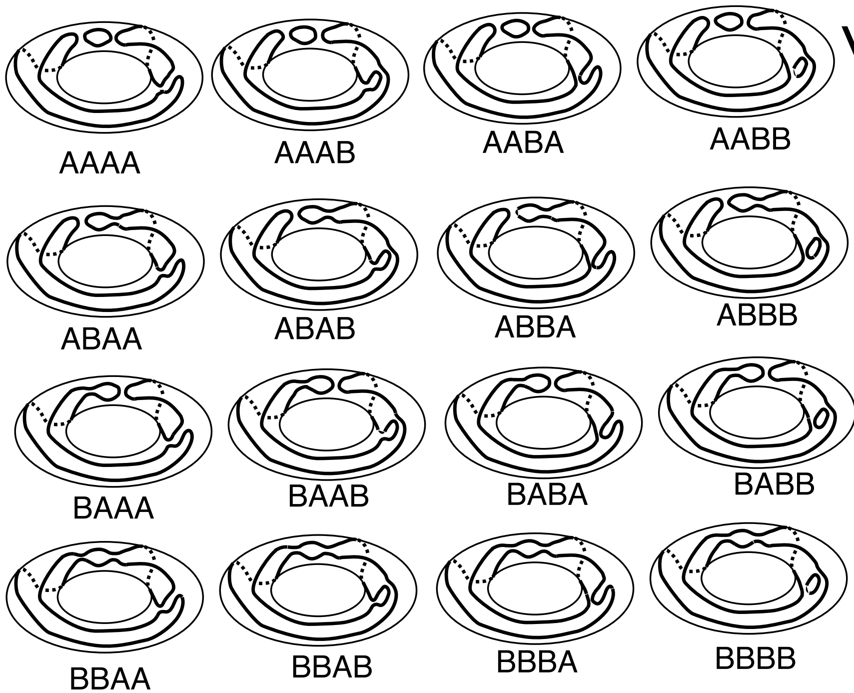


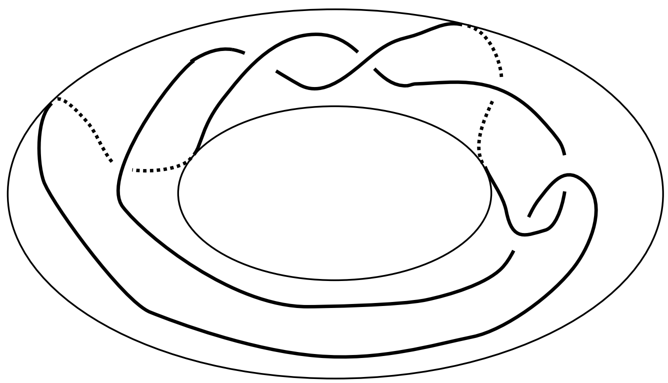
VS

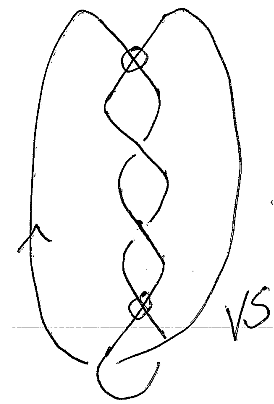


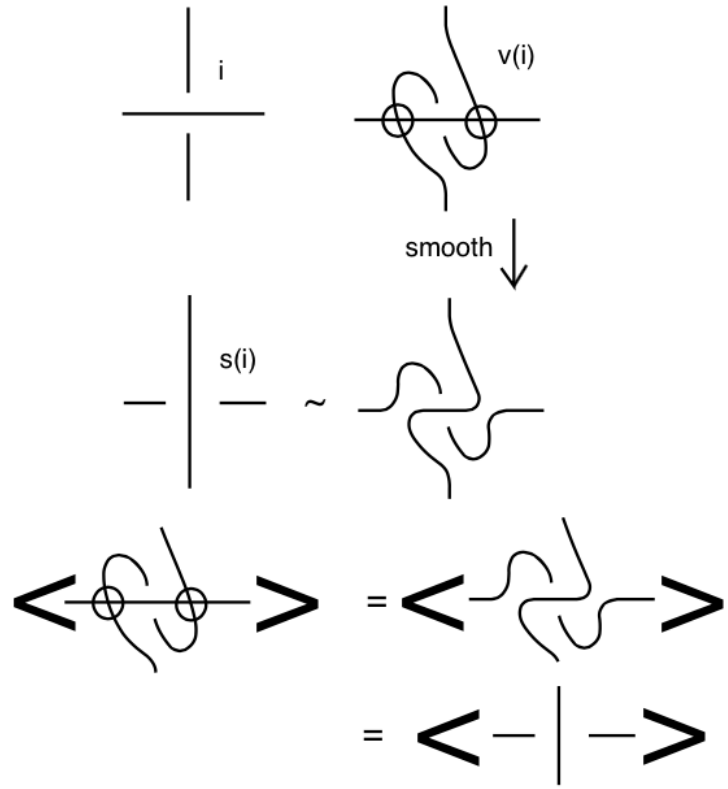
VS on a torus.

Virtual Stevedore
is not
classical.



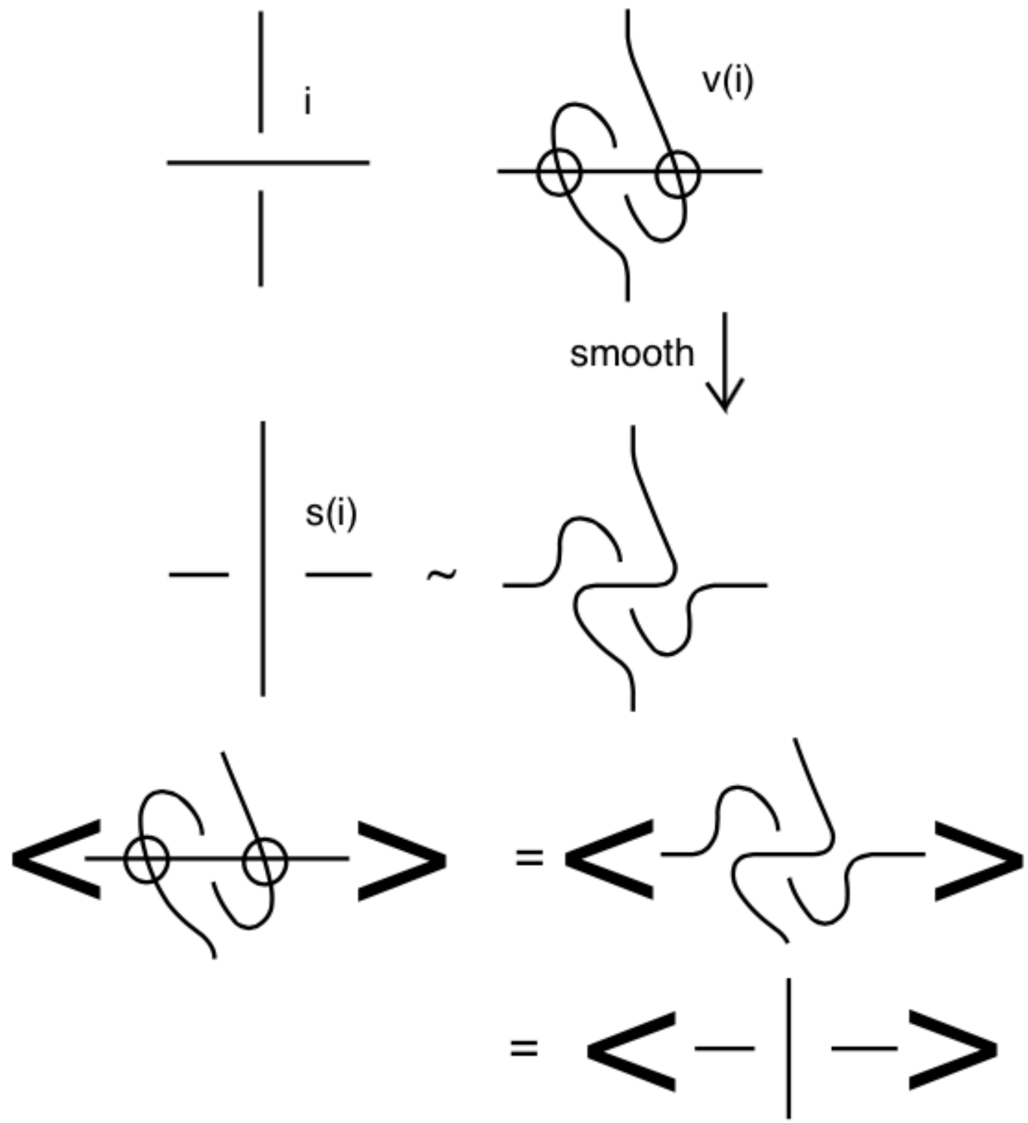




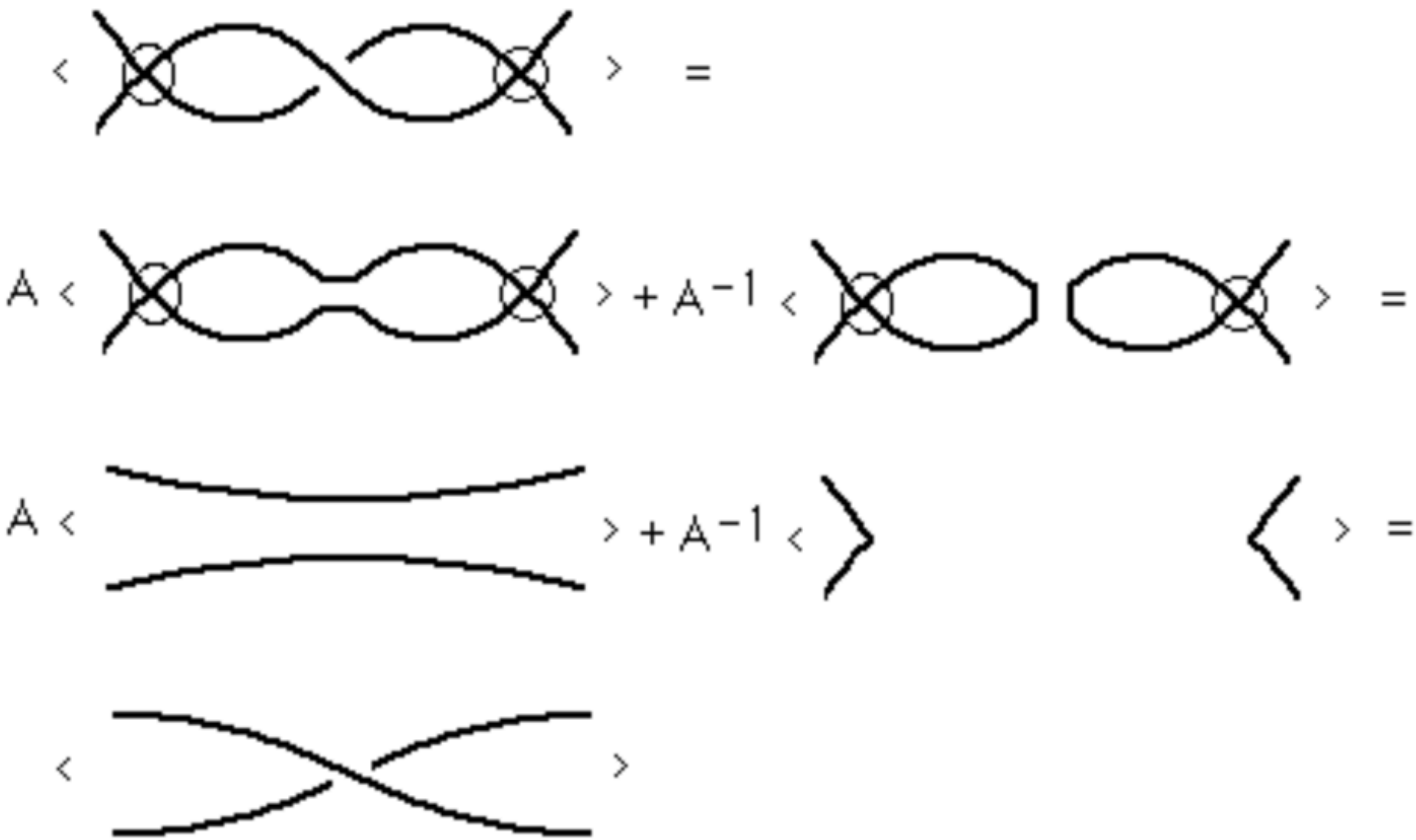


2.5 Non-classicality of Virtual Knots with Unit Jones Polynomial

There is a construction that produces infinitely many non-trivial virtual knot diagrams that have unit Jones polynomial. We shall prove here, that all of these examples of knots with unit Jones polynomial are non-classical. This settles a question raised in [19] and [21]. To clarify our proof we first restate a theorem of Manturov [33] regarding the invariance of Khovanov homology for virtual knots with arbitrary coefficients under Z-equivalence.


i
v(i)
smooth
s(i)
~
=
=

Bracket Polynomial is Unchanged when smoothing flanking virtuals.



Virtualization does not change the IQ(K).

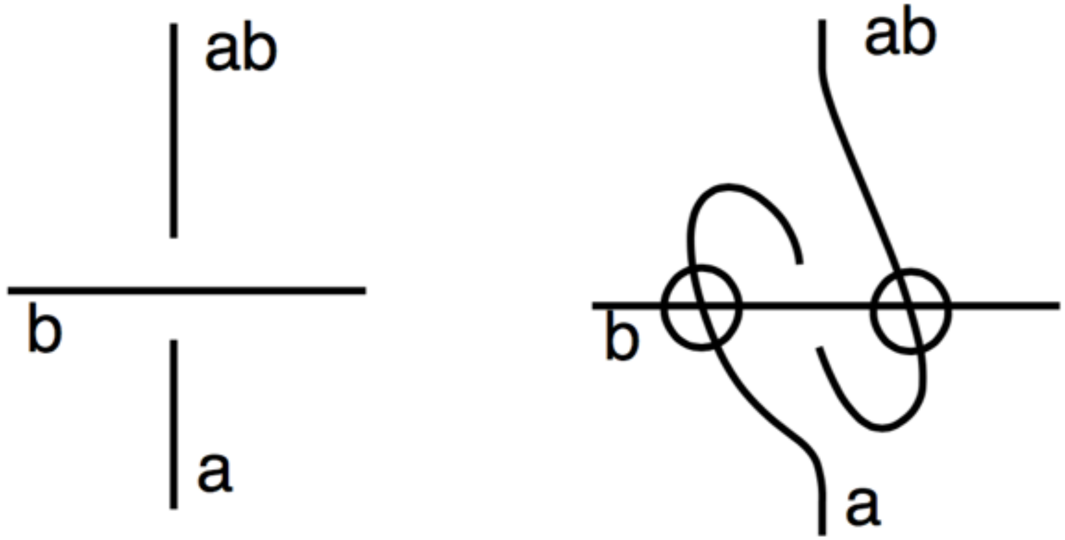
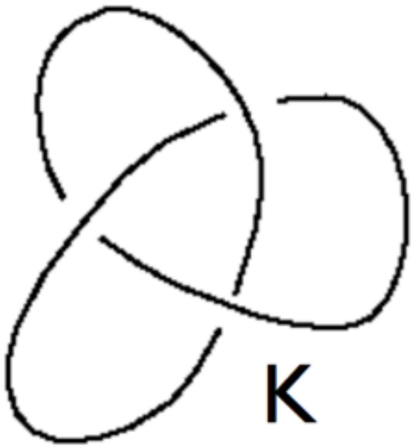


Figure 8. IQ(Virt)

The composition ab can denote a group theoretic operation
For example, let ab = b.a^(-1).b where a.b is group multiplication. The resulting group presentation is, for classical knots, the fundamental group of the two-fold branched covering along the knot.



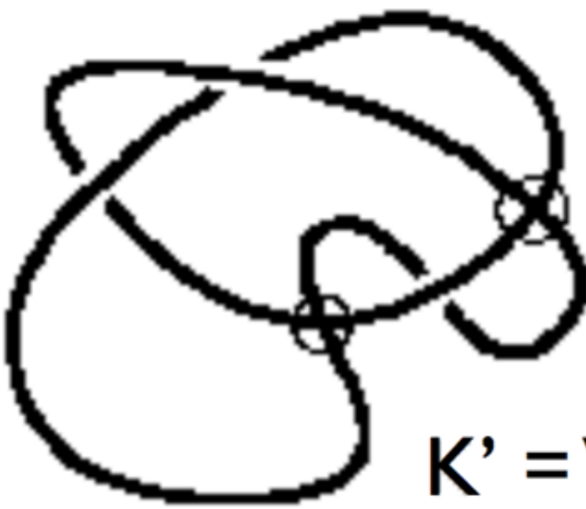
K

<Virt(K)> = <Switch(K)>

and

IQ(Virt(K)) = IQ(K).

Conclusion: There exist infinitely many non-trivial Virt(K) with unit Jones polynomial.



K’ = Virt(K)

U

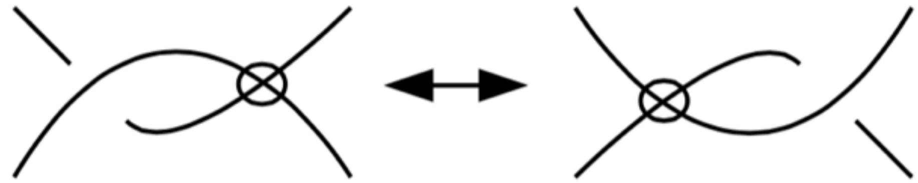


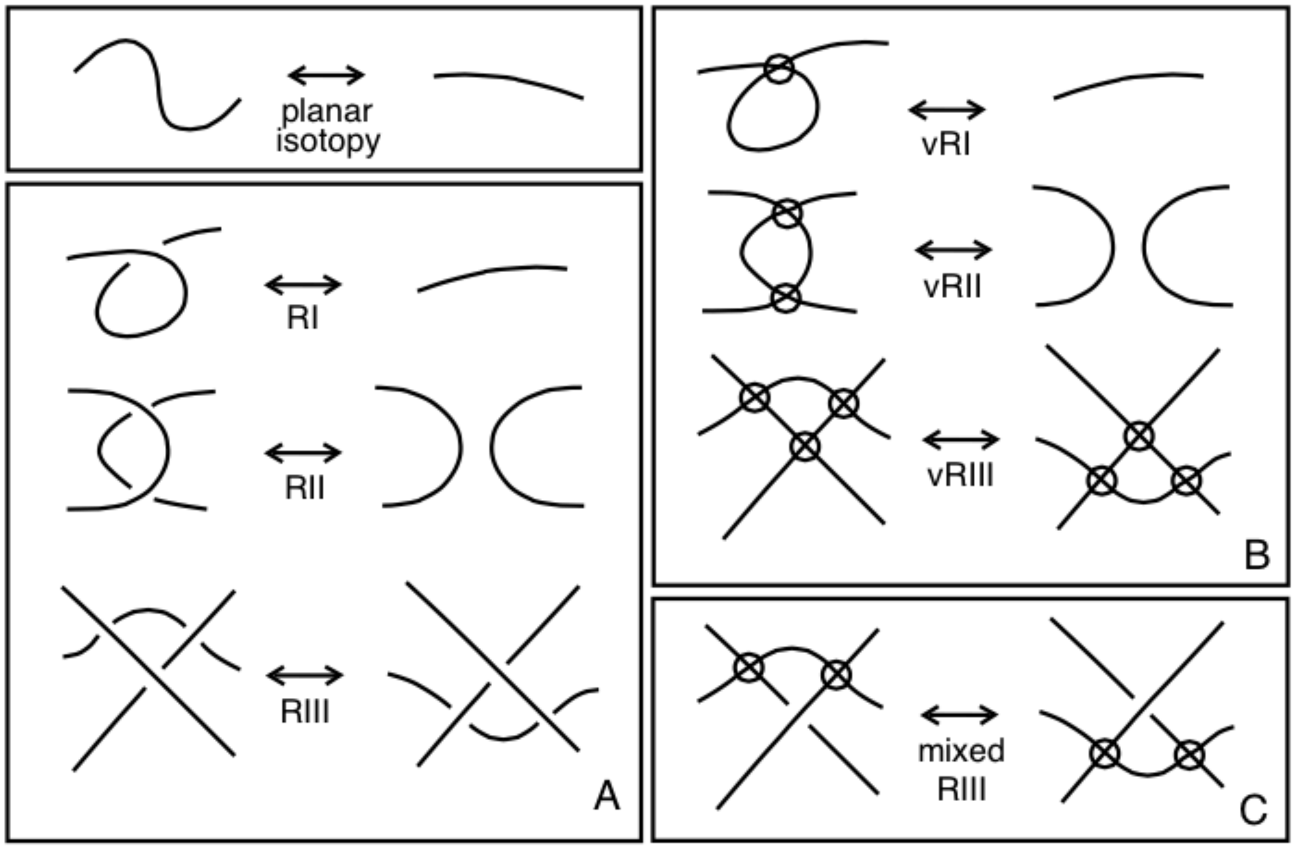
Figure 24: Z-equivalence

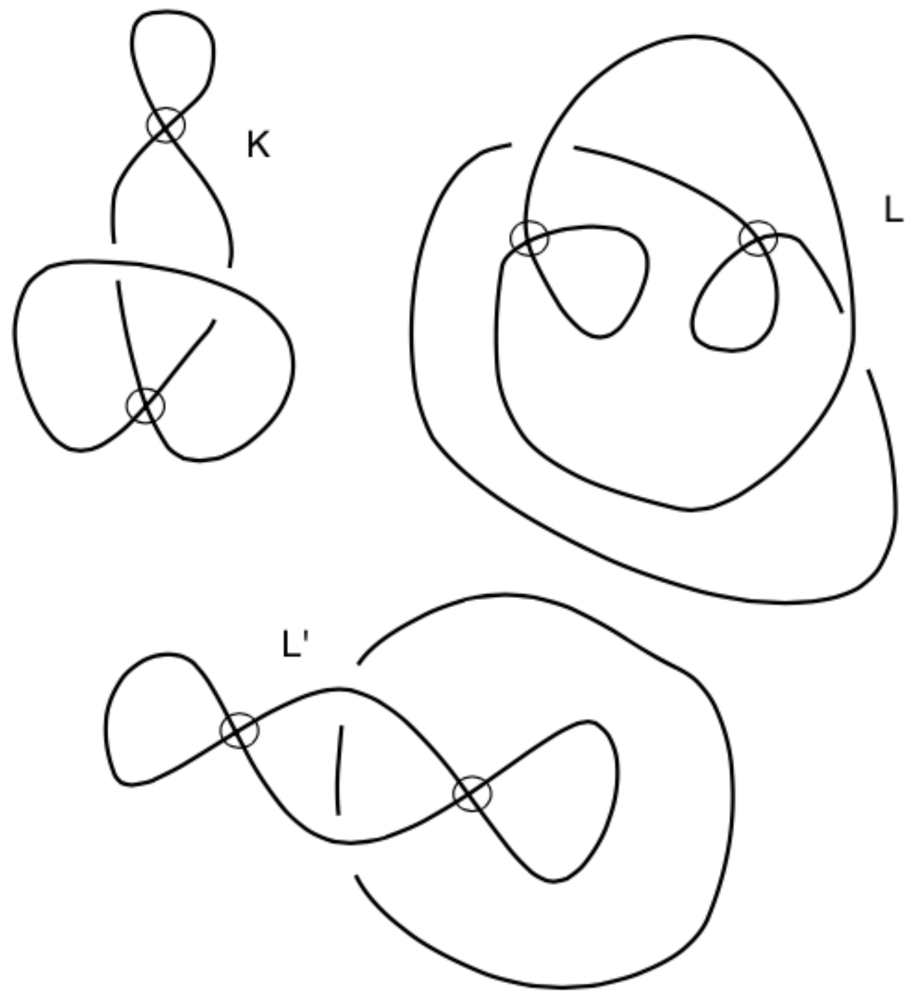
Gauss diagram, *a crossing is virtualized by reversing its sign and leaving its arrow unchanged.* The reader should note this very specific convention for the term virtualization. One checks that the Jones polynomial (via the bracket model [18]) of a knot with a virtualized crossing is the same as the ones polynomial of that same knot with the crossing *switched* (a switch interchanges over and under-crossing lines at the site of the crossing.). Thus, given a classical knot diagram K, one can choose a subset S of the crossings so that switching all of them gives an unknot diagram $U = S(K)$ where $S(K)$ denotes the diagram that results from the diagram K by switching all the crossings in the subset S. Instead of switching, we virtualize all the crossings in S to form a virtual knot diagram $Virt(K)$. It then follows that $Virt(K)$ has unit Jones polynomial, and is a non-trivial knot due to the fact that its un-oriented Gauss code has not been changed (See [19] for the proof of non-triviality).

Theorem 2.4. *If K is a non-trivial classical knot diagram and $Virt(K)$ is the virtual diagram described above with unit Jones polynomial, then $Virt(K)$ is a non-trivial and non-classical virtual diagram.*

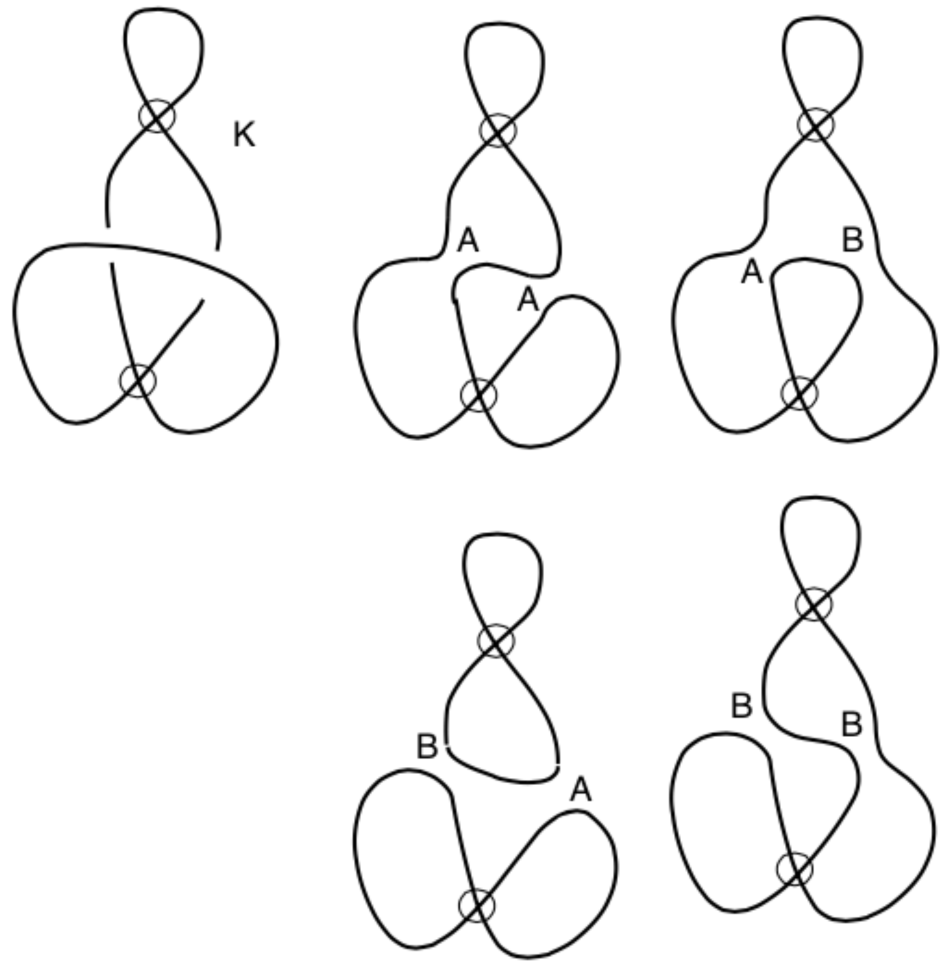
Rotational Virtual Knots

Disallow VR1.



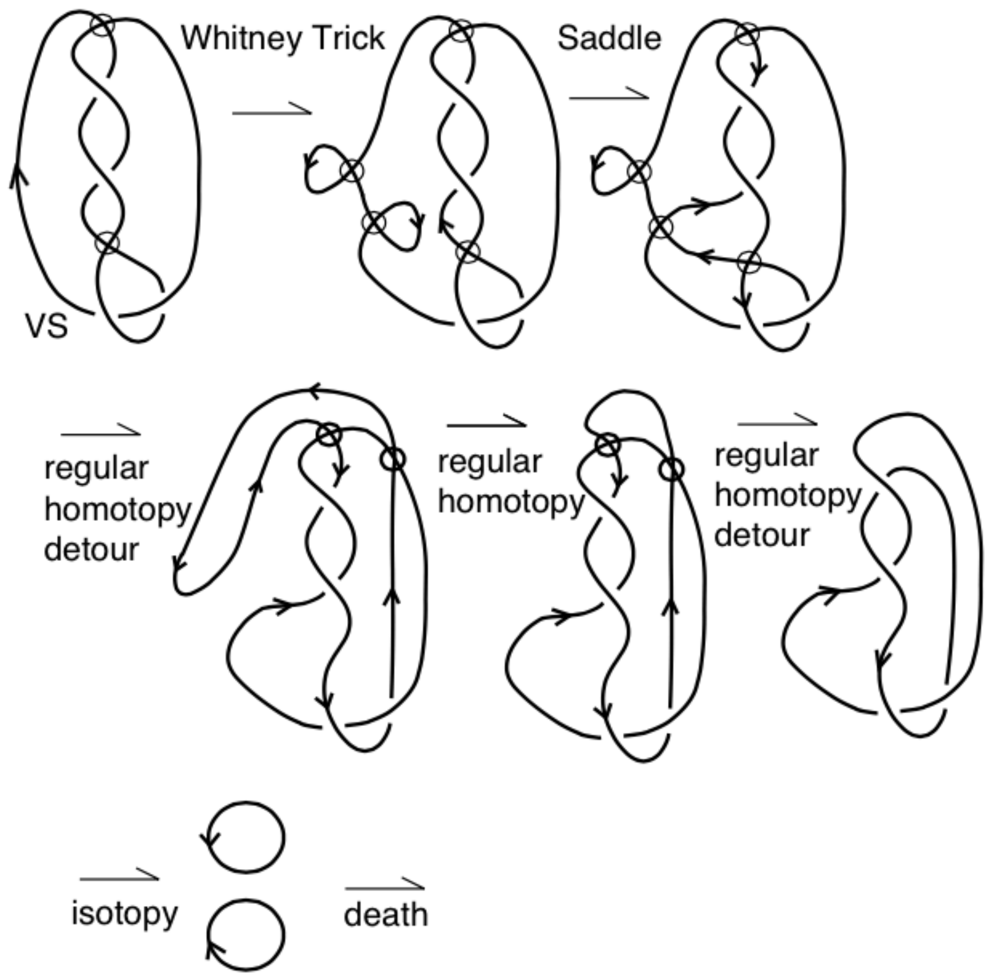


A rotational virtual knot and two rotational virtual links.



[K] = (AA + BB + AB)d + AB []

Bracket Expansion of a Rotational Virtual Knot



The Virtual Stevedore is Rotationally Slice

This talk has been a walk through Khovanov Homology and finding the relevant Frobenius algebras from the categorical chain complex and its associated 4Tu relation. We find the 4Tu relation naturally in looking for relations on surface cobordisms that render the theory invariant under isotopy of links.

There are many questions!

Other directions involve the question of an appropriate homotopy context for Khovanov Homology.

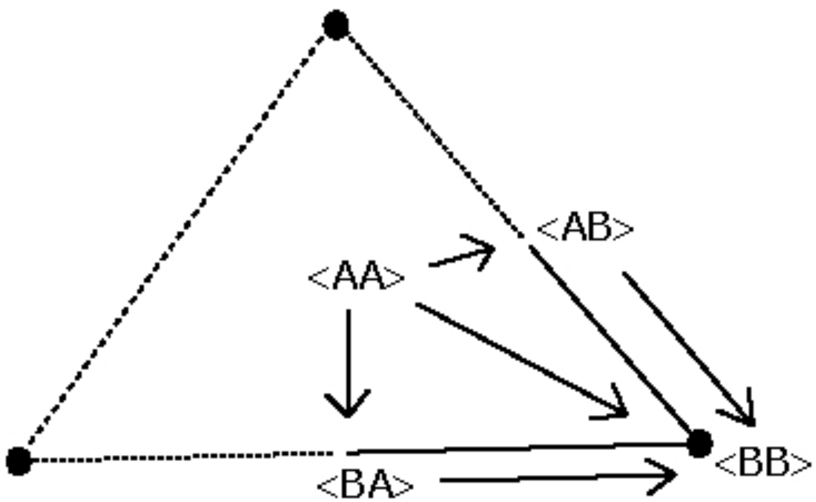
Work of Turner, Everitt, Lipshitz an Sarkar shows how to produce spectra whose homotopy is Khovanov Homology and spaces whose homology is Khovanov homology.

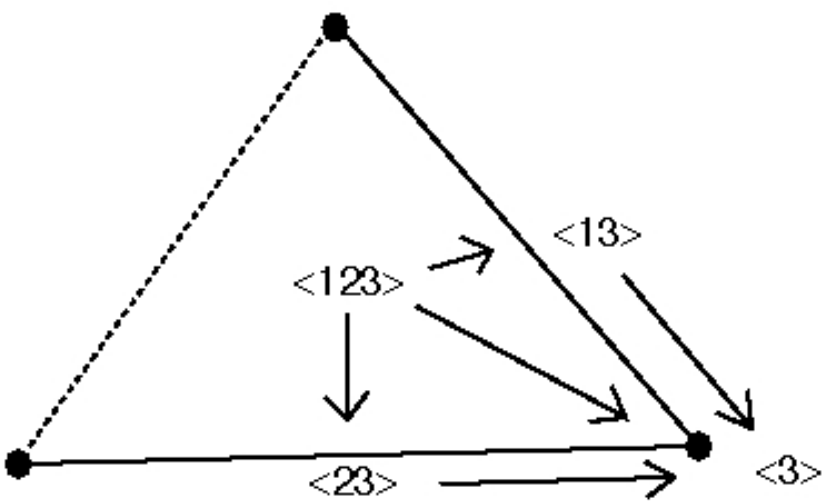
Simplicial Stucture of Khovanov Homology

Work of Chris Gomes and LK shows how to use Dold-Kan construction to make homotpy spectra for link homologies.

Dold-Kan Functor: Chain Compexes -----> Homotopy Simplical Objects

A cube category is the same as a simplex category with one full face removed. The nerve of the cube category is the same as a sub-barycentric subdivision of the remainder part of the deleted simplex.





Moral:
We can work simplicially by using the cube category with a trivial 0-th partial boundary.

One can then make spaces whose homology is Khovanov homology by forming the geometric realization of the corresponding simplicial objects.

By adding degeneracies to the simplicial objects
(as in ,<012> ----> <0112>)
one can do homotopy theory with them.
In particular one can realize spaces whose weak homotopy
type corresponds to the chain homotopy type
of the Khovanov complex.

A related insight (due to Jozef Przytycki) is that Khovanov homology from the cube category can be understood to be the cohomology of the nerve of that category with coefficients via the Frobenius algebra functor.

For a discrete category $\mathbb{C}$, it is well-known how to compute the cohomology of the topos $\mathcal{B}\mathbb{C}$ of presheaves on $\mathbb{C}$ (cf. Chapter I, Section 2). Let A be an abelian group in $\mathcal{B}\mathbb{C}$ (an object of $Ab(\mathcal{B}\mathbb{C})$, in the notation of Section I.4). Using the nerve of $\mathbb{C}$, one can define a cochain complex $C^{\cdot}(\mathbb{C}, A)$, with

$$C^n(\mathbb{C}, A) = \prod_{c_0 \leftarrow \ldots \leftarrow c_n} A(c_n);$$

the coboundary $d : C^{n-1}(\mathbb{C}, A) \to C^n(\mathbb{C}, A)$ is described as

$$(da)_{c_0 \xleftarrow{f_1} \;\ldots\; \xleftarrow{f_n} c_n} = \sum_{i=0}^{n-1} (-1)^i a_{d_i(c_0 \leftarrow \ldots \leftarrow c_n)} + (-1)^n A(f_n) a_{d_n(c_0 \leftarrow \ldots \leftarrow c_n)},$$

where $d_i(c_0 \leftarrow \ldots \leftarrow c_n)$ denotes the familiar simplicial boundary:

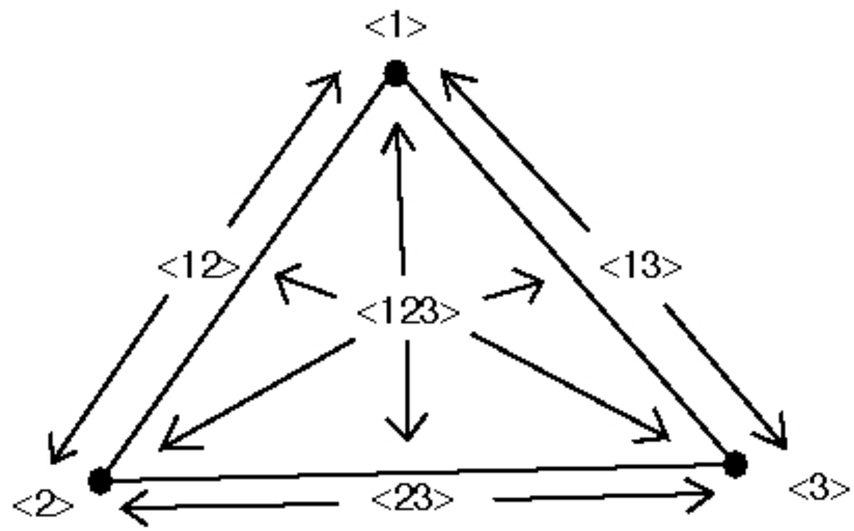
$$d_i(c_0 \xleftarrow{f_1} \ldots \xleftarrow{f_n} c_n) = \begin{cases} c_1 \leftarrow \;\ldots\; \leftarrow c_n & (i = 0) \\ c_0 \leftarrow \;\ldots\; c_{i-1} \xleftarrow{f_i \circ f_{i+1}} c_{i+1} \leftarrow \ldots \leftarrow c_n & (0 < i < n) \\ c_0 \leftarrow \;\ldots\; \leftarrow c_{n-1} & (i = n)\,. \end{cases}$$

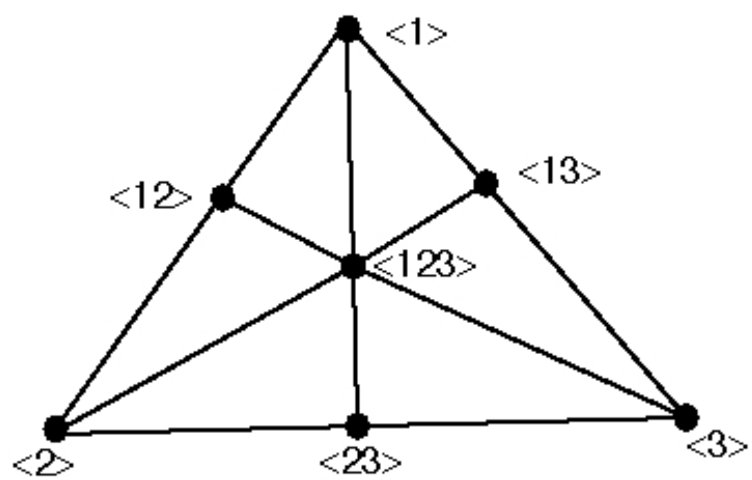
The cohomology of this complex is (usually called) *the cohomology of the category* $\mathbb{C}$ with coefficients in A, and is denoted

$$H^{\cdot}(\mathbb{C}, A).$$

It is the same as the cohomology of the topos $\mathcal{B}\mathbb{C}$:

The morphisms in a simplex category (category based on the face morphisms of a single simplex) have the shape of the barycentric subdivision of that simplex. This implies that simplex homology for a module category that is a functorial image of a simplex category is the same as the homology of the nerve of the functorial image category.





We have been exploring the cohomology of categories.