KNOTS, GRAPHS, SURFACES IN DIMENSIONS 3 AND 4

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1. Basic results on knots, graphs and surfaces in dimensions 3 and 4

We start out with several definitions and fairly elementary statements. Basically all of the following have appeared in one form or another in earlier papers or at least drafts for papers by the author.

Let Σ^3 be an integral homology sphere and let $L \subset \Sigma^3$ be an oriented m-component link. We pick an open tubular neighborhood νL of L and we refer to $X_L = \Sigma^3 \setminus \nu L$ as *the exterior* of L. Let us quickly recall the calculation of the homology groups of $\Sigma^3 \setminus L$. Given any $i \in \mathbb{N}$ we have

$$H_i(\Sigma^3 \setminus L) \cong H_i(\underbrace{\Sigma^3 \setminus \nu L}_{=X_L}) \cong H^{3-i}(X_L, \partial X_L) \cong H^{3-i}(\Sigma^3, \overline{\nu L}) \cong \widetilde{H}^{2-i}(\overline{\nu L}) = \begin{cases} \mathbb{Z}^m, & \text{if } i = 1, \\ \mathbb{Z}^{m-1}, & \text{if } i = 0. \end{cases}$$

We write $H = H_1(X_L; \mathbb{Z})$, which in this case we can identify with $\mathbb{Z}^m$. We consider the Alexander module $H_1(X_L; \mathbb{Z}[H])$. We make the following observations:

(1) Each $H_i(X_L; \mathbb{Z}[H])$ is finitely presented. We deduce this fact from the observation that X_L is a compact manifold and $\mathbb{Z}[H]$ is Noetherian.
(2) We can calculate the $\mathbb{Z}[H]$-homology groups of X_L using a 2-dimensional chain complex. This follows from the observation that X_L is homotopy equivalent to a 2-dimensional CW-complex X_L^{CW} (this can be seen say by pushing in 3-simplices of a triangulation, starting from the boundary), and that the singular homology groups of X_L agree with the cellular homology groups of X_L^{CW}. For example it follows that
$$H_2(X_L; \mathbb{Z}[H]) \;=\; H_2^{\mathrm{CW}}(X_L^{\mathrm{CW}}; \mathbb{Z}[H]) \;\subset\; C_2^{\mathrm{CW}}(X_L^{\mathrm{CW}}; \mathbb{Z}[H])$$
is torsion-free.
(3) We define $\Delta_L \in \mathbb{Z}[H]$ to be the order of $H_1(X_L; \mathbb{Z}[H])$.

At times it is interesting to generalize questions from links to spatial graphs. Recall that a *spatial graph* is an embedded graph Γ in S^3. We pick a regular neighborhood $\nu\Gamma$. With the same argument as above we can calculate the homology groups of $S^3 \setminus \Gamma$. For example for the spatial graph in Figure 1 we have $H_1(S^3 \setminus \Gamma) \cong \mathbb{Z}^2$ and $H_2(S^3 \setminus \Gamma) = 0$. As above we write $H = H_1(S^3 \setminus \nu\Gamma; \mathbb{Z})$. We can study the Alexander module $H_1(S^3 \setminus \nu\Gamma; \mathbb{Z}[H])$ and the same observations as above apply.

In the remainder of the introduction we want to study knots in S^3. Here we write $\mathbb{Z}[H] = \mathbb{Z}[t^{\pm 1}]$. We say that a knot $K \subset S^3$ is *topologically slice* if K bounds a locally flat

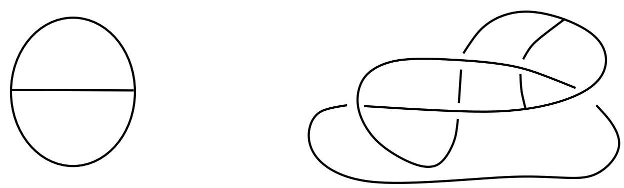

FIGURE 1.

disk $D \subset B^4$. A locally flat disk admits an open tubular neighborhood νD and we refer to $E_D := B^4 \setminus \nu D$ as the *exterior of* D. As above we can make the following observations:

(1) Each $H_i(E_D; \mathbb{Z}[t^{\pm 1}])$ is finitely presented.
(2) We can calculate the $\mathbb{Z}[H]$-homology groups of X_L using a 3-dimensional chain complex. In particular $H_3(E_D; \mathbb{Z}[t^{\pm 1}])$ is torsion-free.

The following proposition gives the most basic obstruction to a knot being topologically slice.

Proposition 1.1. *If K is topologically slice, then $\Delta_K = f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$.*

As an example, for the trefoil we have $\Delta_K = 1 - t + t^2$ we do not have such a factorization which implies that the trefoil is not topologically slice.

Proof. We consider the long exact sequence of the $\mathbb{Z}[t^{\pm 1}]$-homology of the pair (E_D, X_K):

$$0 \to H_2(E_D; \mathbb{Z}[t^{\pm 1}]) \to H_2(E_D; \mathbb{Z}[t^{\pm 1}]) \to H_1(X_K; \mathbb{Z}[t^{\pm 1}]) \to H_1(E_D; \mathbb{Z}[t^{\pm 1}]) \to H_1(E_D; \mathbb{Z}[t^{\pm 1}]) \to 0.$$

Using Poincaré duality one relates the groups to the left and the right and one quickly sees that $\Delta_K = f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$. □

More interesting sliceness obstructions are given by the linking form and the Blanchfield form. These can be viewed as odd-dimensional versions of the more familiar linking forms. We recall the definition of the intersection form of a 4-dimensional manifold W:

$$\begin{array}{rcl} I_W \colon H_2(W; \mathbb{Z}) \times H_2(W; \mathbb{Z}) & \to & \mathbb{Z} \\ (a, b) & \mapsto & \langle \mathrm{PD}_W(a) \cup \mathrm{PD}_W(b), [W] \rangle \end{array}$$

It follows from the (anti-) commutativity of the cup-product that the intersection form is symmetric. This form can be calculated on the homology level by counting algebraic intersection numbers of surfaces that represent the homology classes, or more generally by counting algebraic intersection numbers of singular chains that represent the given homology classes. If we are given an epimorphism $\pi_1(W) \to \mathbb{Z} = \langle t \rangle$ then using the equivariant intersection number of cycles we can also define a $\mathbb{Z}[t^{\pm 1}]$-valued intersection form

$$H_2(W; \mathbb{Z}[t^{\pm 1}]) \times H_2(W; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{Z}[t^{\pm 1}].$$

Now that we have recalled the definition of intersection forms we can give the definition of linking forms. We denote by Y_K the 2-fold branched cover of K. In this case we know that $H_1(Y_K;\mathbb{Z})$ is torsion and we can define the linking form which usually is defined as follows:

$$\begin{array}{rcl}\lambda_K\colon H_1(Y_K;\mathbb{Z})\times H_1(Y_K;\mathbb{Z}) & \to & \mathbb{Q}/\mathbb{Z}\\ ([a],[b]) & \mapsto & \frac{1}{n}a\cdot B\end{array}$$

where a and b are 1-cycles that do not meet, n is chosen such that $n\cdot b$ is null-homologous, i.e. there exists a singular 2-chain B and we denote by $a\cdot B$ the intersection number of the singular chains. The linking form is again symmetric.

Proposition 1.2. *If K is topologically slice, then λ_K is metabolic, i.e. there exists a subgroup $P\subset H_1(Y_K;\mathbb{Z})$ with $P=P^\perp$, where*

$$P^T=\{b\in H_1(Y_K;\mathbb{Z})\,|\,\lambda_K(a,b)=0 \text{ for all } a\in H_1(Y_K;\mathbb{Z})\}$$

denotes the orthogonal complement of P.

Proof. We denote by Z_D the 2-fold cover of B^4, branched along D. By considering the pair (Z_D,Y_K) and again using Poincaré duality one can show that $P:=\ker\{H_1(Y_K;\mathbb{Z})\to H_1(Z_D;\mathbb{Z})\}$ is a metabolizer, i.e. it has the property that $P=P^\perp$. □

The same way that we generalized the $\mathbb{Z}$-valued intersection form to a $\mathbb{Z}[t^{\pm1}]$-valued intersection form we can also generalize the $\mathbb{Q}/\mathbb{Z}$-valued linking form to a $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]$-valued linking form:

$$\begin{array}{rcl}\lambda_K\colon H_1(Y_K;\mathbb{Z}[t^{\pm1}])\times H_1(Y_K;\mathbb{Z}[t^{\pm1}]) & \to & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}]\\ ([a],[b]) & \mapsto & \frac{1}{\Delta_K}a\cdot B\end{array}$$

where $a,b\in C_1(Y_K;\mathbb{Z}[t^{\pm1}])=C_1(\widetilde{Y}_K)$ are chains, $B\in C_(Y_K;\mathbb{Z}[t^{\pm1}])=C_2(\widetilde{Y}_K)$ is a chain with $\partial B=\Delta_K\cdot a$ and $a\cdot B$ denotes the equivariant $\mathbb{Z}[t^{\pm1}]$-valued intersection form of singular chains.

2. Questions and some answers

2.1. **Tubular and regular neighborhoods.** A first question is, what do we mean by a tubular and regular neighborhood? If we go back to the above discussion we see that in the calculation of the homology groups of $\Sigma^3\setminus L$ we used the following three properties:

(a) we used that the closure $\overline{\nu L}$ is a codimension 0 zero submanifold,
(b) we used that $\overline{\nu L}$ is homotopy equivalent to L,
(c) we used that $\Sigma^3\setminus\nu L$ is homotopy equivalent to $\Sigma^3\setminus L$.

We also talked about *the* tubular neighborhood of $L\subset\Sigma^3$, *the* tubular neighborhood of $K\subset\Sigma^3$, $D\subset B^4$ and of $\Gamma\subset S^3$ and correspondingly we talked about *the* exterior of $K\subset S^3$, $D\subset B^4$ and of $\Gamma\subset S^3$. Evidently a tubular neighborhood is not entirely unique, so the question is, what type of uniqueness did we really use? When we talk about *the* exterior then we clearly expect that the exterior is well-defined up to homeomorphism. So we really want the following fourth property:

(d) Given any two tubular/regular neighborhoods νJ and μJ of some $J \subset M$ we need that there exists a homeomorphism $\Phi\colon M \to M$ with $\Phi(\nu J) = \mu J$. (Ideally we want the homeomorphism to be isotopic to the identity.)

How are we doing on this account?

(1) First of all, unfortunately there does not seem to be a universally agreed upon concept of a tubular neighborhood of a submanifold. Nonetheless it seems to me that this can be fixed and that there exists a general notion of a tubular neighborhood of a submanifold that satisfies the obvious generalizations of (a), (b) and (c) and that also satisfies (d). For example in [Fr18, Chapter 57.2] we define a (closed) tubular neighborhood of $L^k \subset M^{k+n}$ as an embedded linear $\overline{B}^n$-bundle over L. Another definition which works well is the notion of an "extendable normal vector bundle" that is introduced in [FQ90, p. 137]. We also refer to [Kos93, Section III.2] or alternatively [Bre93, Theorem II.11.14], [Le02, Theorem 6.24] and [BJ82, Theorem 12.11] for more information on tubular neighborhoods.

(2) For spatial graphs the first question that arises is, what do we mean by an "embedded" graph? Unfortunately often this is not made clear. One possible definition is given in [Fr18, Chapter 1.8]. I could not find a reference that shows that given a spatial graph there exists a notion of a regular neighborhood with properties (a), (b), (c) and (d).

In the case of a PL-manifold M and a "compact polyhedron" X in M a "regular neighborhood" $N(X)$ is introduced in [RS72, p. 33]. By [RS72, Theorem 3.8, Proposition 3.10 and Corollary 3.30] this notion of a regular neighborhood has properties (a), (b) and (d). I am not sure whether (b) is satisfied. Also the notion of a "regular neighborhood" used in [RS72] seems to me slightly artificial for a topologist.

(3) It seems like [FQ90, p. 137] can be used to define a notion of a tubular neighborhood of a topological submanifold of a topological 4-dimensional manifold that satisfies (a), (b), (c) and (d).

2.2. **The topology of compact topological 4-manifolds.** In many text books, e.g. [Bre93, E.5.Corollary] or [Hat02, p. 527] it is shown that the every compact topological manifold is a retract of a finite CW-complex. From this fact it follows immediately that the usual singular homology groups are finitely generated. But this result is not good enough for dealing with twisted coefficients.

Kirby and Siebenmann [KS77, Essay III.2] showed that every *closed* topological manifold of dimension ≥ 6 has a CW-structure. In 1982 Quinn [Qu82] extended this result to the case of 5-dimensional manifolds. We point out that Kirby–Siebenmann often get misquoted, often it is claim that they show that every compact topological manifold of dimension ≥ 6 admits a CW-structure.

What is true is that every compact topological manifold is homotopy equivalent to a finite CW-complex. This follows from the work of Hanner [Han51, Theorem 3.3] who showed in 1950 that every topological manifold is an "absolute neighborhood retract". West [Wes77] (see also [KS77]) showed in 1977 that every compact "absolute neighborhood retract" is homotopy equivalent to a compact CW-complex.

This discussion shows why in the first section the modules $H_*(E_D;\mathbb{Z}[t^{\pm 1}])$ are finitely generated. But the references we have collected so far are not enough to deduce that $H_3(E_D;\mathbb{Z}[t^{\pm 1}])$ is torsion-free.

Using [Wal66, Corollary 5.1] and the computations of homology groups of topological manifolds one obtains the following strengthening of the above results:

(1) Every closed n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional CW-complex.
(2) Every compact, connected n-dimensional topological manifold with non-empty boundary is homotopy equivalent to a finite $(n-1)$-dimensional CW-complex.

2.3. **Cup products and intersecting cycles.** The intersection form of a $2k$-dimensional manifold W is defined via the cup product:

$$\begin{array}{rcl} I_W\colon H_k(W;\mathbb{Z})\times H_k(W;\mathbb{Z}) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \langle \mathrm{PD}_W(a)\cup \mathrm{PD}_W(b),[W]\rangle \end{array}$$

At least for smooth manifolds the intersection form can be calculated using intersection numbers of embedded submanifolds, see e.g. [Bre93, Chapter VI.11] for a rigorous proof. I do not know a reference in the literature for the statement that the cup product can be calculated by "immersed submanifolds" let alone by "intersecting cycles". A proof for the later statement will appear in [Fr18b]. For illustration we give the proof of the following lemma which is a simplified version.

Lemma 2.1. *Let W be an oriented $2k$-dimensional topological manifold. If $a\in H_k(W;\mathbb{Z})$ and $b\in H_k(W;\mathbb{Z})$ can be represented by disjoint cycles, then $I_W(a,b)=0$.*

Proof. By our hypothesis we can find disjoint compact subsets A and B of W such that a lies in the image of $H_k(A)\to H_2(W)$ and b lies in the image of $H_k(B)\to H_k(W)$. Since W is metrizable we can in fact find disjoint open subsets U and V of W that contain A and B. In particular a lies in the image of $H_k(U)\to H_k(W)$ and b lies in the image of $H_k(V)\to H_k(W)$.

We denote by $i\colon U\to W$ the inclusion. We consider $\mathrm{PD}_U(i_*(a))\in H^k_c(U)$. By definition of cohomology with compact support there exists a compact subset K such that $\mathrm{PD}_U(i_*(a))$ lies in the image of the map $H^k(U,U\setminus K)\to H^k_c(U)$. We consider the following commutative diagram:

$$\begin{array}{ccccc}
H^k_c(U) & & \xrightarrow[\cong]{\mathrm{PD}_U} & & H_k(U) \\
 & \nwarrow & & \nearrow{\scriptstyle \cap \mu_{U,K}} & \\
\downarrow{\scriptstyle i_*} & & H^k(U, U \setminus K) & & \downarrow{\scriptstyle i_*} \\
 & & \downarrow{\scriptstyle (i^*)^{-1}} & & \\
H^k(W) & & \xrightarrow[\cong]{\mathrm{PD}_W} & & H_k(W) \\
 & \nwarrow & & \nearrow{\scriptstyle \cap [W]} & \\
 & & H^k(W, W \setminus K). & &
\end{array}$$

We have thus shown that $\mathrm{PD}_W(a)$ lies in the image of $H^k(W, W \setminus K) \to H^k(W)$ for some compact subset $K \subset U$. The same argument shows that $\mathrm{PD}_W(a)$ lies in the image of $H^k(W, W \setminus L) \to H^k(W)$ for some compact subset $L \subset V$. The statement follows from the following commutative diagram

$$\begin{array}{ccc}
H^k(W) \times H^k(W) & \xrightarrow{\cup} & H^n(W) \\
\uparrow & & \uparrow \\
H^k(W, W \setminus K) \times H^k(W, W \setminus L) & \xrightarrow{\cup} & H^n(W, \underbrace{(W \setminus K) \cup (W \setminus L)}_{=W}).
\end{array}$$

and the observation that the group in the lower right corner is zero. □

Things are even worse for the $\mathbb{Z}[t^{\pm 1}]$-valued intersection form

$$H_k(W; \mathbb{Z}[t^{\pm 1}]) \times H_k(W; \mathbb{Z}[t^{\pm 1}]) \ \to \ \mathbb{Z}[t^{\pm 1}].$$

It requires some thought to define this form using cup products. I do not know a reference which proves that such intersection forms can be calculated using embedded submanifolds let alone by intersecting cycles. Presumably with quite some effort one can generalize the above approaches in the untwisted case.

2.4. **Linking forms.** Let M be an $(2n+1)$-dimensional rational homology sphere with $n \geq 1$. In this case the Bockstein homomorphisms in homology and cohomology in dimension n are in fact isomorphisms. We denote by Ω the composition

$$\begin{array}{ccccccc}
H_n(M; \mathbb{Z}) & \xrightarrow{\mathrm{PD}^{\mathbb{Z}}_M} & H^{n+1}(M; \mathbb{Z}) & \xrightarrow{\beta^{-1}} & H^n(M; \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}_{\mathbb{Z}}(H_n(M; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \\
 & & & & \varphi & \mapsto & (\sigma \mapsto \langle \varphi, \sigma \rangle).
\end{array}$$

The *linking form* of M is the form

$$\lambda_M \colon H_n(M; \mathbb{Z}) \times H_n(M; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

defined by $\lambda_M(a, b) = \Omega(a)(b)$. From the definition it is not immediate that linking forms are symmetric. A not entirely rigorous argument was given by Seifert [Se35]. Modern rigorous proofs are given in [Po16, CFH17]. In the introduction I defined the linking form using "intersections of cycles". I do not know a reference for this statement. Using the calculation of the "intersection form" $H^1(N;\mathbb{Z}) \times H^2(N;\mathbb{Q}/\mathbb{Z}) \to H^3(N;\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$ in terms of intersections of cycles one should be able to prove the above formula. I hope to return to this question after [Fr18b].

The Blanchfield form can also be defined using Poincaré duality. With more effort one should be able to push the afore mentioned techniques to show that the two definitions are the same. Again I hope to return to this question later on.

It is natural to like to think of intersection forms as intersections of objects. After all, this is where the inspiration for the definition comes from. It takes a while to get used to working with cup products. But once one got its head around to working with cup products, they are not so hard to work with. For example in [CFH17] we used cup products to determine the linking form of all rational homology 3-spheres in terms of the Heegaard splitting.

To state the result we need some notation. Given an orientation-reversing self-diffeomorphism φ of the genus g surface F_g we write $M(\varphi) := X_g \cup_\varphi Z_g$ where we identify $x \in F_g = \partial X_g$ with $\varphi(x) \in \partial Z_g$. We give $M(\varphi)$ the orientation which turns both the inclusions $X_g \to M(\varphi)$ and $Z_g \to M(\varphi)$ into orientation-preserving embeddings. Furthermore we denote by

$$\begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix}$$

the matrix that represents $\varphi_*\colon H_1(F_g;\mathbb{Z}) \to H_1(F_g;\mathbb{Z})$ with respect to a symplectic basis $a_1,\dots,a_g,b_1,\dots,b_g$.

Theorem 2.2. *Let $g \in \mathbb{N}$ and let $\varphi\colon F_g \to F_g$ be an orientation-reversing diffeomorphism. If $M(\varphi)$ is a rational homology sphere, then $B_\varphi \in M(g \times g,\mathbb{Z})$ is invertible and the linking form of $M(\varphi)$ is isometric to the form*[1]

$$\begin{aligned} \mathbb{Z}^g/B_\varphi^T\mathbb{Z}^g \times \mathbb{Z}^g/B_\varphi^T\mathbb{Z}^g &\to \mathbb{Q}/\mathbb{Z} \\ (v,w) &\mapsto v^T B_\varphi^{-1} A_\varphi w. \end{aligned}$$

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[1]Despite our very best attempts the sign needs to be taken with a grain of salt.

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