Milnor's Question on Transfinite Invariants

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Knotted embeddings in dimensions 3 and 4

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A quick review: Milnor's link invariants

1957 "Isotopy of links"
1954 "Link groups"

Lower central series of a group G:

$G_1 := G$, $G_{k+1} := [G, G_k]$ where $[H,K]$ = subgroup generated by $[a,b] := aba^{-1}b^{-1}$, $a \in H$, $b \in K$

Suppose $L \subset S^3$ is a link with m components, and $F = F\langle x_1, \cdots, x_m\rangle$ is a free group of rank m.

Theorem (Milnor '57)

A meridean homomorphism $F \to \pi := \pi_1(S^3 \smallsetminus L)$, $x_i \mapsto i^{\text{th}}$ meridean, induces

$\pi/\pi_k \cong F/\langle F_k, [w_1, x_1], \cdots, [w_m, x_m]\rangle$ where w_i corresponds to the i^{th} longitude $\lambda_i \in \pi$

$(\Rightarrow \pi/\pi_k \cong F/F_k$ iff $\lambda_i \in \pi_{k-1}$ $\forall i$.)

Suppose $\pi/\pi_k \cong F/F_k$, and consider:

$$\pi/\pi_k \xrightarrow{\cong} F/F_k \xrightarrow{\text{Magnus}} \mathbb{Z}\langle X_1, \cdots, X_m\rangle/\deg \geq k$$
$$x_i \longmapsto 1 + X_i$$
$$x_i^{-1} \longmapsto 1 - X_i + X_i^2 - \cdots$$

formal power series in non-commuting X_i

Milnor's invariant of length k is defined by

$\bar{\mu}_L(i_1 i_2 \cdots i_{k-1} j) :=$ coefficient of $X_{i_1} X_{i_2} \cdots X_{i_{k-1}}$ in the expansion of the j^{th} longitude λ_j.
$(1 \leq i_1, \cdots, i_{k-1}, j \leq m)$

Theorem (Milnor) Let L be a link with $\pi = \pi_1(S^3 \smallsetminus L)$

(i) When $\pi/\pi_k \cong F/F_k$, Milnor invariants of length k are well-defined, and

(ii) all Milnor invariants of length k vanish iff $\pi/\pi_{k+1} \cong F/F_{k+1}$.

That is, Milnor invariants inductively determine whether the link group has the same lower central quotients as the trivial link.

Theorem (Stallings) Milnor invariants are concordance invariants.

More generally, if two links L, L' have homology cobordant exteriors, then $\bar{\mu}_L(i_1 \cdots i_{k-1} j)$ is well-defined iff so is $\bar{\mu}_{L'}(i_1 \cdots i_{k-1} j)$, and whenever they are defined, they are equal.

3-manifolds M and M' with $\partial M \cong \partial M'$ are homology cobordant ($\mathbb{Z}H$-cob.) if $\exists$ 4-manifold W s.t. $\partial W = M \cup_\partial -N$, $H_*(M;\mathbb{Z}) \cong H_*(W;\mathbb{Z}) \cong H_*(M';\mathbb{Z})$

Stallings' Theorem If $\pi \to G$ induces $\cong$ on H_1 and surjection on H_2, then it induces $\pi/\pi_k \cong G/G_k$ for $k < \infty$.

Consequence: $f : X \to Y$ induces $H_*(X) \cong H_*(Y)$

$\Rightarrow f_* : \pi_1 X \to \pi_1 Y$ induces $\cong$ on H_1, $\twoheadrightarrow$ on H_2

$\Rightarrow \pi_1 X/(\pi_1 X)_k \cong \pi_1 Y/(\pi_1 Y)_k$.

A question of Milnor (1957)

Find a method of attacking the transfinite lower central quotients of the fundamental group and extracting information from it.

Definition: transfinite lower central series of a group G (defined for arbitrary ordinals)

$$G_1 := G, \quad G_\kappa := \begin{cases} [G, G_{\kappa-1}] & \text{if } \kappa \text{ is a discrete ordinal}, \\ \bigcap_{\lambda<\kappa} G_\lambda & \text{if } \kappa \text{ is a limit ordinal}. \end{cases}$$

Among earlier attempts toward transfinite Milnor (-type) invariants:

- Orr: link invariant using nilpotent completion $\varprojlim F/F_k$
- Levine: link invariant using "algebraic closures"

Both are unknown to be nontrivial or not!

Our goal: develop a transfinite version of Milnor invariants for 3-manifolds, with the two key features:

- determination of lower central quotients
- invariance under $\mathbb{Z}H$-cobordism

First observation:

while $\pi_1(-)/\pi_1(-)_k$ is invariant under $\left\{\begin{matrix}\mathbb{Z}H\text{-cobordism} \\ \text{concordance}\end{matrix}\right\}$ for $k<\infty$, (due to Stallings' theorem)

$\pi_1(-)/\pi_1(-)_\kappa$ is not for an infinite ordinal κ in general.

e.g. Hillman: $\exists$ slice link L which is not a "homology boundary link", i.e. $\pi := \pi_1(S^3 \smallsetminus L)$ does not admit a surjection onto F.

For $\omega :=$ first infinite ordinal, $F_\omega = \bigcap_{k<\infty} F_k = \{1\}$, and so π/π_ω cannot surject onto $F/F_\omega = F$.

Homology localization (of groups)

In general, localization can be defined for a given collection Ω of morphisms in a category $\mathcal{C}$ (e.g. $\mathcal{C} = \mathcal{G}p$)

$Z \in \mathcal{C}$ is local if
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle \exists\text{ unique}} & \\ Z & & \end{array}$$
whenever $f \in \Omega$, g arbitrary

A functor $E : \mathcal{C} \to \mathcal{C}$ together with $\eta : \mathrm{id}_{\mathcal{C}} \to E$ is a localization if

(i) $E(G)$ is local for every $G \in \mathcal{C}$

(ii)
$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & E(G) \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle \exists\text{ unique}} & \\ Z & & \end{array}$$
whenever Z is local and g is arbitrary.

"$G \xrightarrow{\eta_G} E(G)$ is universal (initial) among morphisms into local objects"

Bousfield: $H\mathbb{Z}$-localization

$$\Omega = \{\phi : G \to \Gamma \mid \phi_* \text{ is } \cong \text{ on } H_1(-;\mathbb{Z}),\ \text{onto on } H_2(-;\mathbb{Z})\}$$

Vogel / Levine: localization through f.p. groups (suitable for $L \subset S^3$)

$$\Omega = \left\{\phi : G \to \Gamma \;\middle|\; \begin{array}{l} G, \Gamma \text{ are f.p.},\ \phi(G) \text{ normally generates } \Gamma, \\ \phi_* \text{ is } \cong \text{ on } H_1,\ \text{onto on } H_2 \end{array}\right\}$$

We will use a modified version (suitable for $H\mathbb{Z}$-cobordism):

C. $\Omega = \{\phi : G \to \Gamma \mid G, \Gamma \text{ are f.p.},\ \phi_* \cong \text{ on } H_1,\ \text{onto on } H_2\}$.

We denote $\widehat{G} :=$ localization of G w.r.t. this Ω.

Proposition [Levine, C.] For every f.p. group G, there is a sequence

$$G = G(1) \to G(2) \to \cdots \to G(i) \to \cdots \quad \text{with each } G(i) \to G(i+1) \text{ in } \Omega$$

s.t. $\hat{G} = \varinjlim G(i)$.

Some properties:

① $\phi : G \to \Gamma$ is in Ω $\Rightarrow$ $\hat{G} \cong \hat{\Gamma}$

[Consequence: $f : X \to Y$, X, Y finite complex, $H_*(X) \overset{f_*}{\cong} H_*(Y)$
$\Rightarrow \widehat{\pi_1(X)} \cong \widehat{\pi_1(Y)}$]

② For finite k and f.p. G, $\hat{G}/\hat{G}_k \cong G/G_k$.

Proof: Each finite composition $G = P_0 \to \cdots \to P_i$ induces $H_1 \cong$, H_2 onto

$\Rightarrow G \to \hat{G} = \varinjlim P_i$ induces $H_1 \cong$, H_2 onto

$\Rightarrow G/G_k \cong \hat{G}/\hat{G}_k$ for finite k, by Stallings

Properties: ① $f : X \to Y$ induces $\cong$ on H_* $\Rightarrow$ $\widehat{\pi_1 X} \cong \widehat{\pi_1 Y}$

② $G/G_k \cong \hat{G}/\hat{G}_k$ for finite k

So, for an arbitrary ordinal κ, $\hat{G}/\hat{G}_\kappa$ is ~~a~~ the "correct" transfinite generalization of the finite lower central quotients G/G_k which is useful for homology cobordism invariance.

From now on, the "transfinite lower central quotient" of G designates $\hat{G}/\hat{G}_\kappa$.

Known localizations:

(1) G is abelian $\Rightarrow \hat{G} = G$ proof:

$$\begin{tikzcd} A \arrow[r, "f \in \Omega"] \arrow[dr] \arrow[dd] & B \arrow[dr] & \\ & H_1A \arrow[r, "\cong"'] \arrow[d] & H_1B \arrow[dl, dashed] \\ G \arrow[r, equal] & H_1G & \end{tikzcd}$$

(2) G is nilpotent $\Rightarrow \hat{G} = G$

proof: replace $H_1(-)$ by $(-)/(-)_k$ for some large k, and invoke Stallings.

In general: Understanding / computing $\hat{G}$ is hard.

e.g. For links L with $\widehat{\pi_1(S^3 \setminus L)} \cong \hat{F}$, Levine defined $\hat{\theta}(L) \in H_3(\hat{F})$.
longitude $\to 1$

Question: $H_3(\hat{F}) \neq 0$?

Main Results: transfinite Milnor invariants for 3-manifolds

Recall that Milnor's link invariants inductively compare lower central quotients of the link group with that of the trivial link $(=F)$.

Fix a closed 3-manifold Y, and let $\Gamma := \pi_1(Y)$.

Fix an ordinal κ.

Suppose M is another closed 3-manifold with $\pi := \pi_1(M)$, which admits an isomorphism $f : \hat{\pi}/\hat{\pi}_\kappa \cong \hat{\Gamma}/\hat{\Gamma}_\kappa$.

__Definition__ $\theta_\kappa(M) :=$ image of the fundamental class $[M]$ under

$$H_3(M) \to H_3(\pi) \to H_3(\hat{\pi}) \to H_3(\hat{\pi}/\hat{\pi}_\kappa) \xrightarrow{\cong} H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa).$$

This depends on the choice of f;

$\theta_\kappa(M)$ in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)/\mathrm{Aut}(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ is independent of f.

__cf.__ Orr's homotopy invariant (of links) $\theta_\kappa(L) \in \pi_3(\text{mapping cone of } \overset{\bigvee^m S^1}{\overset{\shortparallel}{BF}} \to BF/F_\kappa)$

__Definition__ $\theta_\kappa(M) :=$ image of the fundamental class $[M]$ under

$$H_3(M) \to H_3(\pi) \to H_3(\hat{\pi}) \to H_3(\hat{\pi}/\hat{\pi}_\kappa) \xrightarrow{\cong} H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa).$$

__Theorem A.__ $\theta_\kappa(M)$ is a $\mathbb{Z}H$-cobordism invariant.

More precisely: if M and N are $\mathbb{Z}H$-cob., then

(i) $\widehat{\pi_1 M}/\widehat{\pi_1 M}_\kappa \underset{\exists}{\cong} \widehat{\pi_1 N}/\widehat{\pi_1 N}_\kappa$ ($\Rightarrow$ $\theta_\kappa(M)$ is defined iff so is $\theta_\kappa(N)$ using $\widehat{\pi_1 M}/\widehat{\pi_1 M}_\kappa \cong \widehat{\pi_1 N}/\widehat{\pi_1 N}_\kappa \cong \hat{\Gamma}/\hat{\Gamma}_\kappa$)

(ii) $\theta_\kappa(M) = \theta_\kappa(N)$ whenever defined.

Proof: If W is a $\mathbb{Z}H$-cob., i.e. $\partial W = M \cup -N$, $H_*(M) \cong H_*(W) \cong H_*(N)$

$$\Rightarrow \begin{tikzcd} \pi_1 M \arrow[r] \arrow[d] & \pi_1 W \arrow[d] & \pi_1 N \arrow[l] \arrow[d] \\ \widehat{\pi_1 M} \arrow[r, phantom, "\cong"] \arrow[d] & \widehat{\pi_1 W} \arrow[r, phantom, "\cong"] \arrow[d] & \widehat{\pi_1 N} \arrow[d] \\ \widehat{\pi_1 M}/\widehat{\pi_1 M}_\kappa \arrow[r, phantom, "\cong"] & \widehat{\pi_1 W}/\widehat{\pi_1 W}_\kappa \arrow[r, phantom, "\cong"] & \widehat{\pi_1 N}/\widehat{\pi_1 N}_\kappa \arrow[r, phantom, "\cong"] & \hat{\Gamma}/\hat{\Gamma}_\kappa \end{tikzcd}$$

, $\partial[W] = [M] - [N] = 0$

$\Rightarrow [M] = [N]$ in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$.

$\quad\ \ \overset{\shortparallel}{\theta_\kappa(M)}\ \ \overset{\shortparallel}{\theta_\kappa(N)}$

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__Definition__ For M with $\pi := \pi_1 M$ satisfying $\hat{\pi}/\hat{\pi}_\kappa \cong \hat{\Gamma}/\hat{\Gamma}_\kappa$,

$\theta_\kappa(M) :=$ image of the fundamental class $[M]$ under

$$H_3(M) \to H_3(\pi) \to H_3(\hat{\pi}) \to H_3(\hat{\pi}/\hat{\pi}_\kappa) \xrightarrow{\cong} H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa).$$

In general, not all classes in $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ are realized in this way.

__Definition__ (Realizable classes)

$$\mathcal{R}_\kappa(\Gamma) := \{ \theta \in H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa) \mid \exists\, M^3 \text{ with } \widehat{\pi_1 M}/\widehat{\pi_1 M}_\kappa \cong \hat{\Gamma}/\hat{\Gamma}_\kappa \text{ s.t. } \theta_\kappa(M) = \theta \}$$

$\mathcal{R}_\kappa(\Gamma) \subset H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ is not necessarily a subgroup.

(e.g. $H_1(\Gamma)$ is not torsion free $\Rightarrow 0 \notin \mathcal{R}_\kappa(\Gamma)$.)

$\hat{\Gamma}/\hat{\Gamma}_{\kappa+1} \to \hat{\Gamma}/\hat{\Gamma}_\kappa$ induces $\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)$.

Say $\theta \in \mathcal{R}_\kappa(\Gamma)$ __vanishes in__ $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$ if $\theta \in \mathrm{image}(\mathcal{R}_{\kappa+1}(\Gamma))$

__Theorem B__ (Determination of transfinite lower central quotients)

Suppose M has $\pi = \pi_1 M$, $\hat{\pi}/\hat{\pi}_\kappa \cong \hat{\Gamma}/\hat{\Gamma}_\kappa$, κ an arbitrary ordinal.

Then the following are equivalent:

1. There exists a lift $\hat{\pi}/\hat{\pi}_{\kappa+1} \cong \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$, which is an isomorphism.
2. $\theta_\kappa(M)$ vanishes in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$.
3. There is a bordism W over $\hat{\Gamma}/\hat{\Gamma}_\kappa$ s.t. $\partial W = M \cup -N$, $\widehat{\pi_1 N}/\widehat{\pi_1 N}_{\kappa+1} \cong \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ and

$$\begin{tikzcd} \pi = \pi_1 M \arrow[r] \arrow[d] & \pi_1 W \arrow[ddl, dashed, "\exists"] & \pi_1 N \arrow[l] \arrow[d] \\ \hat{\pi}/\hat{\pi}_\kappa \arrow[d, "\cong"'] & & \widehat{\pi_1 N}/\widehat{\pi_1 N}_{\kappa+1} \arrow[d, "\cong"'] \\ \hat{\Gamma}/\hat{\Gamma}_\kappa & & \hat{\Gamma}/\hat{\Gamma}_{\kappa+1} \arrow[ll, "\mathrm{pr.}"'] \end{tikzcd}.$$

Consequently, $\hat{\pi}/\hat{\pi}_{\kappa+1} \cong \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ iff $\theta_\kappa(M) = 0$ in $\mathrm{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \dfrac{\mathcal{R}_\kappa(\Gamma)}{\mathrm{Aut}(\hat{\Gamma}/\hat{\Gamma}_\kappa)}\}$.

Consider towers of transfinite lower central quotients of 3-manifold groups:

$$\begin{array}{l}
\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\kappa+1} \\
\quad | \\
\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\kappa} \\
\quad | \\
\quad \vdots \\
\quad | \\
\widehat{\pi_1 M}/\widehat{\pi_1 M}_{3} = \pi_1 M/(\pi_1 M)_3 \\
\quad | \\
\widehat{\pi_1 M}/\widehat{\pi_1 M}_{2} = \pi_1 M/(\pi_1 M)_2 = H_1(M)
\end{array}$$

Let $\mathcal{L}_{\kappa+1}(\Gamma) := \{$ height $\kappa+1$ towers of 3-manifolds whose height $\leq \kappa$ subtower is identified with that of $\Gamma \}$

$= \{ M$ equipped with $\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\kappa} \cong \widehat{\Gamma}/\widehat{\Gamma}_\kappa \}/\sim$

where $M \sim N$ iff

$$\begin{tikzcd}
\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\kappa+1} \arrow[rr, dashed, "\exists\text{ lift}", "\cong"'] \arrow[d, no head] & & \widehat{\pi_1 N}/\widehat{\pi_1 N}_{\kappa+1} \arrow[d, no head] \\
\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\kappa} \arrow[r, "\cong"] & \widehat{\Gamma}/\widehat{\Gamma}_\kappa & \widehat{\pi_1 N}/\widehat{\pi_1 N}_{\kappa} \arrow[l, "\cong"']
\end{tikzcd}$$

Define an equivalence relation $\sim$ on $\mathcal{R}_\kappa(\Gamma)$ by

$\theta \sim \theta'$ iff $\theta' \in \operatorname{Im}\{ \mathcal{R}_{\kappa+1}(\pi_1 M) \to \mathcal{R}_\kappa(\pi_1 M) \xrightarrow{\cong} \mathcal{R}_\kappa(\Gamma) \}$

when M with $\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\kappa} \cong \widehat{\Gamma}/\widehat{\Gamma}_\kappa$ satisfies $\theta = \theta_\kappa(M)$.

Corollary of Theorem B: $\mathcal{L}_{\kappa+1}(\Gamma) \underset{\text{bij.}}{\approx} \mathcal{R}_\kappa(\Gamma)/\sim$

So, the transfinite lower central quotients of 3-manifold groups are understood in terms of $\mathcal{R}_\kappa(\Gamma)$.

Theorem C (Characterization of realizable classes)

Suppose Γ is a f.p. group, $\theta \in H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$, $\kappa \geq 2$.

Then θ lies in $\mathcal{R}_\kappa(\Gamma)$ iff the following hold:

(a) The cap product $\cap\theta : tH^2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \to tH_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is an isomorphism, where $tH_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \cong tH_1(\Gamma)$.

(b) $\cap\theta : H^1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) / \operatorname{Ker}\{ H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\lambda) \}$ is surjective for all $\lambda < \kappa$, where $H^1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \cong H^1(\Gamma)$.

(when κ is a discrete ordinal, it suffices to check for $\lambda = \kappa - 1$.)

For finite κ (for which $\widehat{\Gamma}/\widehat{\Gamma}_\kappa \cong \Gamma/\Gamma_\kappa$ automatically), Theorem C was shown eariler by Turaev.

Theorem C said: $\theta \in \mathcal{R}_\kappa(\Gamma) \subset H_3(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ iff

(a) $\cap\theta : tH^2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \to tH_1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa)$ is $\cong$, and

(b) $\cap\theta : H^1(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \twoheadrightarrow H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) / \operatorname{Ker}\{ H_2(\widehat{\Gamma}/\widehat{\Gamma}_\kappa) \to H_2(\widehat{\Gamma}/\widehat{\Gamma}_\lambda) \}$ for all $\lambda < \kappa$.

e.g. $\Gamma = F\langle x_1, \dots, x_m \rangle$ free group, $k < \infty$:

$$tH^2(F/F_k) \underset{\text{UCT}}{\cong} \operatorname{Ext}^1(tH_1(F/F_k), \mathbb{Z}) = \operatorname{Ext}(\underbrace{tH_1(F)}_{=0}, \mathbb{Z}) = 0 \implies \text{(a)}$$

$$H_2(F/F_k) = F_k/F_{k+1} \longrightarrow H_2(F/F_{k-1}) = F_{k-1}/F_k \text{ is zero} \implies \text{(b)}$$

$\therefore\ \mathcal{R}_k(F) = H_3(F/F_k)$, and $\operatorname{Coker}\{ \mathcal{R}_{k+1}(F) \to \mathcal{R}_k(F) \}$ is an abelian group.

Consequence: let $L \subset S^3$ a link, $M_L :=$ 0-framed surgery for L.

Then for finite k, $\theta_k(M_L) \in \operatorname{Coker}$ is equal to Orr's $\theta_k(L) \in \operatorname{Coker}$, and so equivalent to the classical Milnor invariants of length $k+1$.

This shows that our invariants generalize Milnor's.

A torus bundle group

let $Y := T^2 \times [0,1] / (h(x), 0) \sim (x, 1)$ where $h = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\Gamma := \pi_1(Y) = \langle x, y, t \mid [x,y] = 1,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1} \rangle$$
$$= (\mathbb{Z}^t)^2 \rtimes \mathbb{Z}, \quad \mathbb{Z}^t := \mathbb{Z}[t^{\pm 1}]/\langle t+1 \rangle$$

Exercise $H_1(\Gamma) = \underset{\substack{\uparrow \\ \text{gen. by } x,y}}{(\mathbb{Z}/2)^2} \times \underset{\substack{\uparrow \\ t}}{\mathbb{Z}}$, $H_2(\Gamma) = H_2(T^2) = \mathbb{Z}$.

Enlarge Γ by adding r^{th} roots u, v of x, y ($r =$ odd):

$$\Gamma \hookrightarrow \Gamma(r) := \langle u, v, s \mid [u^r, v^r] = 1,\ [u,v] \text{ is central},\ sus^{-1} = u^{-1},\ svs^{-1} = v^{-1} \rangle$$

$x \mapsto u^r$

$y \mapsto v^r$

$t \mapsto s$

Observe: if r is odd, $H_1(\Gamma) \cong H_1(\Gamma(r))$ since $r \cdot u = u \pmod 2$

Furthermore, $H_2(\Gamma) \cong H_2(\Gamma(r))$!!!

So, Γ and $\Gamma(r)$ have the same homology localization: $\widehat{\Gamma} \cong \widehat{\Gamma(r)}$.

Theorem [C.-Orr] $\hat{\Gamma} = \varinjlim_{r:\text{odd}} \Gamma(r)$.

(proof uses the notion of "localization of modules".)

$$\Gamma(r) = \langle u, v, s \mid \underset{[u,v]^{r^2}}{[u^r, v^r]} = 1,\ [u,v] \text{ central},\ sus^{-1} = u^{-1},\ svs^{-1} = v^{-1} \rangle$$

$$\rightsquigarrow \quad 1 \to \underset{\left(\frac{1}{r^2}\mathbb{Z}\right)/\mathbb{Z}}{\mathbb{Z}/r^2\mathbb{Z}} \to \Gamma(r) \to \underset{\left(\frac{1}{r}\mathbb{Z}^t\right)^2 \rtimes \mathbb{Z}}{(\overset{u,v}{\mathbb{Z}^t})^2 \rtimes \overset{t}{\mathbb{Z}}} \to 1 \qquad (r: \text{odd})$$

$$\xrightarrow[\text{colimit } r \to \infty]{\text{take}} \quad 1 \to \mathbb{Z}_{(2)}/\mathbb{Z} \to \hat{\Gamma} \to (\mathbb{Z}^t_{(2)})^2 \rtimes \mathbb{Z} \to 1$$

where $\mathbb{Z}_{(2)} = \{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ is odd} \}$.

In particular, $\hat{\Gamma}$ contains torsion of any given odd order

Full computation of: (for every κ)

- $H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$
- $\mathcal{R}_\kappa(\Gamma) \subset H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa)$ (by computing cap products)
- $\operatorname{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$ i.e. image of $\mathcal{R}_{\kappa+1}(\Gamma)$

For finite κ:

$$H_3(\Gamma/\Gamma_\kappa) = (\mathbb{Z}/2^{\kappa-1})^4$$
$$\cup$$
$$\mathcal{R}_\kappa(\Gamma) = \begin{cases} (\mathbb{Z}/2^{\kappa-1})^3 \times \{\pm 1\} & \text{if } \kappa \geq 3 \\ \{(a,b,c,d) \mid ac+bd \equiv 1 \pmod 2\} & \text{if } \kappa = 2 \end{cases}$$
$$\cup$$
$$\text{Image of } \mathcal{R}_{\kappa+1}(\Gamma) = (2 \cdot \mathbb{Z}/2^{\kappa-1})^3 \times \{\pm 1\}$$

$\therefore$ $\operatorname{Coker}\{\mathcal{R}_{\kappa+1}(\Gamma) \to \mathcal{R}_\kappa(\Gamma)\}$ is nontrivial (and finite!)

For M^3 with $\pi_1(M)/\pi_1(M)_\kappa \cong \Gamma/\Gamma_\kappa$, $\exists$ lift $\pi_1(M)/\pi_1(M)_{\kappa+1} \cong \Gamma/\Gamma_{\kappa+1}$ iff $\theta_\kappa(M) = 0$ in Coker.

Transfinite case:

For $\kappa = \omega$:

$$H_3(\hat{\Gamma}/\hat{\Gamma}_\omega) = \mathbb{Z} \quad = \{ \tfrac{a}{k} \in \mathbb{Q} \mid b \text{ odd} \}$$
$$\cup$$
$$\mathcal{R}_\omega(\Gamma) = \{ \tfrac{a}{b} \mid a, b \text{ odd} \}$$
$$\cup$$
$$\text{Image of } \mathcal{R}_{\omega+1}(\Gamma) = \{\pm 1\}$$

$\therefore$ $\operatorname{Coker}\{\mathcal{R}_{\omega+1}(\Gamma) \to \mathcal{R}_\omega(\Gamma)\}$ is large!

For M with $\widehat{\pi_1 M}/\widehat{\pi_1 M}_\omega \cong \hat{\Gamma}/\hat{\Gamma}_\omega$, $\exists$ lift $\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\omega+1} \cong \hat{\Gamma}/\hat{\Gamma}_{\omega+1}$ iff $\theta_\omega(M) = 0$ in the cokernel.

$\therefore$ The transfinite invariant θ_ω detects existence of M s.t. $\widehat{\pi_1 M}/\widehat{\pi_1 M}_\omega \cong \hat{\Gamma}/\hat{\Gamma}_\omega$ but $\widehat{\pi_1 M}/\widehat{\pi_1 M}_{\omega+1} \not\cong \hat{\Gamma}/\hat{\Gamma}_{\omega+1}$

For $\kappa \geq \omega+1$: we have $\hat{\Gamma}_{\omega+1} = \hat{\Gamma}_{\omega+2} = \cdots = \{1\}$, and so $\hat{\Gamma}/\hat{\Gamma}_\kappa = \hat{\Gamma}/\hat{\Gamma}_{\kappa+1}$ $(= \hat{\Gamma})$.

There is no nontrivial transfinite Milnor invariant for $\kappa \geq \omega+1$ (in case of our Γ)

In fact,

$$H_3(\hat{\Gamma}/\hat{\Gamma}_\kappa) = H_3(\hat{\Gamma}) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \mathbb{Z}$$
$$\cup$$
$$\mathcal{R}_\kappa(\Gamma) = (\mathbb{Z}_{(2)}/\mathbb{Z}) \times \{\pm 1\} = \text{image of } \mathcal{R}_{\kappa+1}(\Gamma)$$

Though, $\theta_\kappa(M)$ detects non-$\mathbb{Z}H$-cob. 3-manifolds with $\widehat{\pi_1 M}/\widehat{\pi_1 M}_\kappa \cong \hat{\Gamma}$.