Johnson-Levine homomorphisms and the tree reduction of the LMO functor

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Plan

1. Historical background
2. Idea of the LMO functor
3. Johnson-Levine homomorphisms

Historical background

Quantum invariants

Quantum invariants

1984: Jones polynomial

Quantum invariants

1984: Jones polynomial $\longrightarrow$ 1989: (Witten) Inv. of 3-manifolds

Quantum invariants

$$\begin{tikzcd}
\boxed{\text{1984: Jones polynomial}} \arrow[r] & \boxed{\text{1989: (Witten) Inv. of 3-manifolds}} \\
& \text{Chern-Simons path integral} \arrow[u]
\end{tikzcd}$$

Quantum invariants

1984: Jones polynomial $\rightarrow$ 1989: (Witten) Inv. of 3-manifolds

Chern-Simons path integral

Not math. well defined!

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Chern-Simons path integral

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How to understand this kind of invariants from a math. point of view?

Two aproaches

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Non-perturbative

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$\boxed{\text{Non-perturbative}}$

$\int_{CS} \rightsquigarrow \sum$ Quant. inv.

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$\boxed{\text{TQFT interpretation}}$

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TQFT interpretation

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$\int_{CS} \rightsquigarrow$ Expansion in Feynman diagrams

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- Links: Kontsevich Integral

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• Reshetikhin-Turaev	• Links: Kontsevich Integral
• Turaev-Viro	• 3-manifolds: LMO invariant
TQFT interpretation	LMO functor:
	$\widetilde{Z} : \mathcal{L}Cob \longrightarrow \mathcal{A}$

Idea of the LMO functor

LMO functor (Cheptea, Habiro, Massuyeau): $\widetilde{Z} : \mathcal{LC} \longrightarrow \mathcal{A}$

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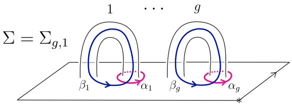

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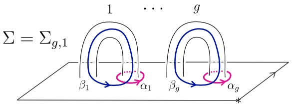

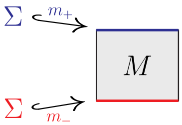

M : 3-manifold
$m : \partial(\Sigma \times [-1,1]) \xrightarrow{\cong} \partial M$
$m_{\pm,*} : H_*(\Sigma) \xrightarrow{\cong} H_*(M)$
$m_{+,*}(A) \subseteq m_{-,*}(A)$, where
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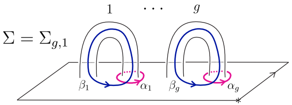

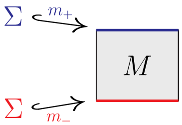

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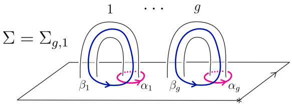

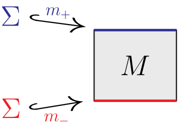

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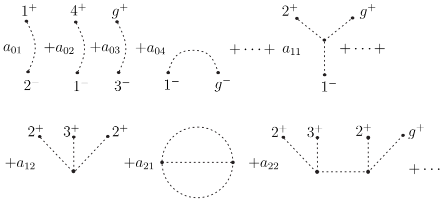

where $a_{ij} \in \mathbb{Q}$. With relations:

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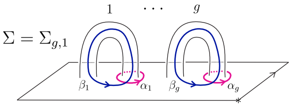

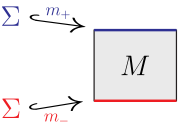

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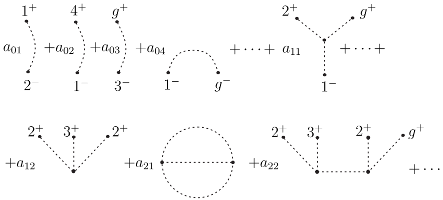

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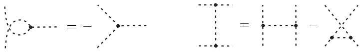
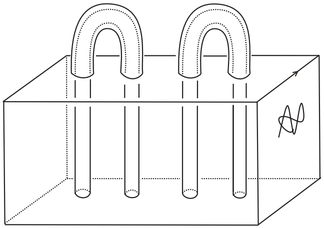
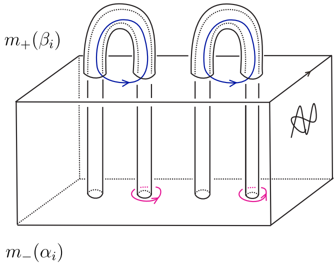
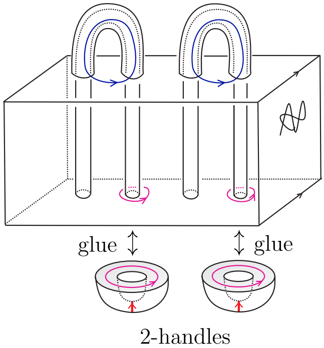
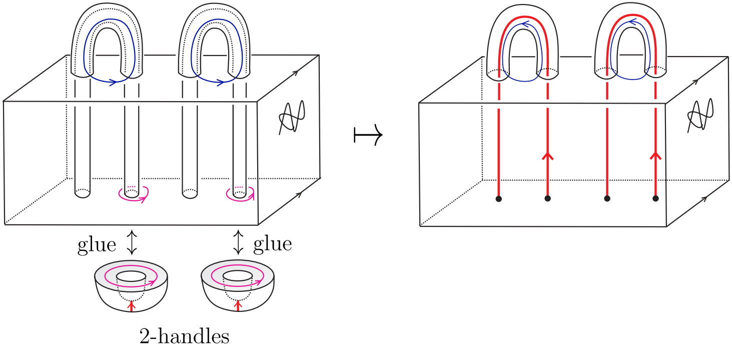


AS IHX




$m_+(\beta_i)$
$m_-(\alpha_i)$


glue
glue
2-handles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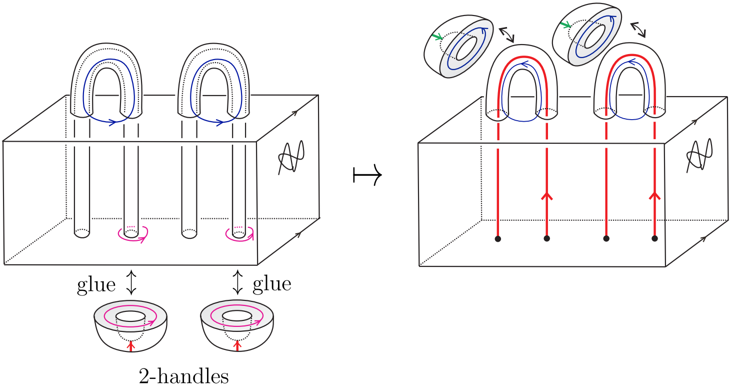
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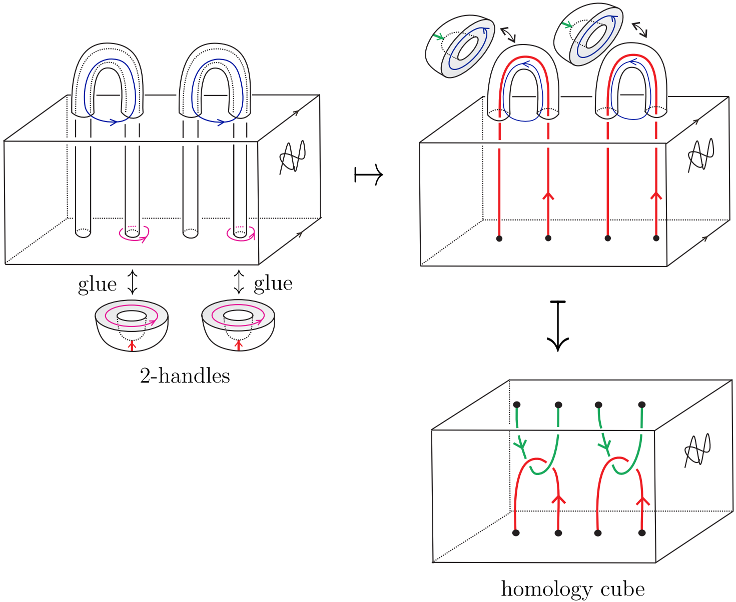
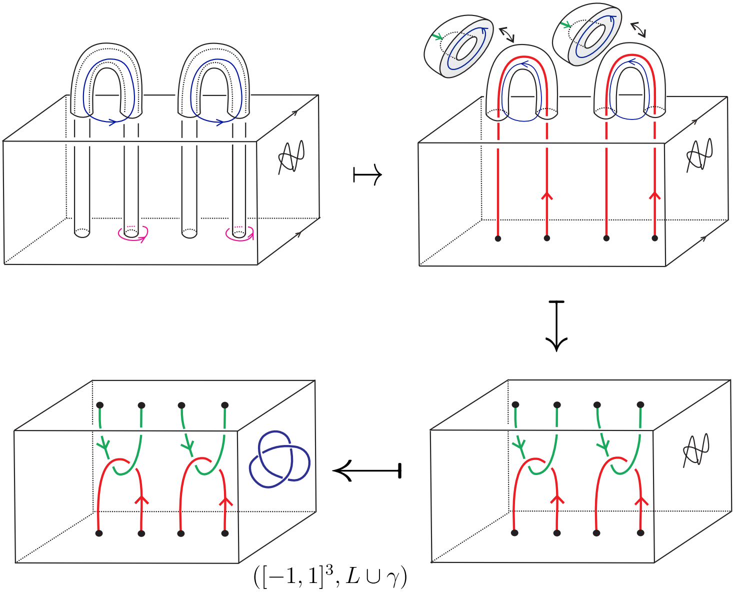
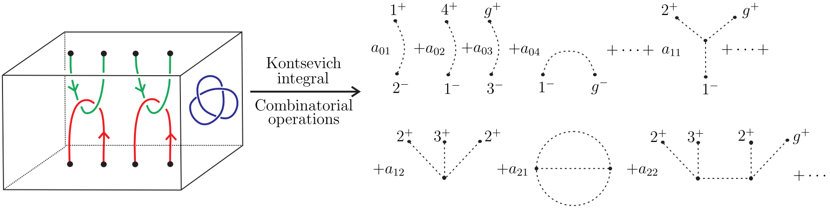
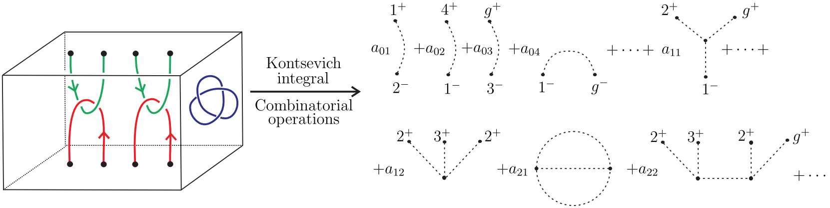

glue
glue
2-handles
homology cube



$([-1,1]^3, L \cup \gamma)$

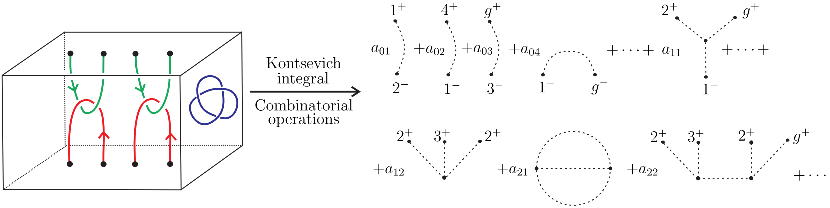


Kontsevich integral
Combinatorial operations
a_{01}
$+a_{02}$
$+a_{03}$
$+a_{04}$
$+\cdots+$
a_{11}
$+\cdots+$
1^+ 4^+ g^+ 2^+ g^+
2^- 1^- 3^- 1^- g^- 1^-
$+a_{12}$
$+a_{21}$
$+a_{22}$
$+\cdots$
2^+ 3^+ 2^+ 2^+ 3^+ 2^+ g^+



Questions

- Topological information encoded by $\widetilde{Z}$?
- Interpretation by using classical invariants?



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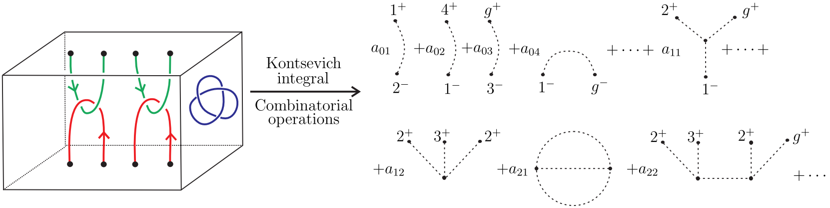

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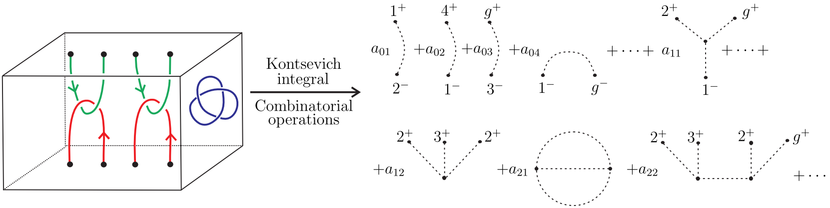
For instance:

- $a_{01}, a_{02}, a_{03}, a_{04}$: Linking numbers.



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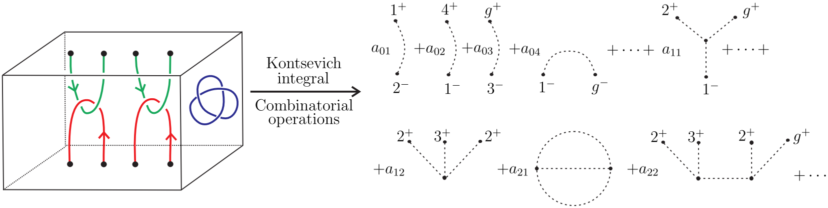
- $a_{01}, a_{02}, a_{03}, a_{04}$: Linking numbers.
- a_{11}, a_{12}, a_{22} : Johnson homomorphisms for a particular subcategory.



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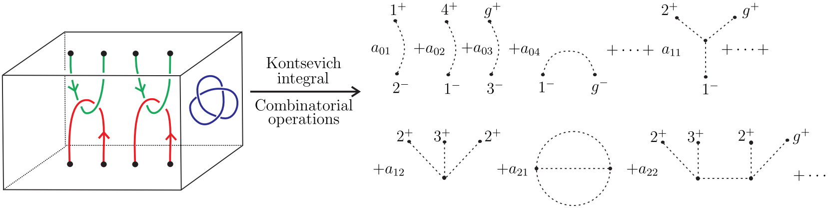

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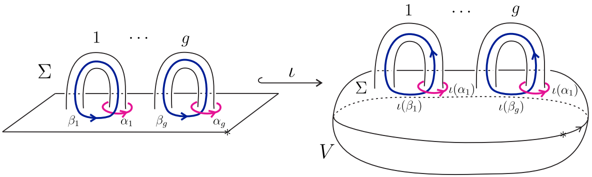
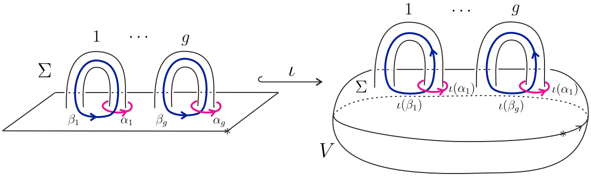
- a_{12}, a_{22} : Johnson-Levine homomorphisms for all $\mathcal{LC}$.

Johnson-Levine homomorphisms


1
⋯
g
Σ
β1
α1
βg
αg
ι
1
⋯
g
Σ
ι(α1)
ι(α1)
ι(β1)
ι(βg)
V



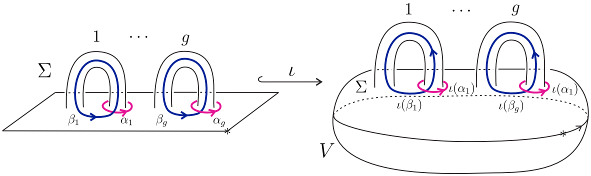
$\pi = \pi_1(\Sigma)$

$\mathbb{A} = \ker(\iota_\# : \pi \to \pi')$

$\pi' = \pi_1(V)$

$\Gamma_1 \pi = \pi$

$\Gamma_{k+1}\pi = [\pi, \Gamma_k \pi]$



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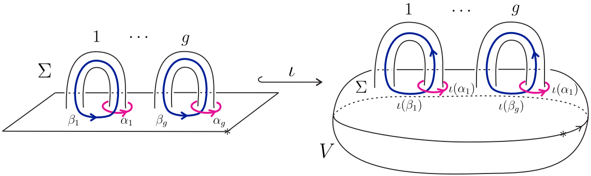

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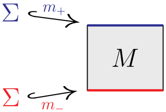

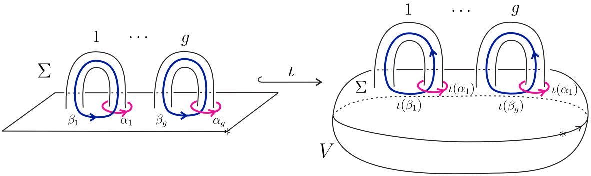

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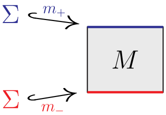
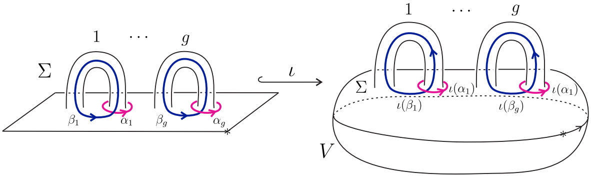


$$\longmapsto \quad m_{-,*}^{-1} \circ m_{+,*}$$



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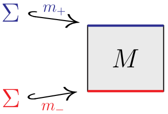

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Johnson-Levine filtration

$$\boxed{J_k^L = \{M \mid \ \iota_\# \rho_k(M)(\mathbb{A}) \subset \Gamma_{k+1}\pi'\}}$$

$$J_1^L \supset J_2^L \supset J_3^L \supset \cdots$$

Johnson-Levine homomorphisms

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Proposition (Levine)

There exist homomorphisms

$$\tau_k^L : J_k^L \longrightarrow D_k(H'),$$

such that $\ker(\tau_k^L) = J_{k+1}^L$.

$\mathcal{A}_k^t(H')$: Tree-like Jacobi diagrams with k triv. vertices and legs colored by H'.

$\mathcal{A}^t_k(H')$: Tree-like Jacobi diagrams with k triv. vertices and legs colored by H'.

Theorem (Kontsevich)

There is an isomorphism

$$\eta_k : \mathcal{A}^t_k(H') \longrightarrow D_k(H') \otimes \mathbb{Q}.$$

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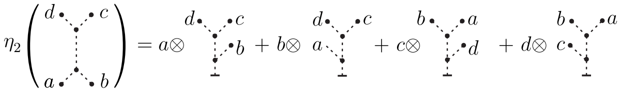
For example,



$$= a \otimes [[d, c], b] + b \otimes [a, [d, c]] + c \otimes [[b, a], d] + d \otimes [c, [b, a]]$$

$$\in D_2(H') \subset H' \otimes L_3(H').$$

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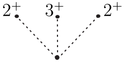
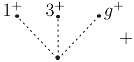


$$+ \; a_{12}$$


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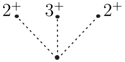
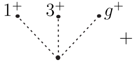
$\eta_k^{-1}\tau_k^L(M)_{|b_j \mapsto j^+} = a_{11}$ 

$+ \; a_{12}$ 

$+ \cdots$

Theorem

For $M \in J_k^L$, we have

$\widetilde{Z}(M) \big/$ loops $\mapsto 0$,  $\mapsto 0$, 

$\mapsto 0$

$$= \emptyset + \eta_k^{-1}\tau_k^L(M)_{|b_j \mapsto j^+} + \text{diagrams with } > k \text{ triv. vertices.}$$