The complex of boundary braids

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joint work with
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Outline

Boundary braids

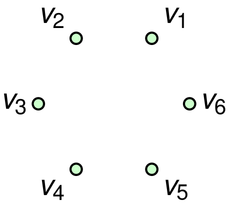
The dual braid complex

Decomposing boundary braids

Braid group setup

Let $n \geq 3$ and let $[n] = \{1, \ldots, n\} \cong \mathbb{Z}/n\mathbb{Z}$.

For $j \in [n]$ let $v_j = \exp(2\pi i j/n) \in \mathbb{C}$ and let $V = \{v_1, \ldots, v_n\}$.



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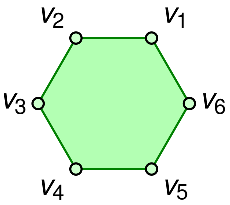
Let $P = \text{CONV}\, V$ and put

$$\begin{aligned} \text{CONF}_n(P) &= \{(x_1, \ldots, x_n) \in P^n \mid i \neq j \Rightarrow x_i \neq x_j\} \\ \text{UCONF}_n(P) &= \text{CONF}_n(P)/S_n \\ \text{BRAID}_n &= \pi_1(\text{UCONF}_n(P), V). \end{aligned}$$



Boundary braids

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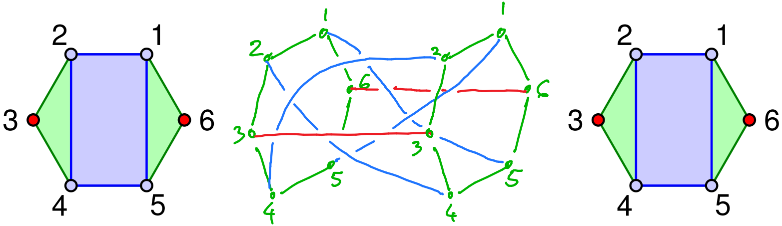

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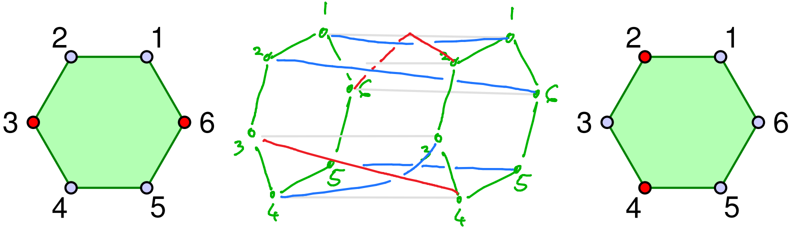

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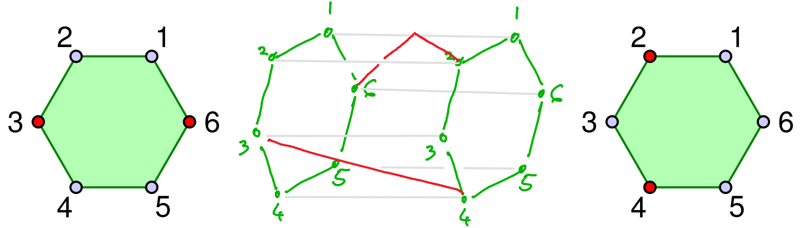
We denote by $\textsc{Move}_n(B)$ the set of paths in $\textsc{UConf}_k(\partial P)$ starting in V_B and ending some $V_{B'}$, and call its elements moves.



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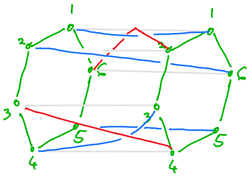
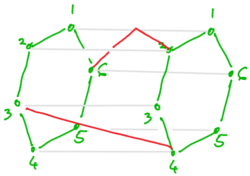
$$\textsc{Fix}_n(B) \hookrightarrow \textsc{Braid}_n(B) \xrightarrow{f} \textsc{Move}_n(B)$$





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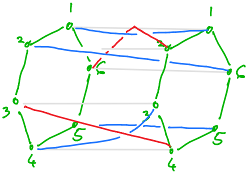
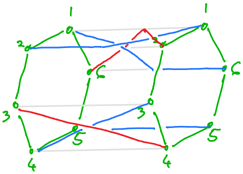
$$\textsc{Fix}_n(B) \hookrightarrow \textsc{Braid}_n(B) \overset{f}{\twoheadrightarrow} \textsc{Move}_n(B), \qquad \sigma : \textsc{Move}_n(B) \hookrightarrow \textsc{Braid}_n(B)$$

- For every move pick a boundary braid that realizes it.





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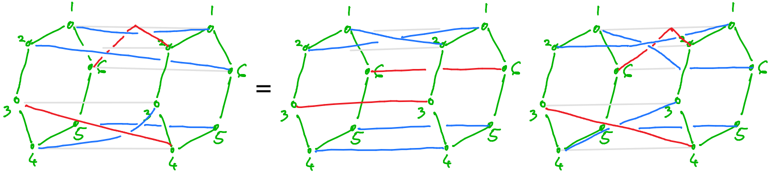

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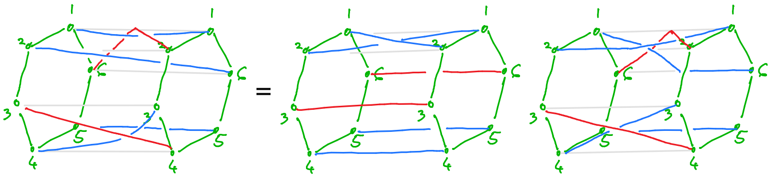

Goal: make a canonical choice.

Motivation: CAT(0)-geometry

A metric space X is CAT(0) if any two points can be connected by a geodesic and if triangles are at most as thick as in euclidean space:

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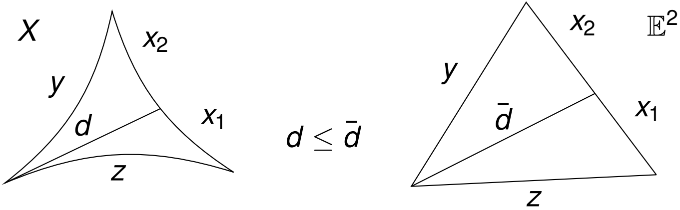

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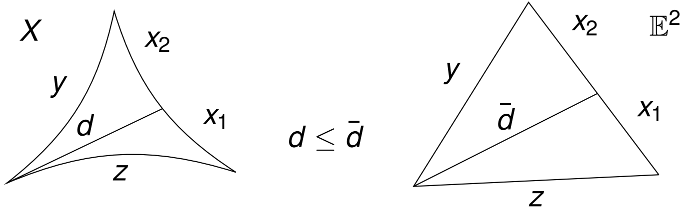

Examples.

- Riemannian manifolds of non-positive sectional curvature.
- Coxeter complexes and buildings.
- Cube complexes whose links are flag.
- Products and convex subsets of CAT(0)-spaces.

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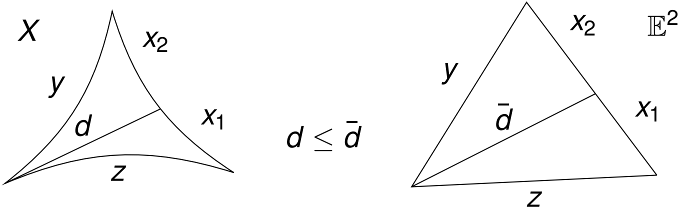

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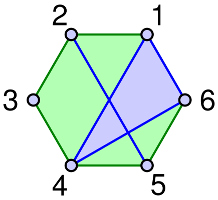
Fact. If G acts freely, properly and cocompactly on a CAT(0)-space, it is torsion-free and has solvable word- and conjugacy problem.

Outline

Non-crossing partitions as generators

For $I \subseteq [n]$ let $V_I = \{v_i, i \in I\}$. A partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$ is non-crossing if $\operatorname{CONV} V_{B_i} \cap \operatorname{CONV} V_{B_j} = \emptyset$ for $i \neq j$.
Non-crossing partitions form a lattice denoted NC_n.

$\{1,4,6\},\{2,5\},\{3\}$



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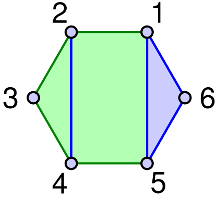

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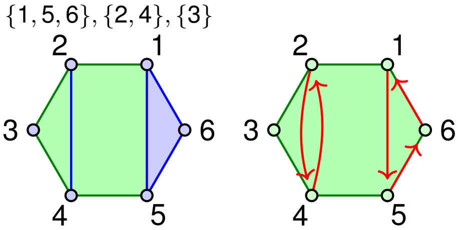
Every non-crossing partition π defines a braid δ_π.
The braids $(\delta_\pi)_{\pi \in \mathrm{NC}_n}$ generate $\textsc{Braid}_n$.



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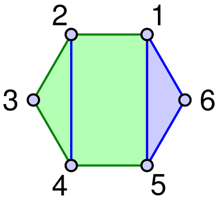

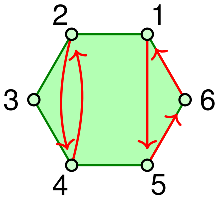

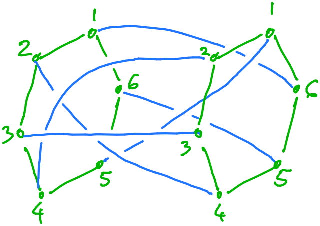

The dual braid complex

Let $\textsc{Cay}(B_n, (\delta_\pi)_{\pi \in \mathrm{NC}_n})$ be the Cayley graph.
The flag complex of $\textsc{Cay}(B_n, (\delta_\pi)_{\pi \in \mathrm{NC}_n})$ is the dual braid complex $\textsc{Cx}(\textsc{Braid}_n)$ of dimension $n - 1$.

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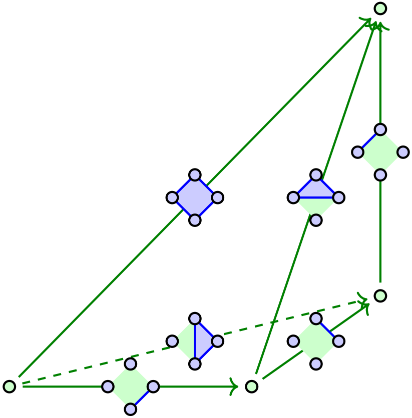

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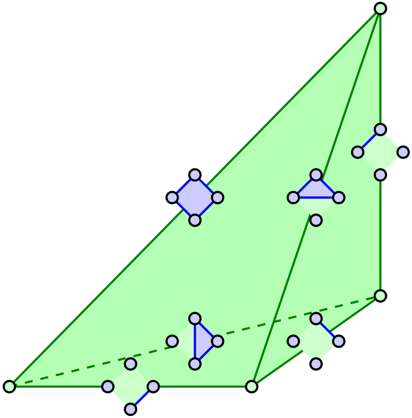

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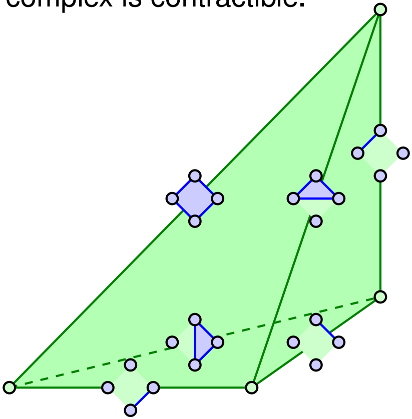
$\textsc{Braid}_n$ acts on it freely and cocompactly.

Theorem (Brady '01). The dual braid complex is contractible.



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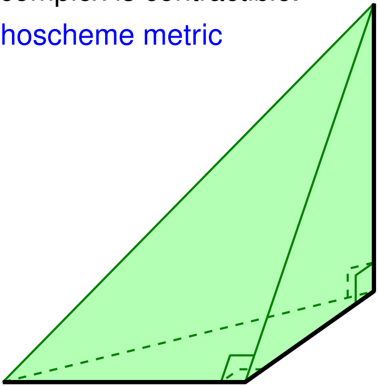

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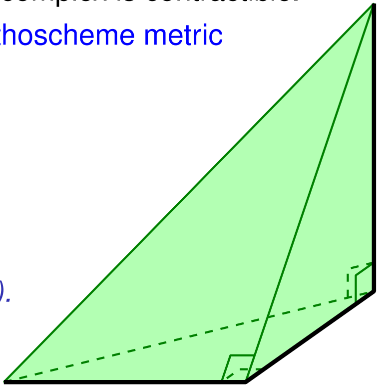
Conjecture (Brady–McCammond '10).
The dual braid complex is CAT(0).

Theorem (Brady–McCammond '10).
$\textsc{Cx}(\textsc{Braid}_n)$ is CAT(0) for $n \leq 5$.

Theorem (Haettel–Kielak–Schwer '16).
$\textsc{Cx}(\textsc{Braid}_n)$ is CAT(0) for $n = 6$.



Outline

Complexes associated to boundary braids

Define $\textsc{Cx}(\textsc{Fix}_n(B)) \leq \textsc{Cx}(\textsc{Braid}_n(B) \leq \textsc{Cx}(\textsc{Braid}_n)$ to be the full subcomplexes supported on $\textsc{Fix}_n(B)$ and $\textsc{Braid}_n(B)$.

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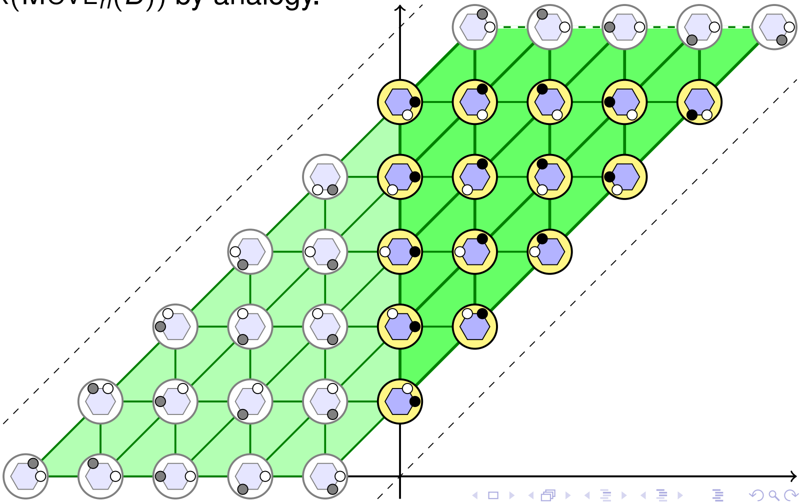

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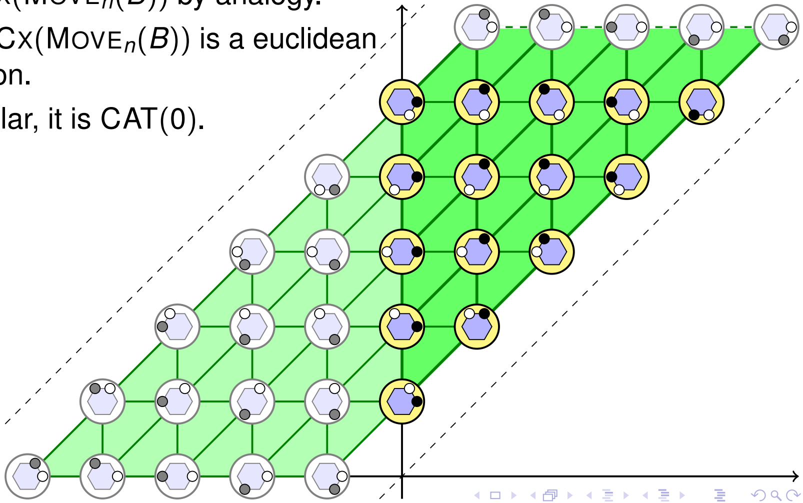
Lemma. $\textsc{Cx}(\textsc{Move}_n(B))$ is a euclidean polyhedron.

In particular, it is CAT(0).



Main result

Theorem (DMW). There is a decomposition of metric spaces

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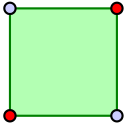

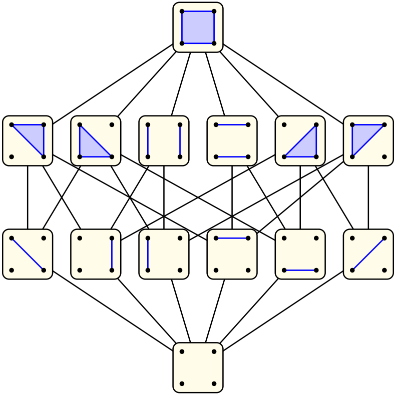

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