KHOVANOV HOMOLOGY OF SEMIADEQUATE LINKS VIA PRESIMPLICIAL SETS

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[JOINT WORK WITH JÓZEF H. PRZYTYCKI]

WINTERBRAIDS VIII

FEBRUARY 6 - LUMINY (FRANCE)

Why to study Khovanov homology?

(Khovanov, 2000)

- It is a bigraded homology $H^{i,j}(L)$.

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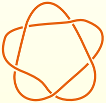

$H^{0,-3}(5_1) = \mathbb{Z}$
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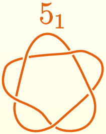

i →, j ↓

j \ i	-5	-4	-3	-2	-1	0
-3						
-5						
-7				$\mathbb{Z}$		
-9						
-11						
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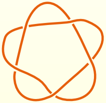

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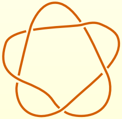

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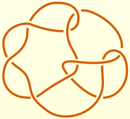

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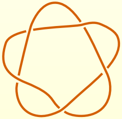

$$V(L) = \sum_{i,j} q^j (-1)^i \operatorname{rk}(H^{i,j}(L)).$$

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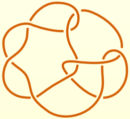

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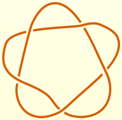

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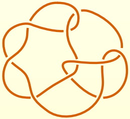

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−5						$\mathbb{Q}$
−7				$\mathbb{Q}$		
−9						
−11		$\mathbb{Q}$	$\mathbb{Q}$			
−13						
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	−7	−6	−5	−4	−3	−2	−1	0
−1							$\mathbb{Q}$	$\mathbb{Q}$
−3								$\mathbb{Q}$
−5					$\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}$		
−7				$\mathbb{Q}$				
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- s-invariant (Rasmussen, 2003)
  - provides a lower bound on the slice genus of a knot.
  - gives a combinatorial proof of the (topological) **Milnor conjecture**: $g_s(T(p,q)) = \frac{1}{2}(p-1)(q-1)$.

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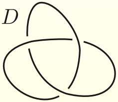

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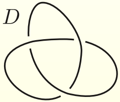

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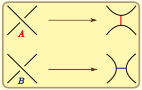

Kauffman states

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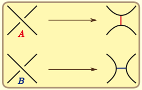

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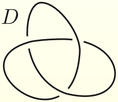

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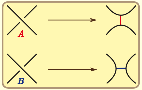

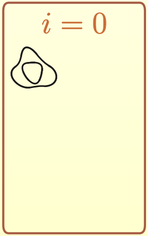

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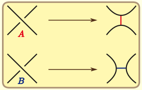

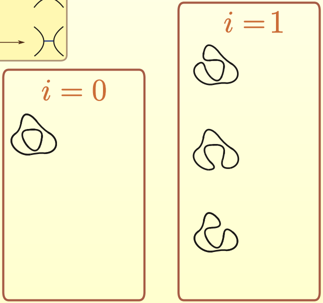

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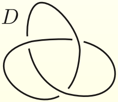

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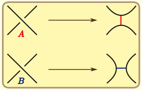

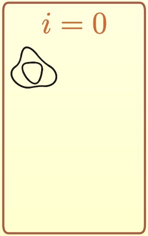

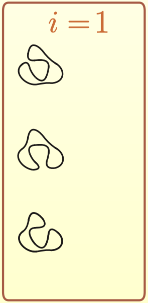

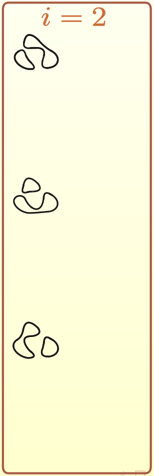

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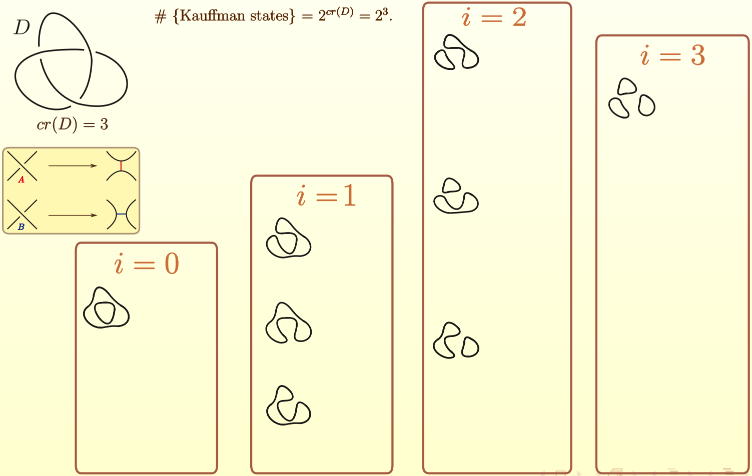

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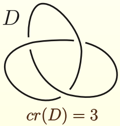

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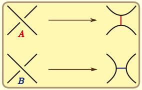

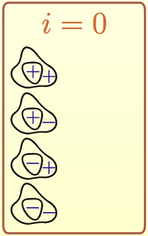

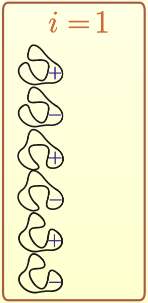

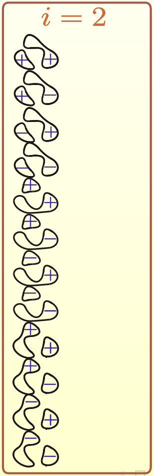

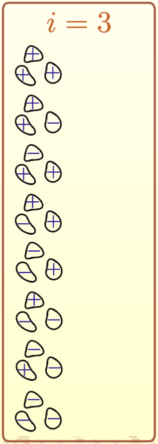

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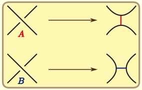

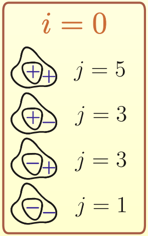

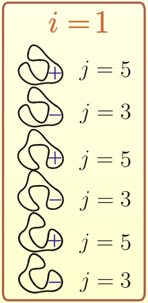

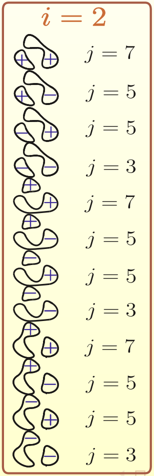

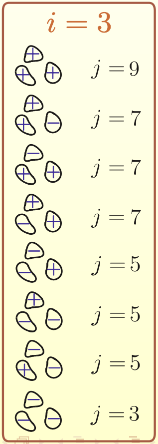

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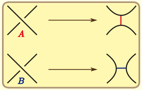
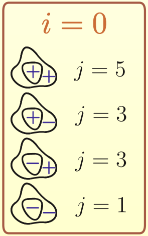
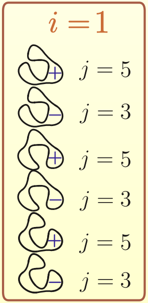
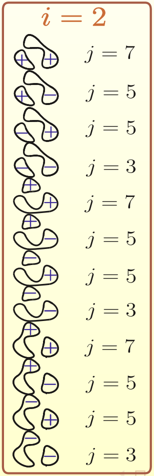
Generators

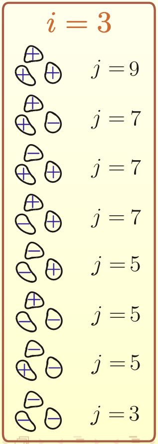
$$C^{i,j}(D) = \bigoplus_{\substack{i(s)=i \\ j(s)=j}} \mathbb{Z}\, s$$











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We look for new approaches to Khovanov homology.

Almost extreme Khovanov complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C_{i+2,j+4}(D) \arrow[r] & C_{i,j+4}(D) \arrow[r] & C_{i-2,j+4}(D) \arrow[r] & \cdots \\
\cdots \arrow[r] & C_{i+2,j}(D) \arrow[r] & C_{i,j}(D) \arrow[r] & C_{i-2,j}(D) \arrow[r] & \cdots \\
\cdots \arrow[r] & C_{i+2,j-4}(D) \arrow[r] & C_{i,j-4}(D) \arrow[r] & C_{i-2,j-4}(D) \arrow[r] & \cdots \\
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\end{tikzcd}$$

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\cdots \arrow[r] & C_{i+2,j_{\max}}(D) \arrow[r] & C_{i,j_{\max}}(D) \arrow[r] & C_{i-2,j_{\max}}(D) \arrow[r] & \cdots & \text{Extreme Khovanov} \\
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & C_{i+2,j+4}(D) \arrow[r] & C_{i,j+4}(D) \arrow[r] & C_{i-2,j+4}(D) \arrow[r] & \cdots & \\
\cdots \arrow[r] & C_{i+2,j}(D) \arrow[r] & C_{i,j}(D) \arrow[r] & C_{i-2,j}(D) \arrow[r] & \cdots & \\
\cdots \arrow[r] & C_{i+2,j-4}(D) \arrow[r] & C_{i,j-4}(D) \arrow[r] & C_{i-2,j-4}(D) \arrow[r] & \cdots & \\
 & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

$$\boxed{j_{\max}(D) = \max_{s \text{ state of } D}\{j(s)\}}$$

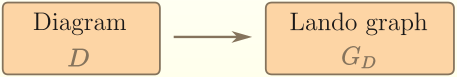
$D \longrightarrow I_D$ simplicial complex / $H_i(I_D) = H^{i,j_{max}}(D)$

Extreme Khovanov homology

Diagram

D

Extreme Khovanov homology



Diagram

D

Lando graph

G_D

Extreme Khovanov homology

Diagram D → Lando graph G_D → Independent simpl. complex I_D

Extreme Khovanov homology

Diagram D → Lando graph G_D → Independent simpl. complex I_D

I

$$D \longrightarrow I_D$$

Extreme Khovanov homology

Diagram D → Lando graph G_D → Independent simpl. complex I_D

THEOREM [González-Meneses, Manchón, S.]

$$D \longrightarrow I_D$$

$$\{C_{i,j_{max}}(D), d_i\} \quad \sim_h \quad \{C_i(I_D), \partial_i\}$$

Extreme Khovanov homology

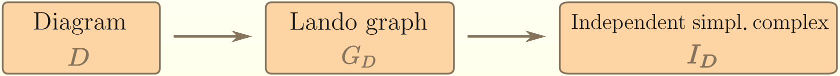


Diagram D → Lando graph G_D → Independent simpl. complex I_D

THEOREM [González-Meneses, Manchón, S.]

$$D \longrightarrow I_D$$

$$\{C_{i,j_{max}}(D), d_i\} \sim_h \{C_i(I_D), \partial_i\}$$

$$\boxed{H_{i,j_{max}}(D) \underset{\text{some shifting}}{=} \widetilde{H}_i(I_D)}$$

Almost extreme Khovanov complex

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+2,j_{\max}}(D) \arrow[r] & C_{i,j_{\max}}(D) \arrow[r] & C_{i-2,j_{\max}}(D) \arrow[r] & \cdots & \text{Extreme Khovanov} \\
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & C_{i+2,j+4}(D) \arrow[r] & C_{i,j+4}(D) \arrow[r] & C_{i-2,j+4}(D) \arrow[r] & \cdots & \\
\cdots \arrow[r] & C_{i+2,j}(D) \arrow[r] & C_{i,j}(D) \arrow[r] & C_{i-2,j}(D) \arrow[r] & \cdots & \\
\cdots \arrow[r] & C_{i+2,j-4}(D) \arrow[r] & C_{i,j-4}(D) \arrow[r] & C_{i-2,j-4}(D) \arrow[r] & \cdots & \\
 & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

$$\boxed{j_{\max}(D) = \max_{s \text{ state of } D}\{j(s)\}}$$

$$D \longrightarrow I_D \ \text{simplicial complex} \ / \ H_i(I_D) = H^{i,j_{max}}(D)$$

Almost extreme Khovanov complex

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+2,j_{\max}}(D) \arrow[r] & C_{i,j_{\max}}(D) \arrow[r] & C_{i-2,j_{\max}}(D) \arrow[r] & \cdots & \text{Extreme Khovanov} \\
\cdots \arrow[r] & C_{i+2,j_{almax}}(D) \arrow[r] & C_{i,j_{almax}}(D) \arrow[r] & C_{i-2,j_{almax}}(D) \arrow[r] & \cdots & \text{Almost-extreme Khovanov} \\
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & C_{i+2,j+4}(D) \arrow[r] & C_{i,j+4}(D) \arrow[r] & C_{i-2,j+4}(D) \arrow[r] & \cdots & \\
\cdots \arrow[r] & C_{i+2,j}(D) \arrow[r] & C_{i,j}(D) \arrow[r] & C_{i-2,j}(D) \arrow[r] & \cdots & \\
\cdots \arrow[r] & C_{i+2,j-4}(D) \arrow[r] & C_{i,j-4}(D) \arrow[r] & C_{i-2,j-4}(D) \arrow[r] & \cdots & \\
 & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

$$j_{\max}(D) = \max_{s \text{ state of } D}\{j(s)\}$$

$$j_{almax}(D) = j_{\max}(D) - 4$$

$D \longrightarrow I_D$ simplicial complex / $H_i(I_D) = H^{i,j_{max}}(D)$

Almost extreme Khovanov complex

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+2,j_{\max}}(D) \arrow[r] & C_{i,j_{\max}}(D) \arrow[r] & C_{i-2,j_{\max}}(D) \arrow[r] & \cdots & \text{Extreme Khovanov} \\
\cdots \arrow[r] & C_{i+2,j_{almax}}(D) \arrow[r] & C_{i,j_{almax}}(D) \arrow[r] & C_{i-2,j_{almax}}(D) \arrow[r] & \cdots & \text{Almost-extreme Khovanov} \\
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & C_{i+2,j+4}(D) \arrow[r] & C_{i,j+4}(D) \arrow[r] & C_{i-2,j+4}(D) \arrow[r] & \cdots & \\
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 & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

$$j_{\max}(D) = \max_{s \text{ state of } D}\{j(s)\} \qquad j_{almax}(D) = j_{\max}(D) - 4$$

$D \longrightarrow I_D$ simplicial complex / $H_i(I_D) = H^{i,j_{max}}(D)$

Is it possible to find something similar for $H^{i,j_{almax}}(D)$?

Almost extreme Khovanov complex

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+2,j_{\max}}(D) \arrow[r] & C_{i,j_{\max}}(D) \arrow[r] & C_{i-2,j_{\max}}(D) \arrow[r] & \cdots & \text{Extreme Khovanov} \\
\cdots \arrow[r] & C_{i+2,j_{almax}}(D) \arrow[r] & C_{i,j_{almax}}(D) \arrow[r] & C_{i-2,j_{almax}}(D) \arrow[r] & \cdots & \text{Almost-extreme Khovanov} \\
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & C_{i+2,j+4}(D) \arrow[r] & C_{i,j+4}(D) \arrow[r] & C_{i-2,j+4}(D) \arrow[r] & \cdots & \\
\cdots \arrow[r] & C_{i+2,j}(D) \arrow[r] & C_{i,j}(D) \arrow[r] & C_{i-2,j}(D) \arrow[r] & \cdots & \\
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 & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

$$j_{\max}(D) = \max_{s \text{ state of } D}\{j(s)\}$$

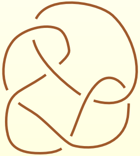
$$j_{almax}(D) = j_{\max}(D) - 4$$

$D \longrightarrow I_D$ simplicial complex / $H_i(I_D) = H^{i,j_{max}}(D)$

Is it possible to find something similar for $H^{i,j_{almax}}(D)$? "YES"

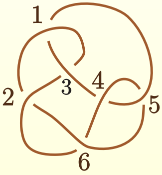
Semiadequate links

D



Semiadequate links

D

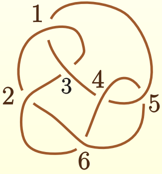


Kauffman state:

$s : cr(D) \to \{A, B\}$

Semiadequate links

D



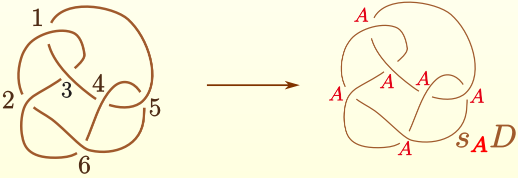
s_A := state where all crossings are marked with A-label.

Kauffman state:

$s : cr(D) \to \{A, B\}$

Semiadequate links

$$D \longrightarrow s_A$$



$s_A :=$ state where all crossings are marked with A-label.

Kauffman state:

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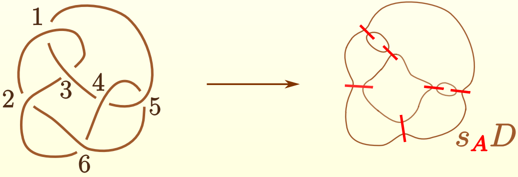
Semiadequate links

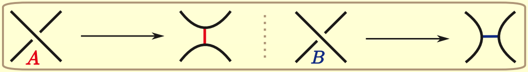
$$D \longrightarrow s_A$$



$s_A :=$ state where all crossings are marked with A-label.

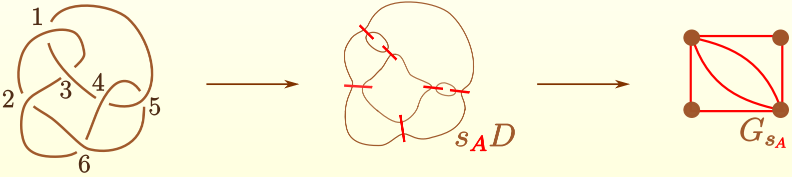
Kauffman state:

$s : cr(D) \to \{A, B\}$



Semiadequate links

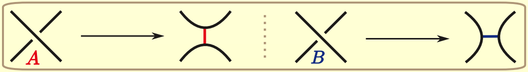
$D \rightarrow s_A \rightarrow G_{s_A}$



s_A := state where all crossings are marked with A-label.

Kauffman state:

$s : cr(D) \rightarrow \{A, B\}$



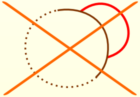
Semiadequate links

$$\begin{tikzcd} D \arrow[r] & s_A \arrow[r] & G_{s_A} \end{tikzcd}$$

Semiadequate links

$$\begin{tikzcd} \boxed{D} \arrow[r] & \boxed{s_A} \arrow[r] & \boxed{G_{s_A}} \end{tikzcd}$$

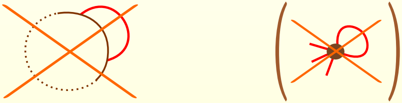
D is A-adequate if



Semiadequate links

$$\begin{tikzcd} \boxed{D} \arrow[r] & \boxed{s_A} \arrow[r] & \boxed{G_{s_A}} \end{tikzcd}$$

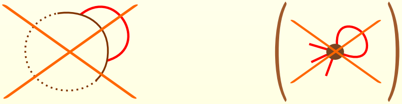
D is A-adequate if



Semiadequate links

$$D \longrightarrow s_A \longrightarrow G_{s_A}$$

D is A-adequate if

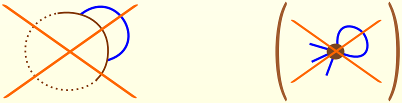


$$D \quad A\text{-adequate} \longrightarrow L \quad A\text{-adequate}$$

Semiadequate links

$$\begin{tikzcd} \boxed{D} \arrow[r] & \boxed{s_B} \arrow[r] & \boxed{G_{s_B}} \end{tikzcd}$$

D is B-adequate if

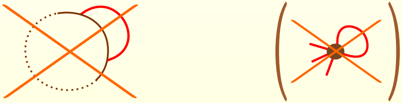


D B-adequate ⟶ L B-adequate

Semiadequate links

$$D \longrightarrow s_A \longrightarrow G_{s_A}$$

D is A-adequate if

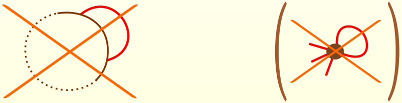


D A-adequate $\longrightarrow$ L A-adequate

Semiadequate links

$$D \longrightarrow s_A \longrightarrow G_{s_A}$$

D is A-adequate if



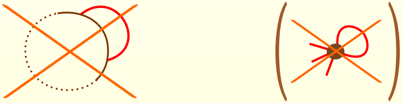
D A-adequate $\longrightarrow$ L A-adequate

Semiadequate := A-adequate or B-adequate

Semiadequate links

$$D \longrightarrow s_A \longrightarrow G_{s_A}$$

D is A-adequate if



D A-adequate $\longrightarrow$ L A-adequate

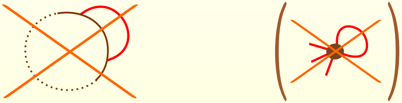
Semiadequate := A-adequate or B-adequate

Adequate := A-adequate and B-adequate

Semiadequate links

$$D \longrightarrow s_A \longrightarrow G_{s_A}$$

D is A-adequate if



D A-adequate $\longrightarrow$ L A-adequate

Semiadequate := A-adequate or B-adequate

Adequate := A-adequate and B-adequate

(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} \\ d_{i,n} = d_i : \quad X_n \quad \to X_{n-1} \qquad 0 \leq \quad i \quad \leq n \\ \end{cases}$$

(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} \\ d_{i,n} = d_i\colon \quad X_n \quad \to X_{n-1} \qquad\qquad 0 \leq i < j \leq n \\ \qquad\qquad d_i d_j = d_{j-1} d_i \end{cases}$$

(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} \\ d_{i,n} = d_i \colon \quad X_n \quad \to X_{n-1} \qquad\qquad 0 \leq i < j \leq n \\ \qquad d_i d_j = d_{j-1} d_i \end{cases}$$

Geometric realization

(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} \\ d_{i,n} = d_i \colon \quad X_n \quad \to X_{n-1} \qquad\qquad 0 \leq i < j \leq n \\ \qquad\quad d_i d_j = d_{j-1} d_i \end{cases}$$

Geometric realization:

$$\Delta^n = \Big\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, \ x_i \geq 0 \Big\}$$

$$d^{i,n} = d^i \colon \Delta^{n-1} \to \Delta^n, \qquad d^i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

(Partial) presimplicial sets

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$$\boxed{|\mathcal{X}| = \bigsqcup_{n\geq 0} X_n \times \Delta^n \big/ \sim}$$

(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} \\ d_{i,n} = d_i\colon \quad X_n \quad \to X_{n-1} & \quad 0 \leq i < j \leq n \\ \qquad d_i d_j = d_{j-1} d_i \end{cases}$$

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"gluing instructions"

↳ $(a, d^i(t)) \sim (d_i(a), t), \qquad a \in X_n, \qquad t \in \Delta^{n-1}$

(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} \\ d_{i,n} = d_i \colon \quad X_n \quad \to X_{n-1} \qquad\qquad 0 \leq i < j \leq n \\ \qquad\quad d_i d_j = d_{j-1} d_i \end{cases}$$

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(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} \\ d_{i,n} = d_i : Dom(X_n) \to X_{n-1} & 0 \leq i < j \leq n \\ \quad d_i d_j = d_{j-1} d_i & \text{as long as both sides are defined} \end{cases}$$

Geometric realization:

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \right\}$$

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$$\boxed{|\mathcal{X}| = \bigsqcup_{n\geq 0} X_n \times \Delta^n \big/ \sim}$$

“gluing instructions”

$$\hookrightarrow (a, d^i(t)) \sim (d_i(a), t), \qquad a \in X_n, \qquad t \in \Delta^{n-1}$$

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$$\boxed{|\mathcal{X}| = \bigsqcup_{n\geq 0} X_n \times \Delta^n \Big/ \sim}$$

$a \in X_n$

$t \in \Delta^{n-1}$

"gluing instructions"

$$\to (a, d^i(t)) \sim (d_i(a), t), \qquad \text{if } d_i(a) \neq 0$$

(Partial) presimplicial sets

$$\mathcal{X} = (X_n, d_i) \begin{cases} \{X_n\}_{n\geq 0} & \\ d_{i,n} = d_i : Dom(X_n) \to X_{n-1} & 0 \leq i < j \leq n \\ \qquad d_i d_j = d_{j-1} d_i & \text{as long as both sides are defined} \end{cases}$$

Geometric realization:

$$\Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \right\}$$

$$d^{i,n} = d^i \colon \Delta^{n-1} \to \Delta^n, \qquad d^i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$

$$\boxed{|\mathcal{X}| = b \cup \bigsqcup_{n\geq 0} X_n \times \Delta^n \big/ \sim}$$

$$a \in X_n \qquad t \in \Delta^{n-1}$$

"gluing instructions"

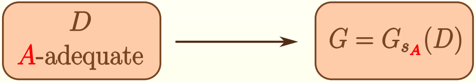
$$(a, d^i(t)) \sim (d_i(a), t), \qquad \text{if } d_i(a) \neq 0$$

$$(a, d^i(t)) \sim \ b, \qquad \text{if } d_i(a) = 0$$

From semiadequate link diagram to presimplicial set

D
A-adequate

From semiadequate link diagram to presimplicial set



D

A-adequate

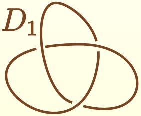
$G = G_{s_A}(D)$

From semiadequate link diagram to presimplicial set

$$\begin{tikzcd}
\boxed{\begin{array}{c} D \\ A\text{-adequate} \end{array}} \arrow[r] & \boxed{G = G_{s_A}(D)} \arrow[r] & \boxed{\mathcal{X}_D = (X_k, d_i)}
\end{tikzcd}$$

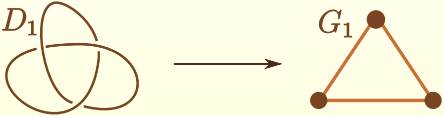
From semiadequate link diagram to presimplicial set

D A-adequate → $G = G_{s_A}(D)$ → $\mathcal{X}_D = (X_k, d_i)$



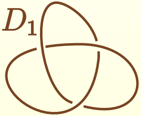
From semiadequate link diagram to presimplicial set

$$\begin{tikzcd} \boxed{\begin{array}{c} D \\ A\text{-adequate} \end{array}} \arrow[r] & \boxed{G = G_{s_A}(D)} \arrow[r] & \boxed{\mathcal{X}_D = (X_k, d_i)} \end{tikzcd}$$

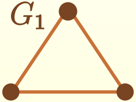


From semiadequate link diagram to presimplicial set

$$\begin{tikzcd} \boxed{\begin{array}{c} D \\ A\text{-adequate} \end{array}} \arrow[r] & \boxed{G = G_{s_A}(D)} \arrow[r] & \boxed{\mathcal{X}_D = (X_k, d_i)} \end{tikzcd}$$



$\longrightarrow$

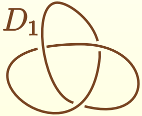


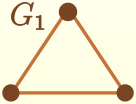
$\longrightarrow$

$$|\mathcal{X}_{D_1}| = \mathbb{R}P^2$$

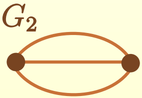
From semiadequate link diagram to presimplicial set

$$\begin{tikzcd} \begin{array}{c} D \\ A\text{-adequate} \end{array} \arrow[r] & G = G_{s_A}(D) \arrow[r] & \mathcal{X}_D = (X_k, d_i) \end{tikzcd}$$





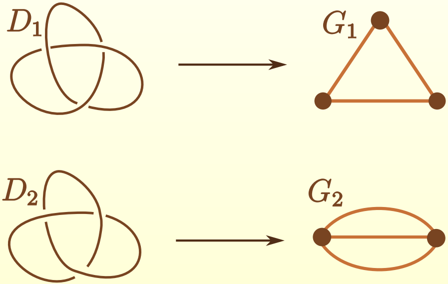
$$|\mathcal{X}_{D_1}| = \mathbb{R}P^2$$



$$|\mathcal{X}_{D_2}| = S^2$$

From semiadequate link diagram to presimplicial set

$$\begin{tikzcd}
\boxed{\begin{array}{c} D \\ A\text{-adequate} \end{array}} \arrow[r] & \boxed{G = G_{s_A}(D)} \arrow[r] & \boxed{\mathcal{X}_D = (X_k, d_i)}
\end{tikzcd}$$



$$\longrightarrow |\mathcal{X}_{D_1}| = \mathbb{R}P^2$$

$$\longrightarrow |\mathcal{X}_{D_2}| = S^2$$

(Construction of $\mathcal{X}_D$ detailed in the poster)

Almost extreme Khovanov homology of semiadequate links

I

D A-adequate

Almost extreme Khovanov homology of semiadequate links

I

$$D \ A\text{-adequate} \xrightarrow{\qquad\qquad} \mathcal{X}_D$$

Almost extreme Khovanov homology of semiadequate links

THEOREM

$$D \ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

Almost extreme Khovanov homology of semiadequate links

THEOREM

$$D \ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some} \\ \text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$

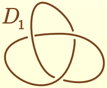
Almost extreme Khovanov homology of semiadequate links

THEOREM

D A-adequate $\longrightarrow$ $\mathcal{X}_D$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some}\\ \text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$



$|\mathcal{X}_{D_1}| = S^2$

$|\mathcal{X}_{D_2}| = \mathbb{R}P^2$

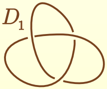
Almost extreme Khovanov homology of semiadequate links

THEOREM

$$D \ \textcolor{red}{A}\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some}\\ \text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$



$$|\mathcal{X}_{D_1}| = S^2$$

j \ i	-3	-1	1	3
7				$\mathbb{Z}$
3				$\mathbb{Z}$
-1		$\mathbb{Z}$		
-5	$\mathbb{Z}/2$			
-9	$\mathbb{Z}$			

$$|\mathcal{X}_{D_2}| = \mathbb{R}P^2$$

j \ i	-3	-1	1	3
9				$\mathbb{Z}$
5			$\mathbb{Z}/2$	
1			$\mathbb{Z}$	
-3	$\mathbb{Z}$			
-7	$\mathbb{Z}$			

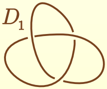
Almost extreme Khovanov homology of semiadequate links

THEOREM

$$D \ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some}\\ \text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$



$$|\mathcal{X}_{D_1}| = S^2$$

i

j	-3	-1	1	3
7				$\mathbb{Z}$
3				$\mathbb{Z}$
-1		$\mathbb{Z}$		
-5	$\mathbb{Z}/2$			
-9	$\mathbb{Z}$			

$$|\mathcal{X}_{D_2}| = \mathbb{R}P^2$$

i

j	-3	-1	1	3
9				$\mathbb{Z}$
5			$\mathbb{Z}/2$	
1			$\mathbb{Z}$	
-3	$\mathbb{Z}$			
-7	$\mathbb{Z}$			

Almost-extreme Khovanov homology II

THEOREM

$$D\ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some}\\ \text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$

Almost-extreme Khovanov homology II

THEOREM

$$D\ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some}\\ \text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$

I

$$D\ A\text{-adequate} \ / \ G = G_{s_A} \text{ simple} \longrightarrow \mathcal{X}_D$$

Almost-extreme Khovanov homology II

THEOREM

$$D \ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\text{some shifting}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$

I

$$\underset{c \text{ crossings}}{D \ A\text{-adequate}} \ / \ \underset{p_1(G) \text{ cyclomatic number}}{G = G_{s_A} \text{ simple}} \longrightarrow \mathcal{X}_D$$

Almost-extreme Khovanov homology II

THEOREM

$$D\ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \sim_h |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some}\\\text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$

I

$$\underset{c\text{ crossings}}{D\ A\text{-adequate}}\ /\ \underset{p_1(G)\text{ cyclomatic number}}{G = G_{s_A}\text{ simple}} \longrightarrow \mathcal{X}_D$$

$$|\mathcal{X}_D| \sim_h \begin{cases} \bigvee_{p_1(G)} S^{c-1} & \text{if } G \text{ is bipartite.} \\ & \end{cases}$$

Almost-extreme Khovanov homology II

THEOREM

$$D \ A\text{-adequate} \longrightarrow \mathcal{X}_D$$

$$\{C_{i,j_{almax}}(D), d_i\} \quad \sim_h \quad |\mathcal{X}_D|$$

$$\boxed{H_{i,j_{almax}}(D) \underset{\substack{\text{some}\\ \text{shifting}}}{=} \widetilde{H}_i(|\mathcal{X}_D|)}$$

THEOREM

$$\underset{c \text{ crossings}}{D \ A\text{-adequate}} \ / \ \underset{p_1(G) \text{ cyclomatic number}}{G = G_{s_A} \text{ simple}} \longrightarrow \mathcal{X}_D$$

$$|\mathcal{X}_D| \sim_h \begin{cases} \bigvee_{p_1(G)} S^{c-1} & \text{if } G \text{ is bipartite.} \\ \bigvee_{p_1(G)-1} S^{c-1} \vee \Sigma^{c-3}\mathbb{R}P^2 & \text{otherwise.} \end{cases}$$

Almost-extreme Khovanov homology III

COROLLARY

D A-adequate / $G = G_{s_A}$ simple

c crossings $p_1(G)$ cyclomatic number

Almost-extreme Khovanov homology III

COROLLARY

D A-adequate / $G = G_{s_A}$ simple

c crossings $p_1(G)$ cyclomatic number

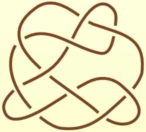
$$H_{i,\underbrace{c+2|s_A|-4}_{j_{almax}}}(D) \simeq \begin{cases} \mathbb{Z}^{p_1} & \text{if } G \text{ is bipartite.} \\ \mathbb{Z}^{p_1-1} \oplus \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

Almost-extreme Khovanov homology III

COROLLARY

$$\underset{c\text{ crossings}}{D\ A\text{-adequate}} \ / \ \underset{p_1(G)\text{ cyclomatic number}}{G = G_{s_A}\text{ simple}}$$

$$H_{i,\underbrace{c+2|s_A|-4}_{j_{almax}}}(D) \simeq \begin{cases} \mathbb{Z}^{p_1} & \text{if } G \text{ is bipartite.} \\ \mathbb{Z}^{p_1-1} \oplus \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$



10_{44}

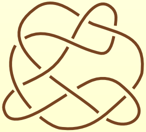
Almost-extreme Khovanov homology III

COROLLARY

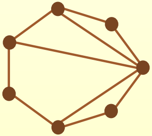
D A-adequate / $G = G_{s_A}$ simple

c crossings $\quad$ $p_1(G)$ cyclomatic number

$$H_{i,\underbrace{c+2|s_A|-4}_{j_{almax}}}(D) \simeq \begin{cases} \mathbb{Z}^{p_1} & \text{if } G \text{ is bipartite.} \\ \mathbb{Z}^{p_1-1} \oplus \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$



10_{44}



G

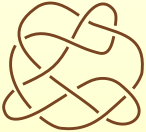
Almost-extreme Khovanov homology III

COROLLARY

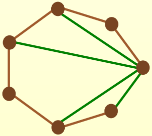
D A-adequate / $G = G_{s_A}$ simple

c crossings $p_1(G)$ cyclomatic number

$$H_{i,\underbrace{c+2|s_A|-4}_{j_{almax}}}(D) \simeq \begin{cases} \mathbb{Z}^{p_1} & \text{if } G \text{ is bipartite.} \\ \mathbb{Z}^{p_1-1} \oplus \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$



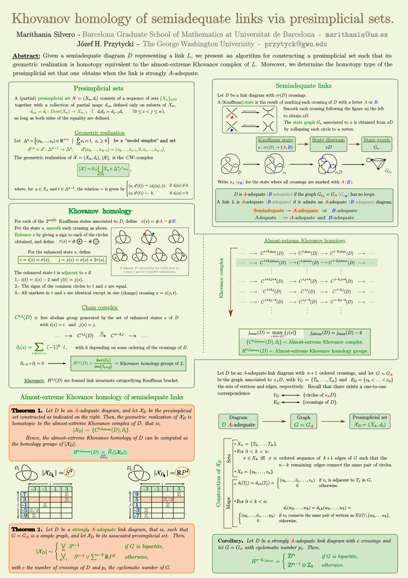
10_{44}



G $p_1 = 4$

$$\boxed{H_{i,20}(10_{44}) = \mathbb{Z}^3 \oplus \mathbb{Z}_2}$$

Poster



Khovanov homology of semiadequate links via presimplicial sets.

Marithania Silvero - Barcelona Graduate School of Mathematics at Universitat de Barcelona - `marithania@us.es`
Józef H. Przytycki - The George Washington Univerisity - `przytyck@gwu.edu`

Abstract: Given a semiadequate diagram D representing a link L, we present an algorithm for constructing a presimplicial set such that its geometric realization is homotopy equivalent to the almost-extreme Khovanov complex of L. Moreover, we determine the homotopy type of the presimplicial set that one obtains when the link is strongly A-adequate.

Presimplicial sets

A (partial) presimplicial set $\mathcal{X} = (X_n, d_i)$ consists of a sequence of sets $\{X_n\}_{n\geq 0}$ together with a collection of partial maps $d_{i,n}$ defined only on subsets of X_n,

$$d_{i,n} = d_i : Dom(X_n) \to X_{n-1} \quad | \quad d_i d_j = d_{j-1} d_i \qquad (0 \leq i < j \leq n),$$

as long as both sides of the equality are defined.

Geometric realization

Let $\Delta^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0\right\}$ be a "model simplex" and set

$$d^{i,n} = d^i : \Delta^{n-1} \to \Delta^n, \qquad d^i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}).$$

The geometric realization of $\mathcal{X} = (X_n, d_i)$, $|\mathcal{X}|$, is the CW-complex

$$\boxed{|\mathcal{X}| = b \cup \bigsqcup_{n\geq 0} X_n \times \Delta^n / \sim},$$

where, for $a \in X_n$ and $t \in \Delta^{n-1}$, the relation $\sim$ is given by $\begin{cases} (a, d^i(t)) \sim (d_i(a), t), & \text{if } d_i(a) \neq 0. \\ (a, d^i(t)) \sim \ b, & \text{if } d_i(a) = 0. \end{cases}$

Khovanov homology

For each of the $2^{cr(D)}$ Kauffman states associated to D, define $\sigma(s) = \#A - \#B$.

For the state s, smooth each crossing as above.

Enhance s by giving a sign to each of the circles obtained, and define $\tau(s) = \#\oplus - \#\ominus$.

For the enhanced state s, define

$$\boxed{i = i(s) = \sigma(s), \qquad j = j(s) = \sigma(s) + 2\tau(s).}$$

A diagram D representing the trefoil knot 3_1, a state s and its 4 possible enhancements.

The enhanced state t is adjacent to s if

1.- $i(t) = i(s) - 2$ and $j(t) = j(s)$.
2.- The signs of the common circles to t and s are equal.
3.- All markers in t and s are identical except in one (change) crossing $x = x(s,t)$.

Chain complex

$C^{i,j}(D) \equiv$ free abelian group generated by the set of enhanced states s of D with $i(s) = i$ and $j(s) = j$.

$$\cdots \longrightarrow C^{i,j}(D) \xrightarrow{\partial_i} C^{i-2,j} \longrightarrow \cdots$$

$$\partial_i(s) = \sum_{t \text{ adjacent to } s} (-1)^k \ t,$$ with k depending on some ordering of the crossings of D.

$$\partial_{i-2} \circ \partial_i = 0 \longrightarrow \boxed{H^{i,j}(D) = \frac{ker(\partial_i)}{im(\partial_{i+2})} =: \text{Khovanov homology groups of } L.}$$

Khovanov: $H^{i,j}(D)$ are framed link invariants categorifying Kauffman bracket.

Almost-extreme Khovanov homology of semiadequate links

Theorem 1. *Let D be an A-adequate diagram, and let $\mathcal{X}_D$ be the presimplicial set constructed as indicated on the right. Then, the geometric realization of $\mathcal{X}_D$ is homotopic to the almost-extreme Khovanov complex of D, that is,*

$$|\mathcal{X}_D| \sim \{C^{i,j_{almax}}(D), \partial_i\}.$$

Hence, the almost-extreme Khovanov homology of D can be computed as the homology groups of $|\mathcal{X}_D|$:

$$H^{i,j_{almax}}(D) \underset{\text{some shifting}}{=} \widetilde{H}_i(|\mathcal{X}_D|).$$

D_1 $|\mathcal{X}_{D_1}| = S^2$

j \ i	-3	-1	1	3
7				$\mathbb{Z}$
3				$\mathbb{Z}$
-1		$\mathbb{Z}$		
-5	$\mathbb{Z}/2$			
-9	$\mathbb{Z}$			

D_2 $|\mathcal{X}_{D_2}| = \mathbb{R}P^2$

j \ i	-3	-1	1	3
9				$\mathbb{Z}$
5			$\mathbb{Z}/2$	
1			$\mathbb{Z}$	
-3	$\mathbb{Z}$			
-7	$\mathbb{Z}$			

Theorem 2. *Let D be a strongly A-adequate link diagram, that is, such that $G = G_A$ is a simple graph, and let $\mathcal{X}_D$ be its associated presimplicial set. Then,*

$$|\mathcal{X}_D| \sim \begin{cases} \bigvee_{p_1} S^{c-1} & \textit{if } G \textit{ is bipartite,} \\ \bigvee_{p_1 - 1} S^{c-1} \vee \sum^{c-3} \mathbb{R}P^2 & \textit{otherwise,} \end{cases}$$

with c the number of crossings of D and p_1 the cyclomatic number of G.

Semiadequate links

Let D be a link diagram with $cr(D)$ crossings.

A (Kauffman) state is the result of marking each crossing of D with a letter A or B.

Smooth each crossing following the figure on the left to obtain sD.

The state graph G_s associated to s is obtained from sD by collapsing each circle to a vertex.

$$\begin{tikzcd} \text{Kauffman state: } s : cr(D) \to \{A, B\} \arrow[r] & \text{State diagram } sD \arrow[r] & \text{State graph } G_s \end{tikzcd}$$

Write s_A (s_B) for the state where all crossings are marked with A (B).

D is A-adequate (B-adequate) if the graph $G_{s_A} = G_A$ (G_{s_B}) has no loops.

A link L is A-adequate (B-adequate) if it admits an A-adequate (B-adequate) diagram.

Semiadequate := A-adequate or B-adequate
Adequate := A-adequate and B-adequate

Almost-extreme Khovanov homology

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & C^{i+2,j_{max}}(D) \arrow[r] & C^{i,j_{max}}(D) \arrow[r] & C^{i-2,j_{max}}(D) \arrow[r] & \cdots \\ \cdots \arrow[r] & C^{i+2,j_{almax}}(D) \arrow[r] & C^{i,j_{almax}}(D) \arrow[r] & C^{i-2,j_{almax}}(D) \arrow[r] & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots \arrow[r] & C^{i+2,j+4}(D) \arrow[r] & C^{i,j+4}(D) \arrow[r] & C^{i-2,j+4}(D) \arrow[r] & \cdots \\ \cdots \arrow[r] & C^{i+2,j}(D) \arrow[r] & C^{i,j}(D) \arrow[r] & C^{i-2,j}(D) \arrow[r] & \cdots \\ \cdots \arrow[r] & C^{i+2,j-4}(D) \arrow[r] & C^{i,j-4}(D) \arrow[r] & C^{i-2,j-4}(D) \arrow[r] & \cdots \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

$$j_{max}(D) = \max_{s \text{ state of } D} \{j(s)\} \qquad j_{almax}(D) = j_{max}(D) - 4$$

$\{C^{i,j_{almax}}(D), \partial_i\} =:$ Almost-extreme Khovanov complex.

$H^{i,j_{almax}}(D) =:$ Almost-extreme Khovanov homology groups.

Let D be an A-adequate link diagram with $n+1$ ordered crossings, and let $G = G_A$ be the graph associated to s_AD, with $V_G = \{T_0, \ldots, T_m\}$ and $E_G = \{v_0 < \ldots < v_n\}$ the sets of vertices and edges, respectively. Recall that there exists a one-to-one correspondence

$$V_G \longleftrightarrow \{\text{circles of } s_AD\}.$$
$$E_G \longleftrightarrow \{\text{crossings of } D\}.$$

$$\begin{tikzcd} \text{Diagram } D \ A\text{-adequate} \arrow[r] & \text{Graph } G = G_A \arrow[r] & \text{Presimplicial set } \mathcal{X}_D = (X_k, d_i) \end{tikzcd}$$

Construction of $\mathcal{X}_D$

Sets
- $X_n = \{T_0, \ldots, T_m\}$.
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- $d_i(T_j) = d_{i,n}(T_j) = \begin{cases} (v_0, \ldots, \hat{v}_i, \ldots, v_n) & \text{if } v_i \text{ is adjacent to } T_j \text{ in } G, \\ 0 & \text{otherwise.} \end{cases}$
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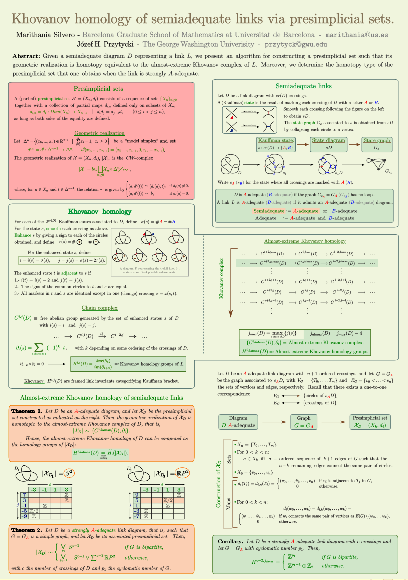

Khovanov homology of semiadequate links via presimplicial sets.

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Presimplicial sets

A (partial) presimplicial set $\mathcal{X} = (X_n, d_i)$ consists of a sequence of sets $\{X_n\}_{n\geq 0}$ together with a collection of partial maps $d_{i,n}$ defined only on subsets of X_n,

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Let $\Delta^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ \middle| \ \sum_{i=0}^{n} x_i = 1, \ x_i \geq 0\right\}$ be a "model simplex" and set

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$$|\mathcal{X}| = b \cup \bigsqcup_{n\geq 0} X_n \times \Delta^n \big/ \sim \ ,$$

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For each of the $2^{cr(D)}$ Kauffman states associated to D, define $\sigma(s) = \#A - \#B$.
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Almost-extreme Khovanov homology of semiadequate links

Theorem 1. *Let D be an A-adequate diagram, and let $\mathcal{X}_D$ be the presimplicial set constructed as indicated on the right. Then, the geometric realization of $\mathcal{X}_D$ is homotopic to the almost-extreme Khovanov complex of D, that is,*

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i

j	-3	-1	1	3
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$$|\mathcal{X}_D| \sim \begin{cases} \bigvee_{p_1} S^{c-1} & \textit{if } G \textit{ is bipartite,} \\ \bigvee_{p_1 - 1} S^{c-1} \vee \sum^{c-3} \mathbb{R}P^2 & \textit{otherwise,} \end{cases}$$

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Let D be a link diagram with $cr(D)$ crossings.
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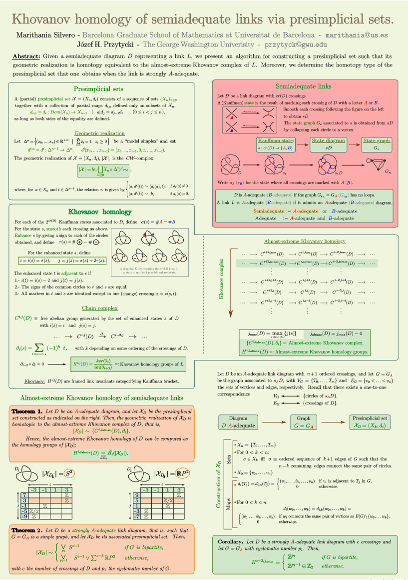

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7				$\mathbb{Z}$
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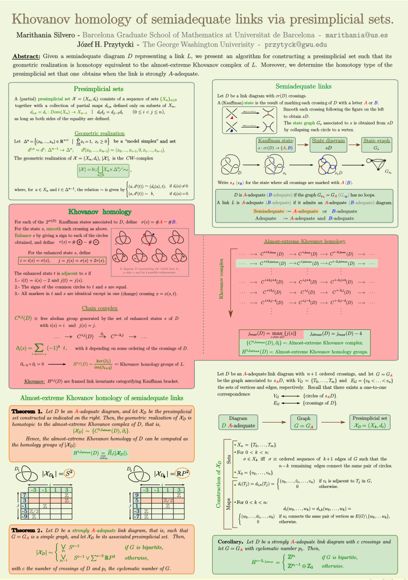

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$$|\mathcal{X}_D| \sim \begin{cases} \bigvee_{p_1} S^{c-1} & \text{if } G \text{ is bipartite,} \\ \bigvee_{p_1-1} S^{c-1} \vee \sum^{c-3} \mathbb{R}P^2 & \text{otherwise,} \end{cases}$$

with c the number of crossings of D and p_1 the cyclomatic number of G.

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Almost-extreme Khovanov homology

Khovanov complex

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{i+2,j_{max}}(D) & \longrightarrow & C^{i,j_{max}}(D) & \longrightarrow & C^{i-2,j_{max}}(D) & \longrightarrow & \cdots \\
\cdots & \longrightarrow & C^{i+2,j_{almax}}(D) & \longrightarrow & C^{i,j_{almax}}(D) & \longrightarrow & C^{i-2,j_{almax}}(D) & \longrightarrow & \cdots \\
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\end{array}$$

$$j_{max}(D) = \max_{s \text{ state of } D} \{j(s)\} \qquad j_{almax}(D) = j_{max}(D) - 4$$

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Let D be an A-adequate link diagram with $n+1$ ordered crossings, and let $G = G_A$ be the graph associated to $s_A D$, with $V_G = \{T_0, \ldots, T_m\}$ and $E_G = \{v_0 < \ldots < v_n\}$ the sets of vertices and edges, respectively. Recall that there exists a one-to-one correspondence

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Corollary. *Let D be a strongly A-adequate link diagram with c crossings and let $G = G_A$ with cyclomatic number p_1. Then,*

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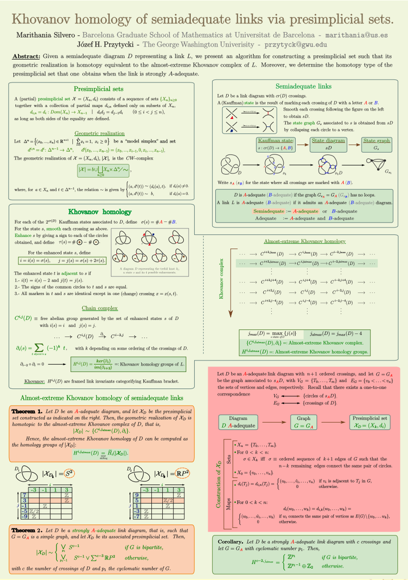

Khovanov homology of semiadequate links via presimplicial sets.

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Presimplicial sets

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Let $\Delta^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0\right\}$ be a "model simplex" and set

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$$\boxed{|\mathcal{X}| = b \cup \bigsqcup_{n\geq 0} X_n \times \Delta^n \big/ \sim},$$

where, for $a \in X_n$ and $t \in \Delta^{n-1}$, the relation $\sim$ is given by $\begin{cases} (a, d^i(t)) \sim (d_i(a), t), & \text{if } d_i(a) \neq 0. \\ (a, d^i(t)) \sim \ b, & \text{if } d_i(a) = 0. \end{cases}$

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For each of the $2^{cr(D)}$ Kauffman states associated to D, define $\sigma(s) = \#A - \#B$.

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$C^{i,j}(D) \equiv$ free abelian group generated by the set of enhanced states s of D with $i(s) = i$ and $j(s) = j$.

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Theorem 1. *Let D be an A-adequate diagram, and let $\mathcal{X}_D$ be the presimplicial set constructed as indicated on the right. Then, the geometric realization of $\mathcal{X}_D$ is homotopic to the almost-extreme Khovanov complex of D, that is,*

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$|\mathcal{X}_{D_1}| = S^2$

$j \backslash i$	-3	-1	1	3
7				$\mathbb{Z}$
3				$\mathbb{Z}$
-1		$\mathbb{Z}$		
-5	$\mathbb{Z}/2$			
-9	$\mathbb{Z}$			

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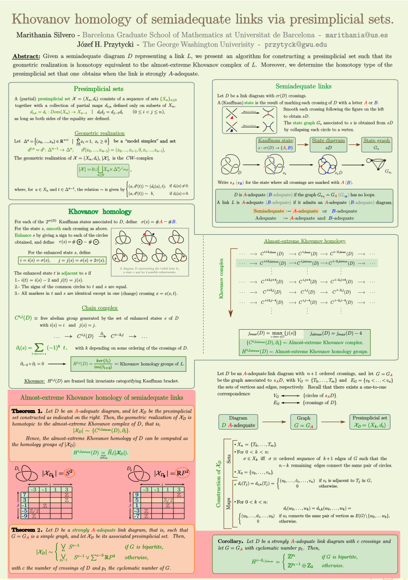

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i

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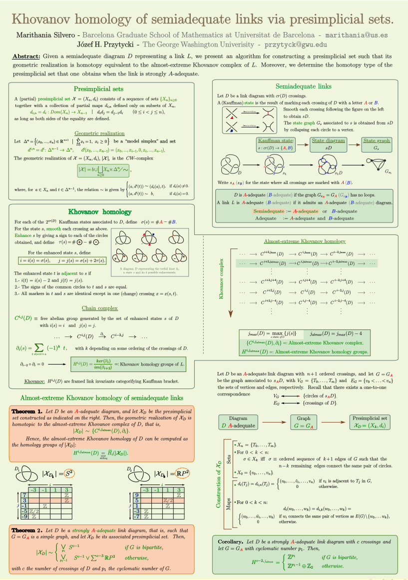

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Geometric realization

Let $\Delta^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0\right\}$ be a "model simplex" and set

$$d^{i,n} = d^i : \Delta^{n-1} \to \Delta^n, \qquad d^i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}).$$

The geometric realization of $\mathcal{X} = (X_n, d_i)$, $|\mathcal{X}|$, is the CW-complex

$$\boxed{|\mathcal{X}| = b \cup \bigsqcup_{n\geq 0} X_n \times \Delta^n \big/ \sim},$$

where, for $a \in X_n$ and $t \in \Delta^{n-1}$, the relation $\sim$ is given by $\begin{cases} (a, d^i(t)) \sim (d_i(a), t), & \text{if } d_i(a) \neq 0. \\ (a, d^i(t)) \sim b, & \text{if } d_i(a) = 0. \end{cases}$

Khovanov homology

For each of the $2^{cr(D)}$ Kauffman states associated to D, define $\sigma(s) = \#A - \#B$.

For the state s, smooth each crossing as above.

Enhance s by giving a sign to each of the circles obtained, and define $\tau(s) = \# \oplus - \# \ominus$.

For the enhanced state s, define

$$\boxed{i = i(s) = \sigma(s), \qquad j = j(s) = \sigma(s) + 2\tau(s).}$$

A diagram D representing the trefoil knot 3_1, a state s and its 4 possible enhancements.

The enhanced state t is adjacent to s if

1.- $i(t) = i(s) - 2$ and $j(t) = j(s)$.
2.- The signs of the common circles to t and s are equal.
3.- All markers in t and s are identical except in one (change) crossing $x = x(s,t)$.

Chain complex

$C^{i,j}(D) \equiv$ free abelian group generated by the set of enhanced states s of D with $i(s) = i$ and $j(s) = j$.

$$\dots \longrightarrow C^{i,j}(D) \xrightarrow{\partial_i} C^{i-2,j} \longrightarrow \dots$$

$$\partial_i(s) = \sum_{t \text{ adjacent to } s} (-1)^k \; t, \qquad \text{with } k \text{ depending on some ordering of the crossings of } D.$$

$$\partial_{i-2} \circ \partial_i = 0 \longrightarrow \boxed{H^{i,j}(D) = \frac{ker(\partial_i)}{im(\partial_{i+2})} =: \text{Khovanov homology groups of } L.}$$

Khovanov: $H^{i,j}(D)$ are framed link invariants categorifying Kauffman bracket.

Semiadequate links

Let D be a link diagram with $cr(D)$ crossings.

A (Kauffman) state is the result of marking each crossing of D with a letter A or B.

Smooth each crossing following the figure on the left to obtain sD.

The state graph G_s associated to s is obtained from sD by collapsing each circle to a vertex.

Kauffman state: $s : cr(D) \to \{A, B\}$ → State diagram sD → State graph G_s

Write s_A (s_B) for the state where all crossings are marked with A (B).

D is A-adequate (B-adequate) if the graph $G_{s_A} = G_A$ (G_{s_B}) has no loops.

A link L is A-adequate (B-adequate) if it admits an A-adequate (B-adequate) diagram.

Semiadequate := A-adequate or B-adequate
Adequate := A-adequate and B-adequate

Almost-extreme Khovanov homology

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i+2,j_{max}}(D) \arrow[r] & C^{i,j_{max}}(D) \arrow[r] & C^{i-2,j_{max}}(D) \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{i+2,j_{almax}}(D) \arrow[r] & C^{i,j_{almax}}(D) \arrow[r] & C^{i-2,j_{almax}}(D) \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{i+2,j+4}(D) \arrow[r] & C^{i,j+4}(D) \arrow[r] & C^{i-2,j+4}(D) \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{i+2,j}(D) \arrow[r] & C^{i,j}(D) \arrow[r] & C^{i-2,j}(D) \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{i+2,j-4}(D) \arrow[r] & C^{i,j-4}(D) \arrow[r] & C^{i-2,j-4}(D) \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Khovanov complex

$$j_{max}(D) = \max_{s \text{ state of } D} \{j(s)\} \qquad j_{almax}(D) = j_{max}(D) - 4$$

$\{C^{i,j_{almax}}(D), \partial_i\} =:$ Almost-extreme Khovanov complex.

$H^{i,j_{almax}}(D) =:$ Almost-extreme Khovanov homology groups.

Almost-extreme Khovanov homology of semiadequate links

Theorem 1. *Let D be an A-adequate diagram, and let $\mathcal{X}_D$ be the presimplicial set constructed as indicated on the right. Then, the geometric realization of $\mathcal{X}_D$ is homotopic to the almost-extreme Khovanov complex of D, that is,*

$$|\mathcal{X}_D| \sim \{C^{i,j_{almax}}(D), \partial_i\}.$$

Hence, the almost-extreme Khovanov homology of D can be computed as the homology groups of $|\mathcal{X}_D|$:

$$H^{i,j_{almax}}(D) \underset{\text{some shifting}}{=} \widetilde{H}_i(|\mathcal{X}_D|).$$

D_1 $|\mathcal{X}_{D_1}| = S^2$

j \ i	-3	-1	1	3
7				$\mathbb{Z}$
3				$\mathbb{Z}$
-1		$\mathbb{Z}$		
-5	$\mathbb{Z}/2$			
-9	$\mathbb{Z}$			

D_2 $|\mathcal{X}_{D_2}| = \mathbb{R}P^2$

j \ i	-3	-1	1	3
9				$\mathbb{Z}$
5			$\mathbb{Z}/2$	
1			$\mathbb{Z}$	
-3	$\mathbb{Z}$			
-7	$\mathbb{Z}$			

Theorem 2. *Let D be a strongly A-adequate link diagram, that is, such that $G = G_A$ is a simple graph, and let $\mathcal{X}_D$ be its associated presimplicial set. Then,*

$$|\mathcal{X}_D| \sim \begin{cases} \bigvee_{p_1} S^{c-1} & \textit{if } G \textit{ is bipartite,} \\ \bigvee_{p_1 - 1} S^{c-1} \vee \sum^{c-3} \mathbb{R}P^2 & \textit{otherwise,} \end{cases}$$

with c the number of crossings of D and p_1 the cyclomatic number of G.

Let D be an A-adequate link diagram with $n+1$ ordered crossings, and let $G = G_A$ be the graph associated to $s_A D$, with $V_G = \{T_0, \dots, T_m\}$ and $E_G = \{v_0 < \dots < v_n\}$ the sets of vertices and edges, respectively. Recall that there exists a one-to-one correspondence

$$V_G \longleftrightarrow \{\text{circles of } s_A D\}.$$
$$E_G \longleftrightarrow \{\text{crossings of } D\}.$$

Diagram D A-adequate → Graph $G = G_A$ → Presimplicial set $\mathcal{X}_D = (X_k, d_i)$

Construction of $\mathcal{X}_D$

Sets

- $X_n = \{T_0, \dots, T_m\}$.
- For $0 < k < n$: $\sigma \in X_k$ iff $\sigma \equiv$ ordered sequence of $k+1$ edges of G such that the $n-k$ remaining edges connect the same pair of circles.
- $X_0 = \{v_0, \dots, v_n\}$.

Maps

- $d_i(T_j) = d_{i,n}(T_j) = \begin{cases} (v_0, \dots, \hat{v}_i, \dots, v_n) & \text{if } v_i \text{ is adjacent to } T_j \text{ in } G, \\ 0 & \text{otherwise.} \end{cases}$
- For $0 < k < n$:

$$d_i(w_0, \dots, w_k) = d_{i,k}(w_0, \dots, w_k) = \begin{cases} (w_0, \dots, \hat{w}_i, \dots, w_k) & \text{if } w_i \text{ connects the same pair of vertices as } E(G) \setminus \{w_0, \dots w_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary. *Let D be a strongly A-adequate link diagram with c crossings and let $G = G_A$ with cyclomatic number p_1. Then,*

$$H^{c-2, j_{almax}} = \begin{cases} \mathbb{Z}^{p_1} & \textit{if } G \textit{ is bipartite,} \\ \mathbb{Z}^{p_1 - 1} \oplus \mathbb{Z}_2 & \textit{otherwise.} \end{cases}$$

Merci beaucoup.