The profinite completions of knot groups and twisted Alexander polynomials

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Theorem 1. [U. arXiv:1702.03819]

Let $J, K \subset S^3$ be knots. If there is an isomorphism $\widehat{\pi}_J \cong \widehat{\pi}_K$ on the profinite completions of their knot groups, then the equality $\Delta_J(t) \doteq \Delta_K(t)$ of their Alexander polynomials holds (up to multiplication by units of $\mathbb{Z}[t^{\mathbb{Z}}]$).

Question.

Let $F/\mathbb{Q}$ be a finite extension, S a finite set of prime ideals of the integer ring of F. Let $O = O_{F,S}$ denote the ring of S-integers of F and suppose that O is a UFD. Consider the set of knot group representations $\rho : \pi_K \to \mathrm{GL}_n(O)$. Does the map $\{(\pi_K, \rho)\}/\cong \ \longrightarrow \ \{\Delta_{K,\rho}(t) \in O[t^{\mathbb{Z}}]\}/\doteq$ factor through the set $\{(\widehat{\pi}_K, \widehat{\rho})\}/\cong$?

Background 1. Alexander polynomials

K: knot (the image of $S^1 \hookrightarrow S^3$)
$X := S^3 - K \quad \pi := \pi_1(X)$
$X_\infty \to X$ ($\mathbb{Z}$-cover)

$1 \to [\pi, \pi] \to \pi \to \pi^{\rm ab} \to 1$ exact
$=: \pi' \qquad \cong t^{\mathbb{Z}}$ (t: meridian)

$t \curvearrowright \pi'^{\rm ab} \cong H_1(\pi') \underset{\rm Hur.}{\cong} H_1(X_\infty)$: fin.gen. $\Lambda = \mathbb{Z}[t^{\mathbb{Z}}]$-module

${\rm Fitt}_\Lambda H_1(X_\infty) = ({}^\exists \Delta_K(t)) \subset \Lambda$
Such $\Delta_K(t)$ is called the Alexander polynomial of K.

$X_n \to X$ ($\mathbb{Z}/n\mathbb{Z}$-cover)
$\rightsquigarrow M_n \to S^3$ (branched $\mathbb{Z}/n\mathbb{Z}$-cover) by the Fox completion.

By the Crowell exact sequence, $H_1(M_n) \cong H_1(X_\infty)/(t^n - 1)H_1(X_\infty)$

Fox' formula: $|H_1(M_n)| = |\prod_{\zeta^n=1} \Delta_K(\zeta)| = |{\rm Res}(\Delta_K(t), t^n - 1)|$

Here, if G is a finite group, then $|G|$ denotes the order of G.
If G is an infinite group. then we put $|G| = 0$.

The resultant of $f(t), g(t) \in O[t^{\mathbb{Z}}]$ is defined by
${\rm Res}(f(t), g(t)) := \det {\rm Syl}(f(t), g(t)) \in O$, where ${\rm Syl}(f(t), g(t)) \in {\rm M}_{m+n}(O)$.

Background 2. Profinite completions

Let π be a knot group (or a 3-manifold group).
The profinite completion of π is a topological group given by $\widehat{\pi} := \varprojlim_{\text{finite quotient}} \pi/N$.

π is a residually finite group (i.e., $^\exists$ natural injection $\pi \hookrightarrow \widehat{\pi}$)
by [Hempel1987] + [Perelman2002-03].

[Grothendieck1970]'s question: Is finite type residually finite π determined by $\widehat{\pi}$?
$\rightsquigarrow$ *No.* $^\exists$ example satisfying $\pi \not\cong \Gamma$ and $\widehat{\pi} \cong \widehat{\Gamma}$ by [BridsonGrunewald2004].

Question: What topology of K does $\widehat{\pi}_K$ know?

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Former results:
[BoileauFriedl2015] If $\Delta_K(t)$ does not vanish at any root of unity, then OK.
[BridsonReid2015] The group of figure eight knot can be distinguished from any other 3-manifold group by comparing $\widehat{\pi}$'s.

Proof of Thm 1: If $\Delta_K(t)$ does not vanish at any root of unity ([BF2015])

[Fried1988]'s proposition. （See also [Hillar2005]）

Let $\Delta(t) \in \mathbb{R}[t]$ is a *reciprocal* (symmetric) polynomial with $\Delta(0) \neq 0$ and suppose that it does not vanish at any root of unity. Then $\Delta(t)$ is determined by the absolute values of cyclic resultants $b_n := |\mathrm{Res}(\Delta(t), t^n - 1)|$ $(= \prod_{\zeta^n=1} |\Delta(\zeta)|)$.

Idea of proof: $B(z) := \exp \sum_{n=1}^{\infty} b_n \frac{t^n}{n}$ extends to a rational function on $\mathbb{C}$ by analytic continuation, and the roots of $\Delta(t)$ appear there.
(Such $B(z)$ is called a dynamical zeta function of [ArtinMazur1965]).

Proof of Thm 1: If $\Delta_K(t)$ does not vanish on roots of unity, then $H_1(M_n)$ is finite.
Since $H_1(M_n) \cong \widehat{H}_1(M_n) \cong \mathrm{Ker}(\widehat{\pi} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z})^{\mathrm{ab}}/t^{n\widehat{\mathbb{Z}}}$, $H_1(M_n)$ is determined by $\widehat{\pi}$.
Their order is given by cyclic resultants as $|H_1(M_n)| = |\mathrm{Res}(\Delta_K(t), t^n - 1)|$.
By [Fried1988]'s proposition, Thm 1 follows. ■

Examples which cannot be distinguished by the above method [Fried1988]+

Let $\Phi_m = \Phi_m(t)$ denote the m-th cyclotomic polynomials for each $m \in \mathbb{N}$ and let p, q be distinct prime numbers.

- The pair of $f := \Phi_{pq}\Phi_{p^2q}\Phi_{pq^2}$ and $g := \Phi_{pq}^2\Phi_{p^2q^2}$.
- The pair of $F := f^2g$ and $G := fg^2$.

Proof of Thm 1: If $\Delta_K(t)$ does vanish at roots of unity [U.]

Consider the completed group ring $\widehat{\Lambda} := \widehat{\mathbb{Z}}[[t^{\widehat{\mathbb{Z}}}]] = \varprojlim_{m,n} \mathbb{Z}/n\mathbb{Z}[t^{\mathbb{Z}/m\mathbb{Z}}]$.

$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ is not an integral domain,
while the p-adic integer ring $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ is an integral domain.
The decomposition $\widehat{\Lambda} \cong \prod_p \mathbb{Z}_p[[t^{\widehat{\mathbb{Z}}}]]$ is useful.

Step 1. $\widehat{\pi}_J \cong \widehat{\pi}_K \quad \rightsquigarrow$ isomorphism $\widehat{H}_1(X_{J,\infty}) \cong \widehat{H}_1(X_{K,\infty})$ of $\widehat{\Lambda}$-modules
$\rightsquigarrow$ equality $(\Delta_J(t^v)) = (\Delta_K(t))$ of ideals of $\widehat{\Lambda}$
Note that if we write $\pi_J^{\rm ab} = s^{\mathbb{Z}}$, then the induced isomorphism $s^{\widehat{\mathbb{Z}}} = \widehat{\pi}_J^{\rm ab} \cong \widehat{\pi}_K^{\rm ab} = t^{\widehat{\mathbb{Z}}}$ sends $s \mapsto t^v$ for some unit v of $\widehat{\mathbb{Z}}$.

Step 2. Let $\Phi_m(t) \in \mathbb{Z}[t]$ denote the m-th cyclotomic polynomial.
($\Phi_m(t)$ is the minimal polynomial over $\mathbb{Q}$ of a primitive m-th root ζ_m of unity, and known to be in $\mathbb{Z}[t]$. It can also be defined inductively by $t^n - 1 = \prod_{m|n} \Phi_m(t)$.)

Lemma. $\Phi_m(t)$ is not a zero divisor of $\widehat{\Lambda}$.
Proof: Let M, I denote the images of the ideals $\cup_k {\rm Ann}(\Phi_m(t)^k)$ and $(\Phi_m(t))$ in $A = \mathbb{Z}_p[t^{\mathbb{Z}/n\mathbb{Z}}]$. Prove $M = IM$ and apply NAK's lemma.

Lemma. If $v \in \widehat{\mathbb{Z}}^*$, then $\Phi_m(t^v)/\Phi_m(t) \in \widehat{\Lambda}$.

Proof of Thm 1: By these Lemmas, we can cancel common cyclotomic divisors. ■

Results on Twisted Alexander polynomials

Question.

Let $F/\mathbb{Q}$ be a finite extension, S a finite set of prime ideals of the integer ring of F. Let $O = O_{F,S}$ denote the ring of S-integers of F and suppose that O is a UFD. Consider the set of knot group representations $\rho : \pi_K \to \mathrm{GL}_n(O)$. Does the map $\{(\pi_K, \rho)\}/\cong \ \longrightarrow \ \{\Delta_{K,\rho}(t) \in O[t^{\mathbb{Z}}]\}/\doteq$ factor through the set $\{(\widehat{\pi}_K, \widehat{\rho})\}/\cong$?

If the answer is Yes, we say $(\widehat{\pi}_K, \widehat{\rho})$ *determines* $\Delta(t) = \Delta_{K,\rho}(t)$.

Theorem 2. [U.]

1. If $\Delta(t)$ is reciprocal, then *Yes*, up to conjugate of $F/\mathbb{Q}$.
2. If $\mathbb{Q}(\zeta_n)$ is the maximal cyclotomic field contained in the minimal decomposition field of $\Delta(t)$, then for almost all prime numbers p with $n|(p-1)$, the roots of $\Delta(t)$ is determined up to multiplication by roots of $(p-1)$-th roots of unity.
3. If the cyclotomic polynomial $\Phi_m(t)$ decomposes in $O[t]$ and its divisors are partially contained in $\Delta(t)$, then we cannot distinguish them.

$\Delta(t)$ is not necessarily reciprocal, but the criterion is known (cf. [Hillman2012]).
E.g., if ρ is an SL_2 representation then $\Delta(t)$ is reciprocal.
The proof of Theorem 2 reduces to

"does $(\Delta(t^v)) \subset \widehat{O}[[t^{\widehat{\mathbb{Z}}}]]$ with unknown unit v of $\widehat{\mathbb{Z}}$ determine $\Delta(t) \in O[t]$?"

Proofs of Theorem 2 on twisted Alexander polynomials

By the universality of profinite completions, ρ induces $\widehat{\rho}: \widehat{\pi}_K \to \mathrm{GL}_n(\widehat{O})$.
The continuous homology $\widehat{H}_1(\widehat{\pi}_K, \widehat{\rho})$ is a finitely generated $\widehat{\Lambda}_O := \widehat{O}[[t^{\widehat{\mathbb{Z}}}]]$-module, and we obtain $\mathrm{Fitt}_{\widehat{\Lambda}_O} = (\Delta_{K,\rho}(t^v))$ for some unit v of $\widehat{\mathbb{Z}}$.

Thus The proof reduces to "does $(\Delta(t^v)) \subset \widehat{O}[[t^{\widehat{\mathbb{Z}}}]]$ determine $\Delta(t) \in O[t]$?"
Assrsion ① follows from Theorem 1 by the norm map $\mathrm{Nr}_{F/\mathbb{Q}}$ on $O[t]$.
(Recall that Fried's proposition needs $\Delta(t) \in \mathbb{R}[t]$.) □

Fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ for each prime number p.
The following Lemmas prove Assersion ②.

Lemma. If $0 \neq \alpha \in \overline{\mathbb{Q}}$, then $|\alpha|_p = 1$ for almost all p. If $|\alpha|_p = 1$, then there exists a unique root of unity ζ with $|\alpha - \zeta|_p < 1$.
Let $\mathbb{Q}(\zeta_n)$ be the maximal cyclotomic field contained in the normal closure of $\mathbb{Q}(\alpha)$, then for almost all p with $n|(p-1)$, $\alpha \in \mathbb{Q}_p$, $|\alpha|_p = 1$ and $|\alpha^{p-1} - 1| < 1$.

Lemma. If an ideal of $\widehat{O}[[t^{\widehat{\mathbb{Z}}}]]$ is generated by unknown $f(t) \in O[t^{\mathbb{Z}}]$, then for each $m \in \mathbb{N}$, we can detect all the α^m's satisfying $|\alpha^m - 1|_p < 1$ with multiplicity for roots $\alpha \in \overline{\mathbb{Q}}$ of $f(t)$.

Idea: To consider the completed Alexander module for $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p$-cover.
We have $\mathbb{Z}_p[[t^{\widehat{\mathbb{Z}}}]] \twoheadrightarrow \mathbb{Z}_p[[t^{\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p}]]$. If we put $g(t') = g(t^m) = \prod_{\zeta^m=1} f(t\zeta)$, then we have a correspondence of principal ideals of $\mathbb{Z}_p[[t^{\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p}]]$ and $\mathbb{Z}_p[[t'^{\mathbb{Z}_p}]]$ by $(f(t)) \mapsto (g(t'))$. Recall the Iwasawa isomorphism $\mathbb{Z}_p[[t'^{\mathbb{Z}_p}]] \cong \mathbb{Z}_p[[T]]; t' \mapsto 1 + T$.
Since $\mathbb{Z}_p[[T]]$ has convergence radius 1, we obtain α^m's with $|\alpha^m - 1|_p < 1$. ■

Results and question

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Questions

1. Does $\rho : \widehat{\pi}_K \to \mathrm{GL}_n(\widehat{O})$ appear in any natural topological setting?
2. Can we find a representation with any topological meaning in the set $\{\widehat{\rho} : \widehat{\pi}_K \to \mathrm{GL}_n(\widehat{O})\}$? (E.g. the $\widehat{\rho}$ of a holonomy representation?)