Combinatorial Knot Theory
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A (general) **knot theory** is a groupoid with diagrams as objects and morphisms generated by a set of R**-moves**.

A **knot** is a component of the theory.

For example planar knot diagrams and Reidemeister moves define

Classical Knot Theory

But there is more.

As well as planar diagrams, there are arrow diagrams, simple polygonal circuits in $\mathbb{R}^3$, immersed hypersurfaces in $\mathbb{R}^n$, etc, etc.

Planar diagrams are immersed 1-manifolds in the plane, in general position, This covers the usual diagrams for classical knots as well as braids and knotoids.

We extend this with a "tag" at each crossing.

Positive and negative classical crossings

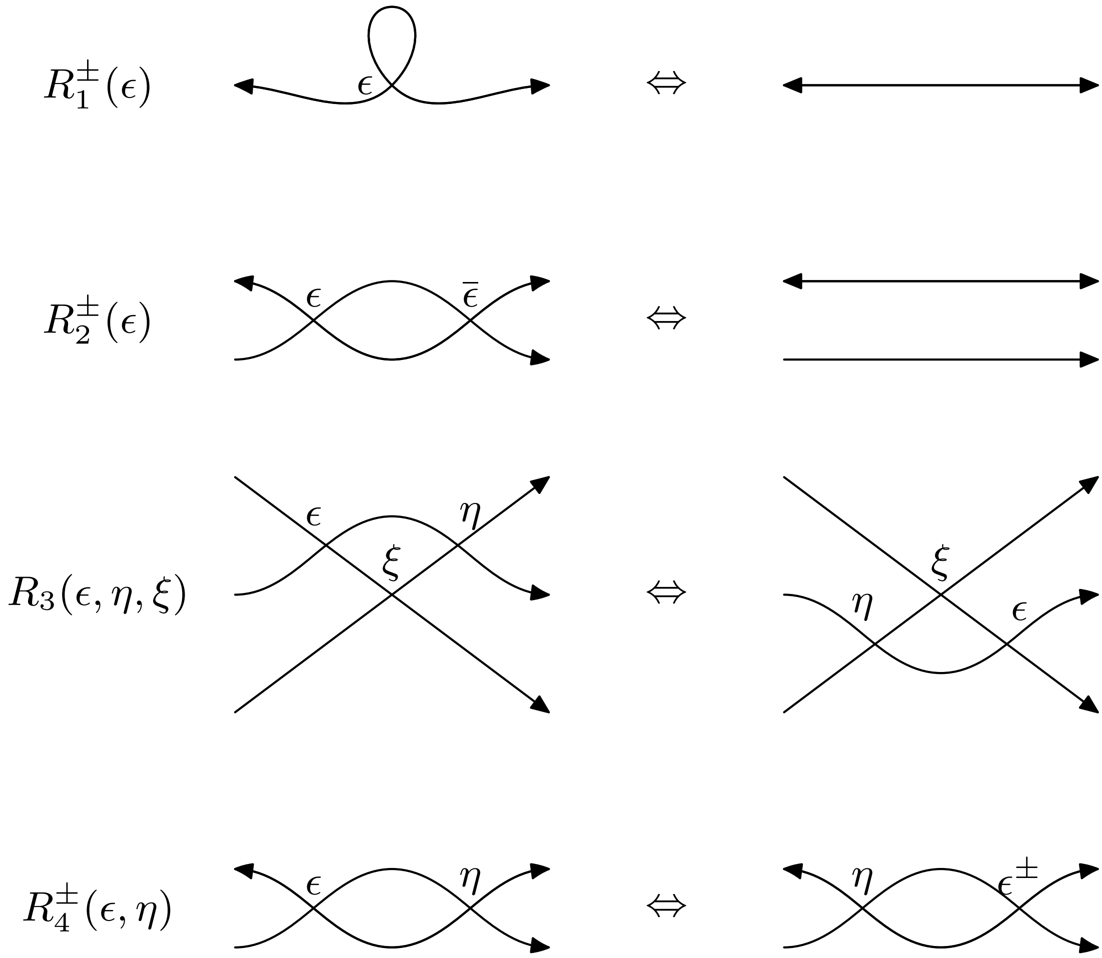
flat and virtual /weld crossings

singular crossings

We can now generalize to arbitary tags as follows.

general crossing tag

The general R-moves look like the following. They are the analogues of the Reidemeister moves except R_4 which is a newcomer.



general Reidemeister moves

$R_1, R_2,$ and R_4 are signed according to the ambiguity of the orientation. R_3 can always be oriented from left to right, if R_2 is allowed, by a trick due to Turaev and we shall always assume this.

If $R_3(\epsilon, \eta, \xi)$ we say that the pair (ϵ, η) **dominates** ξ and sometimes write $(\epsilon, \eta) \succ \xi$.

If $(\epsilon, \bar{\epsilon}) \succ \xi$ then we say that ϵ **dominates** ξ .

If this is true for all ξ then we say that ϵ **dominates** the theory.

The R-moves for generalized planar diagrams are a set of rules which may be allowed or forbidden.

For example, in classical knot theory, all the R_1 and R_2 moves are allowed and both c and $\bar{c}$ dominate.

In virtual knot theory, all the R_1 and R_2 moves are allowed and v dominates every thing. However neither c or $\bar{c}$ are allowed to dominate v .

In welded knot theory w has all the properties of v but in addition c dominates w but $\bar{c}$ is not allowed to dominate w .

The various tags c, v, w etc have known properties within themselves but have rules for interactions between two. So a knot theory can be represented as a sort of presentation.

For example classical knot theory has the "free" presentation

$$< c, \bar{c}| >$$

because we know the properties of c.

Virtual knot theory has the presentation

$$< c, v|v \succ c, \bar{c} >$$

Welded knot theory allows the "first forbidden move"

$$< c, \bar{c}, w|c \succ w >$$

The "second forbidden move" $\bar{c} \not\succ v$ is forbidden.

Doodles have a presentation

$$< f, v|R_1(f), R_2(f), v \succ f >$$

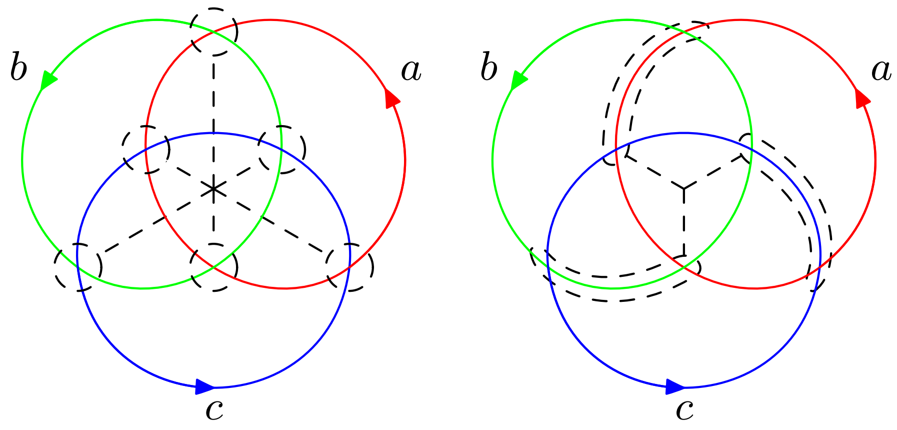
Free knots have the same generators as doodles but more relations

$$< f, v|R_1(f), R_2(f), v \succ f, f \succ f, R_4(f, v) >$$

etc

Doodles and commutator identities

Doodles give rise to commutator identities in the free group



The left hand side yields the defining identity

$$(a,c)^b(b,c)(b,a)^c(c,a)(c,b)^a(a,b) \equiv 1$$

where for example, $(a,c)^b = b^{-1}(a^{-1}c^{-1}ac)b$, and the right hand side yields

$$(bc,a)(ca,b)(ab,c) \equiv 1.$$

Another which can be extracted from the borromean doodle is the Hall-Witt identity,

$$((a,b),c^a)((c,a),b^c)((b,c),a^b) \equiv 1$$

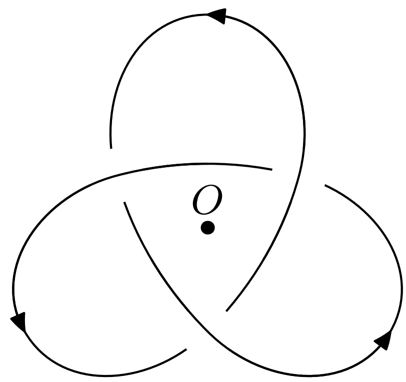
This is a group-theoretic analogue of the Jacobi identity for Lie algebras.

Braid like Planar Diagrams

Let $f : S^1 \longrightarrow \mathbb{R}^2$ be a component of a planar diagram. Let O be a point not on the diagram. Then the component is **braid like** from O if the map $\hat{f} : S^1 \longrightarrow S^1$ defined by

$$\hat{f}(x) = \frac{f(x) - O}{||f(x) - O||}$$

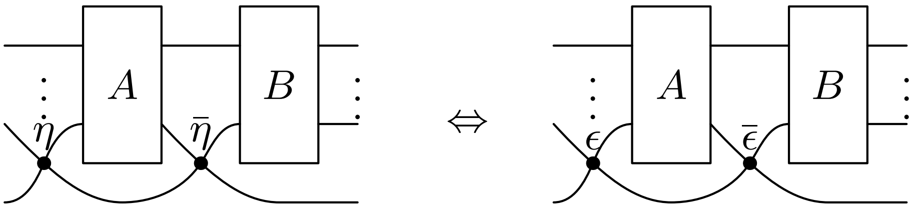
is a covering.



For example the Trefoil

Generalized Alexander Theorem (Bartholemew, F) Every knot in a theory which has R_2 has a braid like planar representative.

Generalized Markov Theorem (Bartholemew, F) Consider a knot theory which is dominated by the tag ϵ. Let D_1 and D_2 be braid like planar diagrams in the theory which represent the same knot. Then D_1 and D_2 are related by the following moves which preserve the braid like structure: R_2, R_3, Markov extension and the K-move.

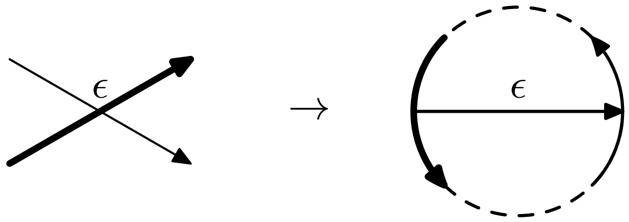


The K-move

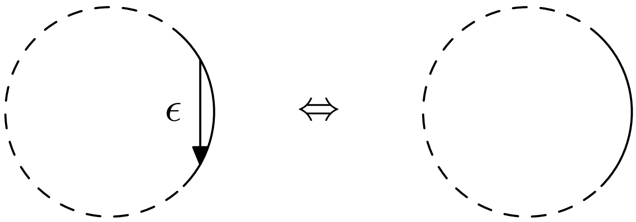
In the case of classical and welded knots, the K move is not necessary but for virtual knots it is.

Arrow Diagrams

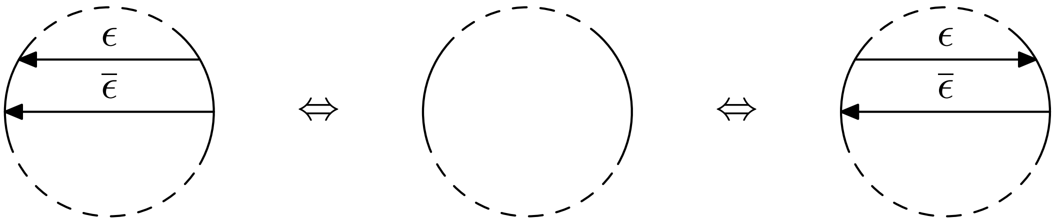
Where a component crosses itself we represent this by an arrow forming the chord of a circle.



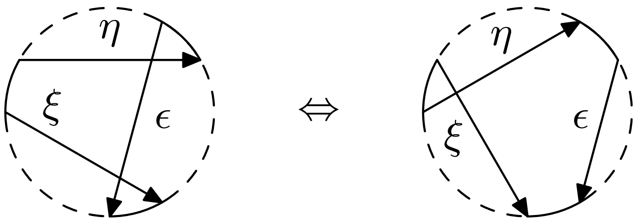
R moves for Arrow Diagrams



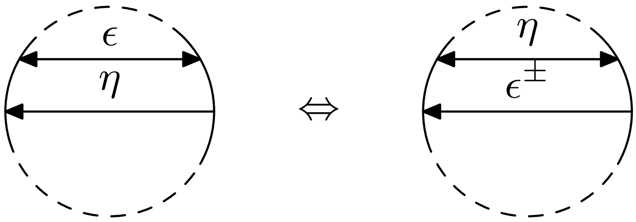
arrow diagrams under R1



arrow diagrams under R2



arrow diagrams under R3

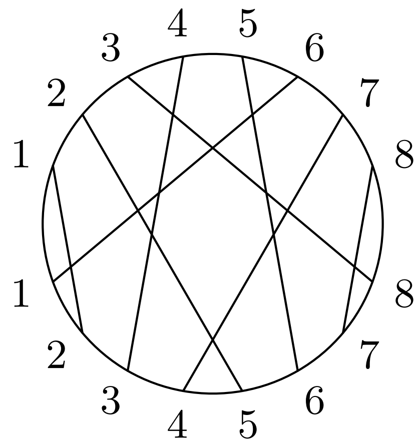


arrow diagrams under R4

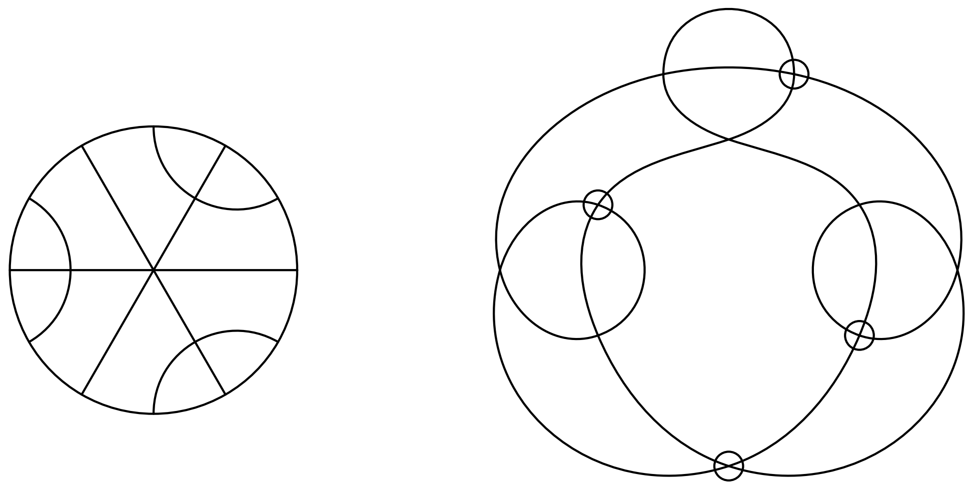
Free Knots

Free knots are represented by arrow diagrams without tags and without arrow heads.

They satisfy all the R-moves above and can sometimes be defined by a permutation.



A free knot defined by a permutation

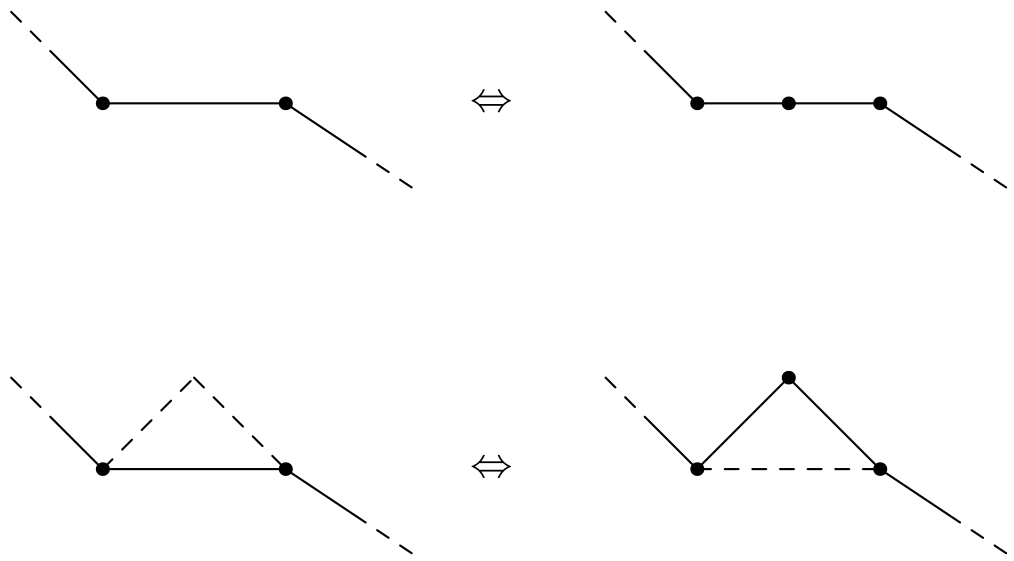


The 3-morning star. This is not defined by a permutation.

PL diagrams

The diagrams are simple polygonal circuits in $\mathbb{R}^3$ up to homeomorphisms which preserve the polygonal structure.

The R-moves are of two types.



Although the first is a consequence of the second.

A knot is **ordered** if each diagram has an order which is changed by an R move.

A knot is **uniradical** if every diagram has a descending path to a unique root (sink).

For example a classical knot, represented by a polygonal loop, could be ordered by the number of edges (vertices), the **stick number**.

However if we order a polygonal representative of the unknot by the minimum number of triangles in a triangulated spanning disk then we see that the unknot is uniradical.

There are other uniradical statements for doodles (Bartholemew, N. Kamada, S. Kamada, F), virtual knots (Kuperberg), singular knots (Jordan, Rourke, F), etc.

Theorem (Newman) An ordered knot is uniradical iff
1. There are no infinite descending paths
2. Every path of the form ascending/descending ($\nearrow\searrow$) can be repaced by one descending/ascending ($\searrow\nearrow$).

The R-complex of a knot

The groupoid of a knot, K, with diagrams as objects and morphisms generated by R-moves can be made into a simplicial set, RK, in a standard way.

The n-simplexes of RK are composible sequences of morphisms of length n

$$(D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} D_n) = (f_0, \ldots, f_{n-1})$$

The i-th face, d_i is

$$(f_0, \ldots, f_i \circ f_{i+1}, \ldots, f_{n-1})$$

if $0 < i < n-1$

$$(f_1, \ldots, f_{n-1}),\ (f_0, \ldots, f_{n-2})$$

for $i = 0, n-1$

The i-th degeneracy, s_i is

$$(f_0, \ldots, 1_{D_i}, f_i, f_{i+1}, \ldots, f_{n-1})$$

The k **-th horn** Λ_k^n for each $k \leq n$, is the subcomplex of the boundary of the standard n-simplex, Δ^n with the k-th face removed.

A simplicial set, S, is **Kan** if any functor $\Lambda_k^n \longrightarrow S$ extends to $\Delta^n \longrightarrow S$.

The R-complex of a knot, RK, is Kan. This means that the homotopy groups of RK can be defined.

For any knot, RK is a $K(\pi, 1)$.

Some Questions

Since RK is a $K(\pi, 1)$ what is π for say the trefoil, the figure eight . . . ?

Are the homology groups of RK interesting?

What's Khovanov homology got to do with RK ?

Is RK , for a uniradical knot K , contractible?