Braided Embeddings

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Introduction

A *braid* is a collection of strands in 3-space going from left to right which are allowed to pass under or over each other.

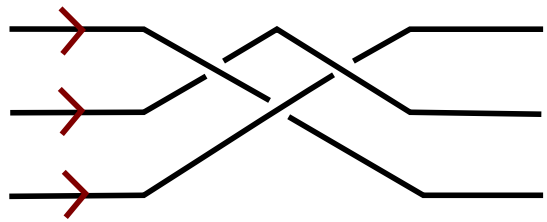

Figure : A braid

If we close up the ends of a braid, then we get a link.

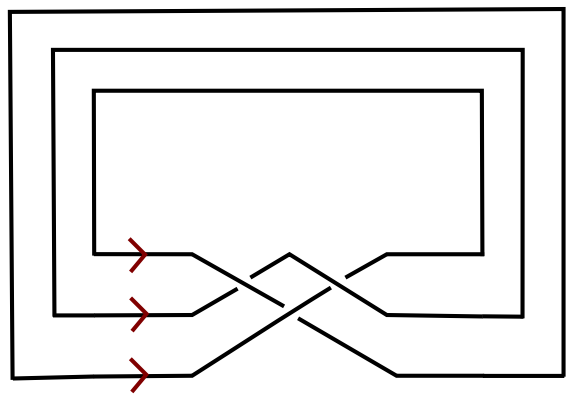

Figure : Closure of the above braid

Introduction

Closing up the ends of a braid gives a link, called a *closed braid*.

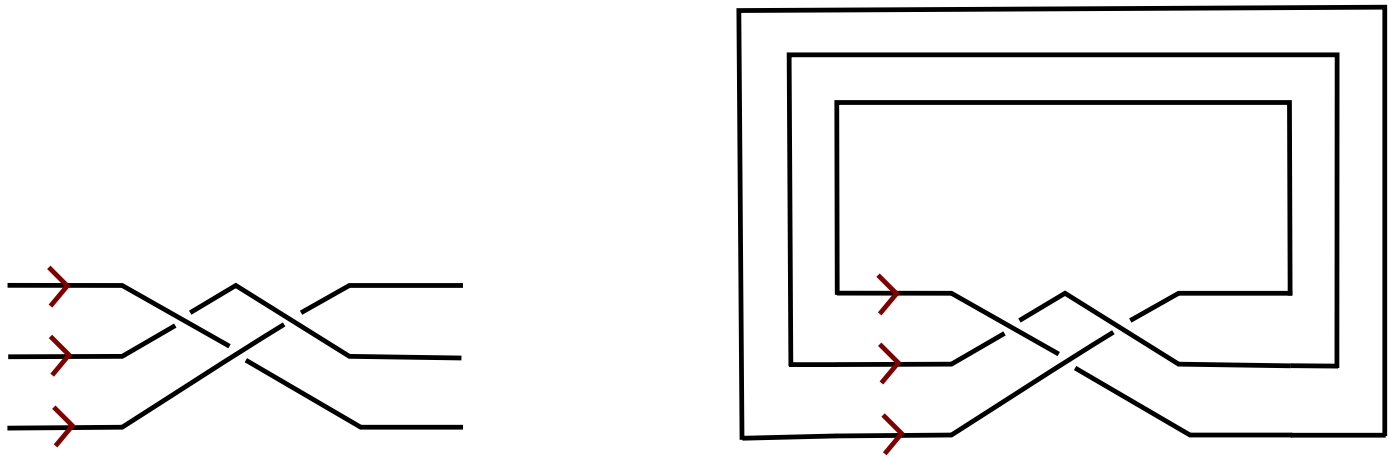

Figure : Closure of a braid

Question

Is every link a closed braid?

Introduction

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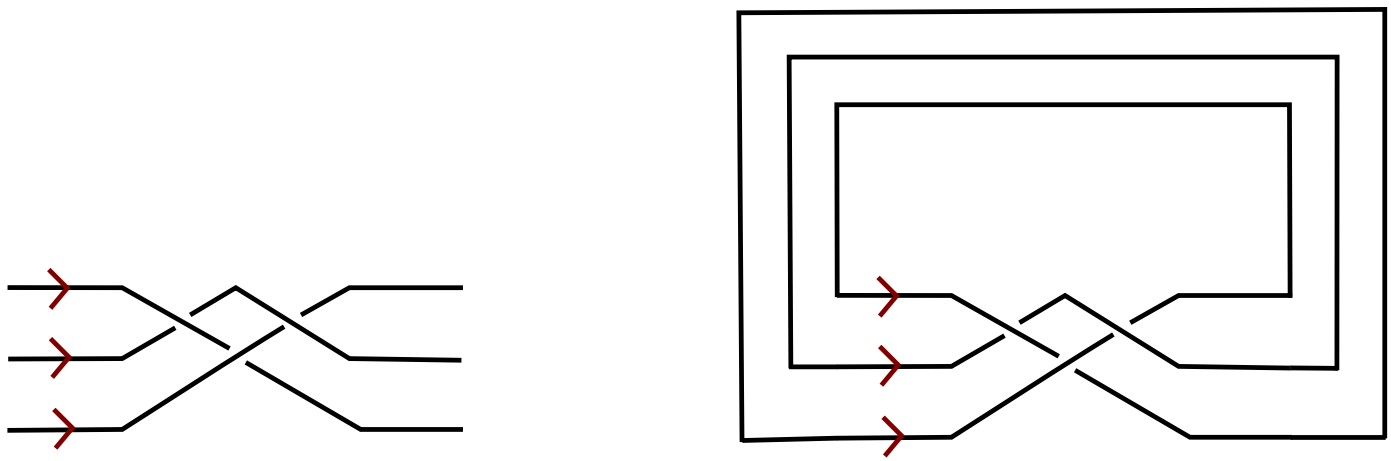


Figure : Closure of a braid

Alexander's Theorem (1923)

Every oriented link in $\mathbb{R}^3$ is isotopic to a closed braid.

The Braid Group B_n (Artin, 1925)

- Group structure on braids with n-strands:
  - multiplication: concatenation
  - identity: braid with no crossings
  - inverse: mirror image

Presentation of braid group

The braid group B_n on n-strands has a presentation

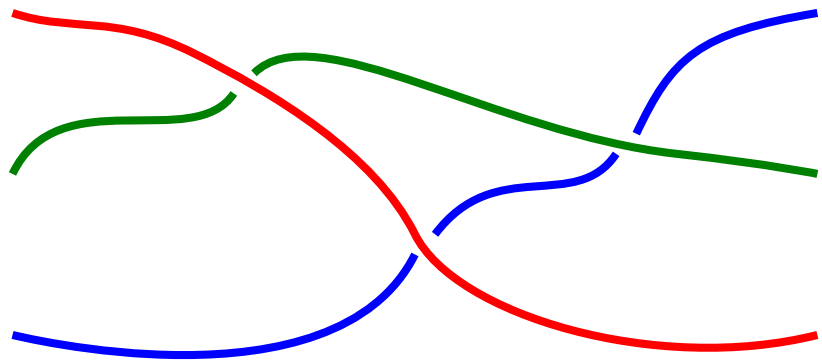
$$B_n = \{\sigma_1, ..., \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad for\ 1 \leq i \leq n-2,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \ if\ |i-j| > 1\}$$

- Forget: $B_n \to S_n$ is a group homomorphism.

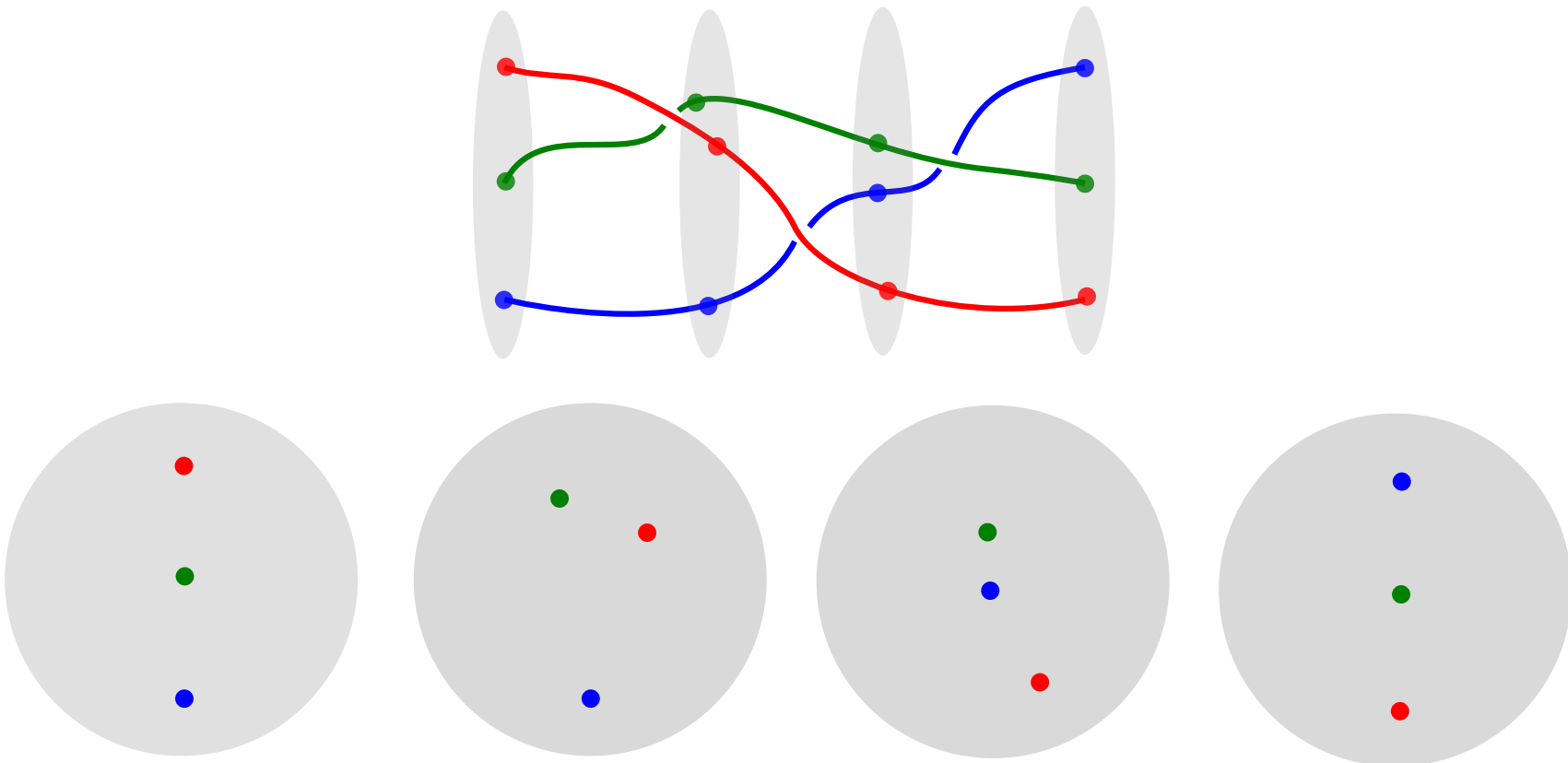
The Braid Group II

Configuration space point of view
(Hurwitz, 1891; Fox-Neuwirth, 1962)



The Braid Group II

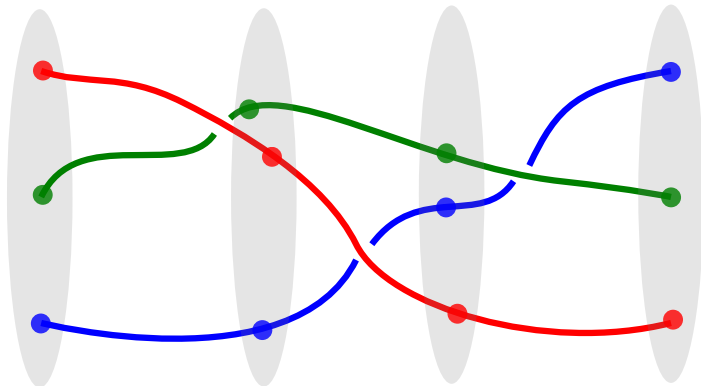
Let's slide a disk along the stands



The Braid Group II

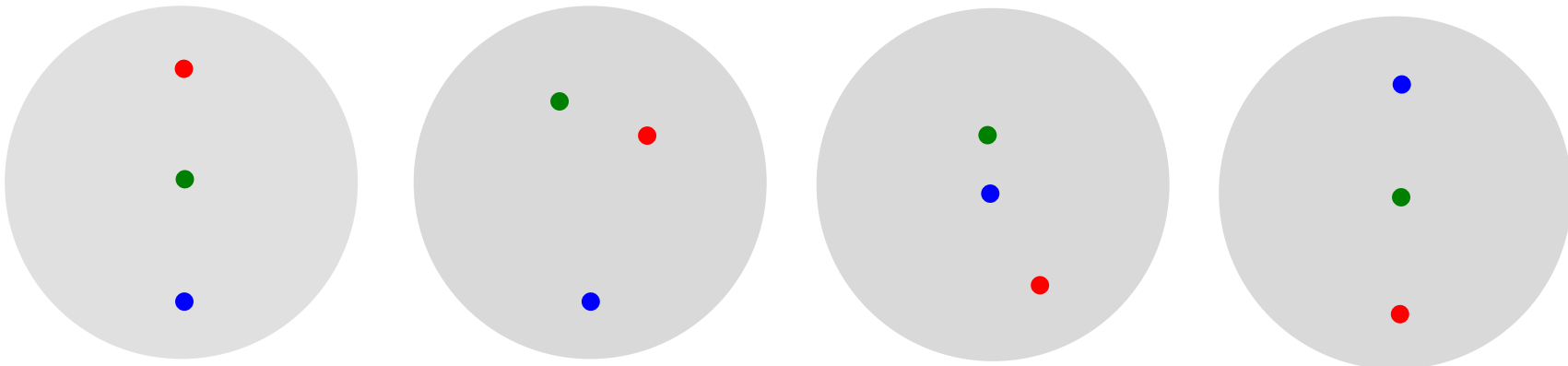
Fox-Neuwirth (1962) moreover showed that:

$UConf_n(D^2) =$
{unordered n-tuple of distinct points in D^2}



$\pi_1(UConf_n(D^2)) = B_n$

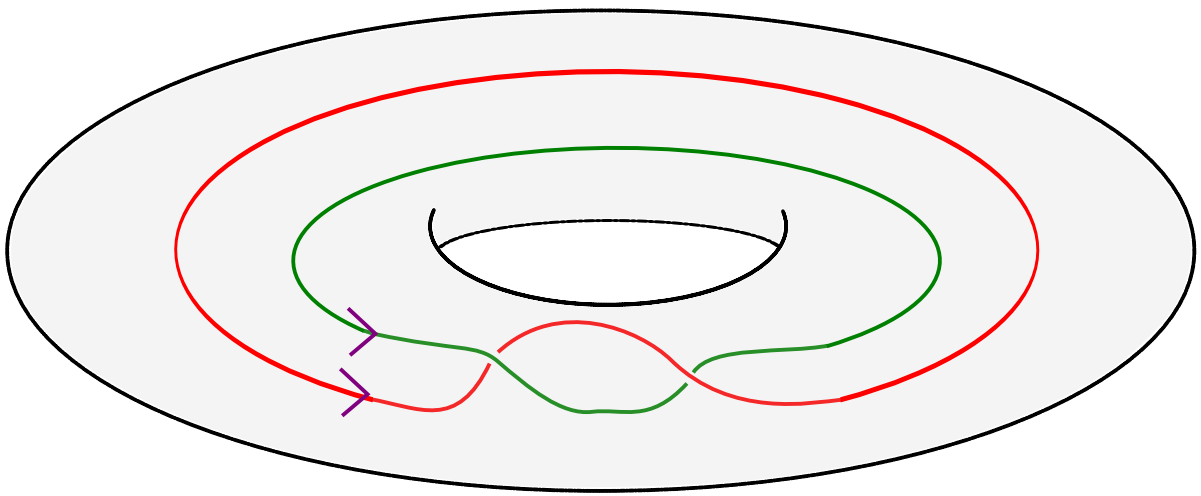
$\pi_i(UConf_n(D^2)) = 1$ for $i \geq 2$



Braided embedding

Definition

An embedding $f : M^k \hookrightarrow N^k \times D^d$ so that $pr_1 \circ f$ is an (oriented) branched covering map will be called a *braided embedding*, and denoted $f : M^k \rightsquigarrow N^k \times D^d$.



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Braided embeddings have been studied by

- Etnyre
- Furukawa
- Carter
- Kamada
- Hilden
- Lozano
- Montesinos
- Viro
- Rudolph
- Hansen
- Petersen
- Melikhov

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Natural Questions

1. (Existence) Is there a braided embedding of M^k in $N^k \times D^d$?
2. (Lifting) Which branched covers can be lifted to braided embeddings?
3. (Isotopy) When can a given embedding of M^k a closed oriented manifold in $\mathbb{R}^{k+2}$ be isotoped to a braided embedding?

Existence

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Theorem (Hilden-Lozano-Montesinos, 1983)

$M^3 \rightsquigarrow S^3 \times D^2$ *for every closed oriented manifold* M^3.

Lifting Covers

Question

What is the smallest d such that a given branched cover $\pi : M^k \to N^k$ lifts to $M^k \rightsquigarrow N^k \times D^d$?

Covering map	smallest such d

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Lifting Covers, $d = 2$

Question

When can a cover $M^k \to N^k$ *be lifted to* $M^k \rightsquigarrow N^k \times D^2$*?*

Lifting Covers, $d = 2$

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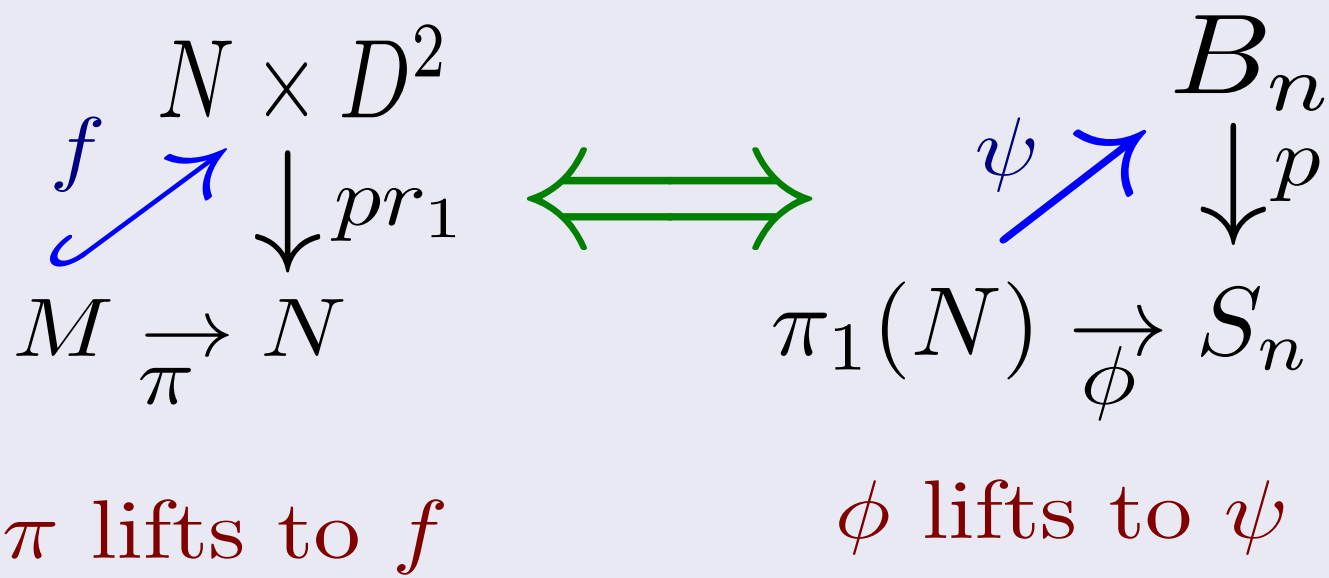
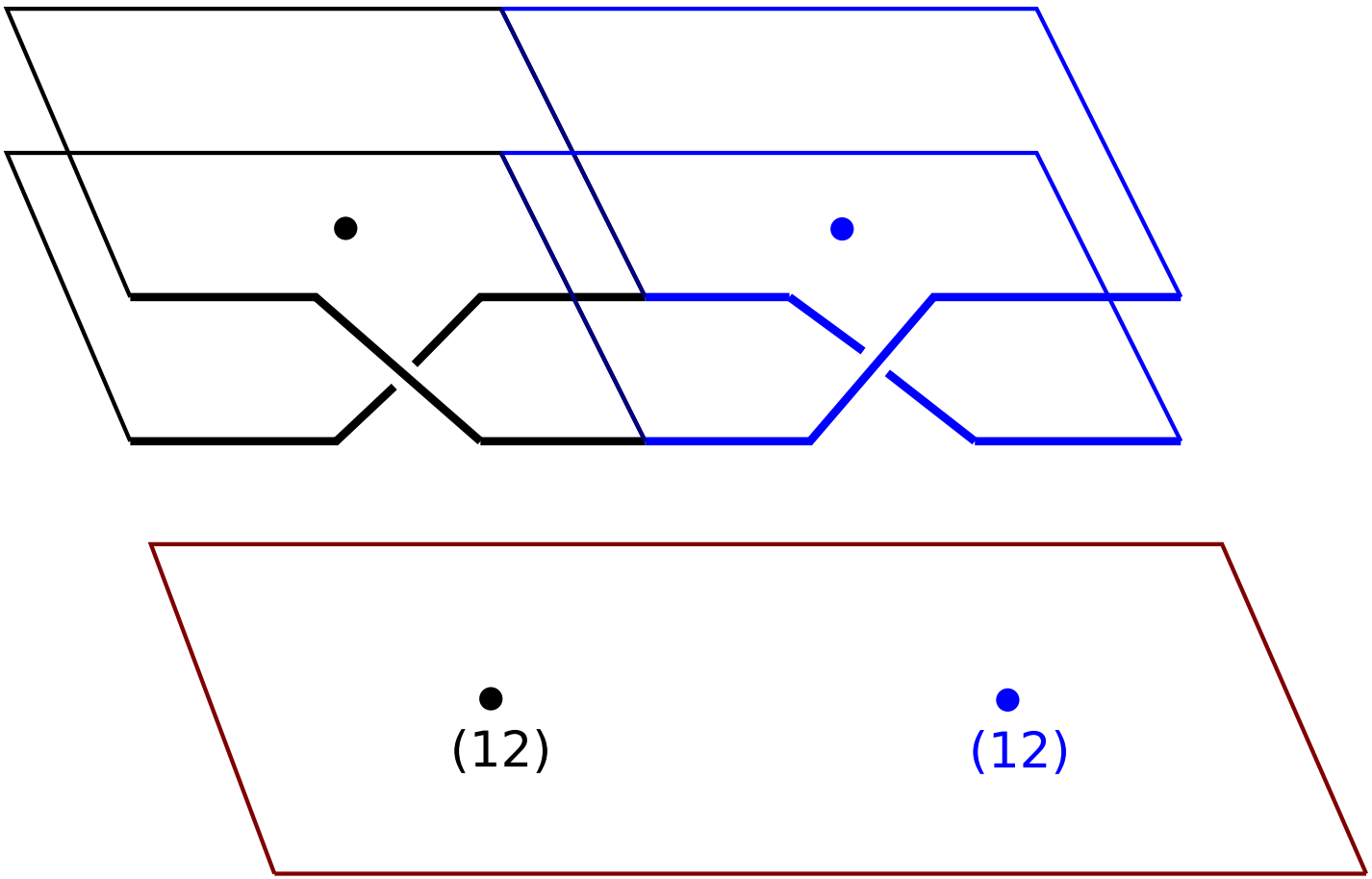
When can a cover $M^k \to N^k$ be lifted to $M^k \rightsquigarrow N^k \times D^2$?

Theorem (Hansen, 1978)



PL Braided Embeddings over S^2

(Kamada) A braid tuple $(\alpha_1, ..., \alpha_n)$ with $\alpha_1 ... \alpha_n = 1$ and each $\hat{\alpha_i}$ is an unknot, descibes a PL braided embeddings over S^2.



Lifting Branched Covers over S^2

Theorem

In the PL category, every branched cover $\pi : M^2 \to S^2$ lifts to $f : M^2 \rightsquigarrow S^2 \times D^2$.

- (Carter and Kamada, 2013) for simple branched covers.
- (K., *2018*) in general.

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This is not true in the smooth category, for example the branched cover $((123), (123), (123))$ does not lift.

Lifting to Braided immersions

Question

When can a branched cover $M^k \to N^k$ be lifted to a co-dimension 2 braided immersion?

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Theorem (Etnyre-Furukawa, 2017)

Let $\pi : M \to N$ be a nice (= branch locus is a submanifold with trivial normal bundle) smooth branched cover. Then π always lifts to a braided immersion $f : M \to N \times D^2$.

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Corollary (Etnyre-Furukawa, 2017)

- *For $k > 1$, $\mathbb{CP}^k$ cannot be a nice branched cover over S^{2k}.*
- *For $k > 7$, $\mathbb{RP}^k$ cannot be a nice branched cover over S^k.*

Braided Embeddings in $\mathbb{R}^{k+2}$

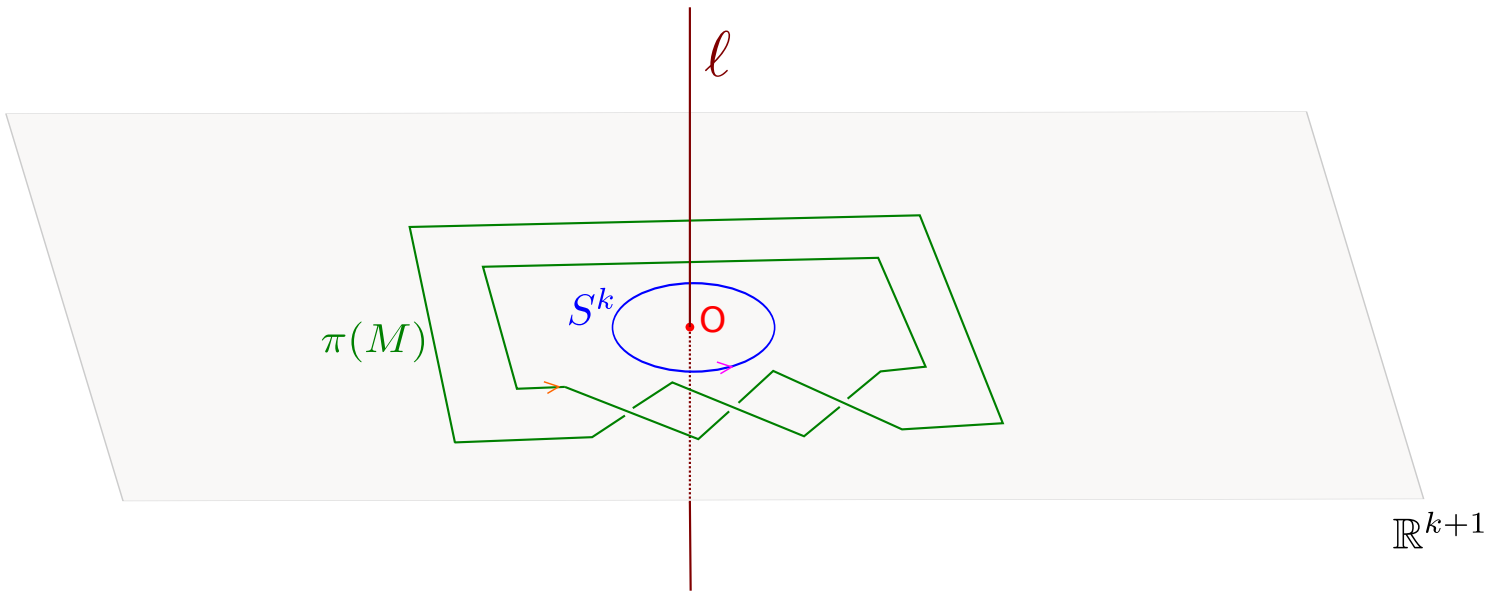
$f : M^k \hookrightarrow \mathbb{R}^{k+2}$ is an embedding of a closed oriented k-manifold.

Definition

$f(M)$ is a *closed braid* if it misses ℓ and the composition

$$M^k \xrightarrow{f} \mathbb{R}^{k+2} \setminus \ell \xrightarrow{\pi} \mathbb{R}^{k+1} \setminus O \xrightarrow{p} S^k$$

is an oriented branched covering map.



Isotopy

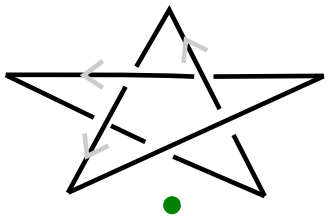
Question

When can a k-link in $\mathbb{R}^{k+2}$ be isotoped to a closed braid?

The answer is affirmative in the following cases:

- $k = 1$, Alexander (1923).
- smooth ribbon surfaces in $\mathbb{R}^4$, Rudolph (1983).
- $k = 2$ Viro (1990), Kamada (1994).
- $k = 3$ in the PL category, K. (*2017*).

$k = 1$: an example



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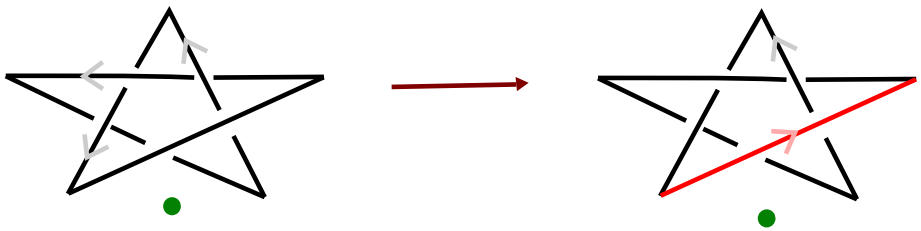

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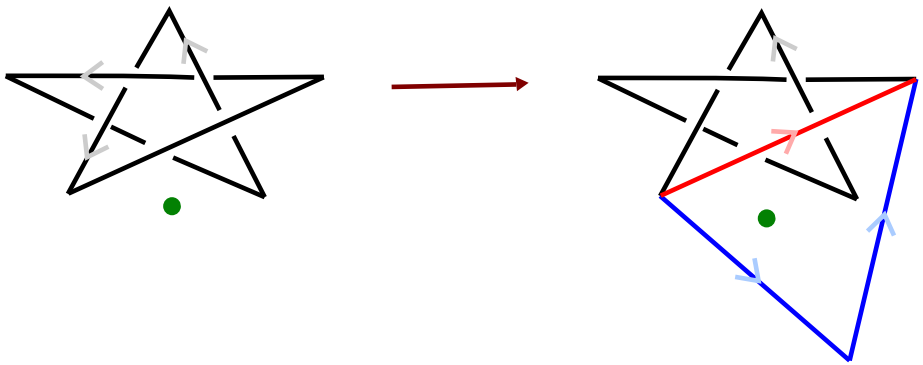

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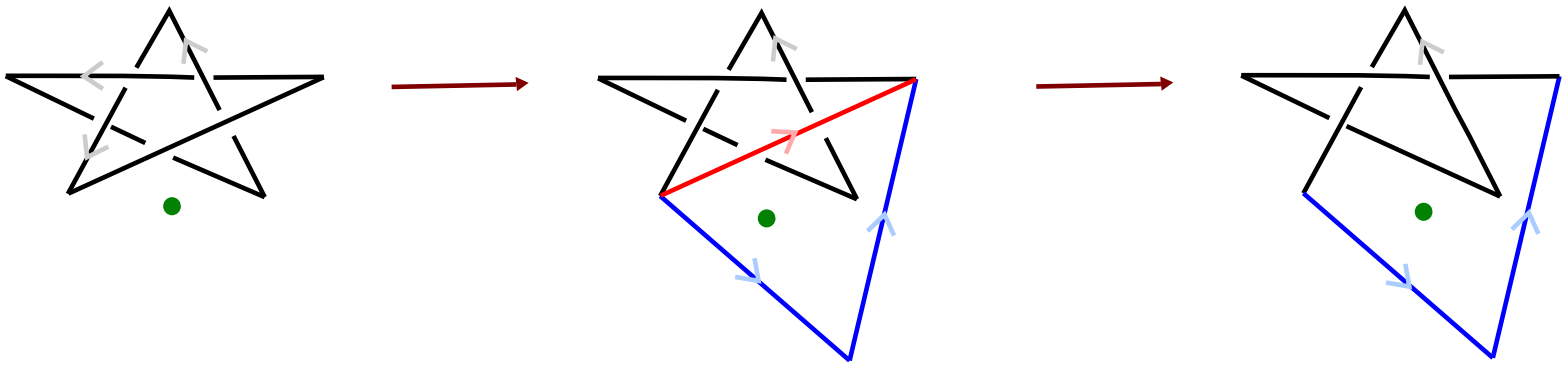

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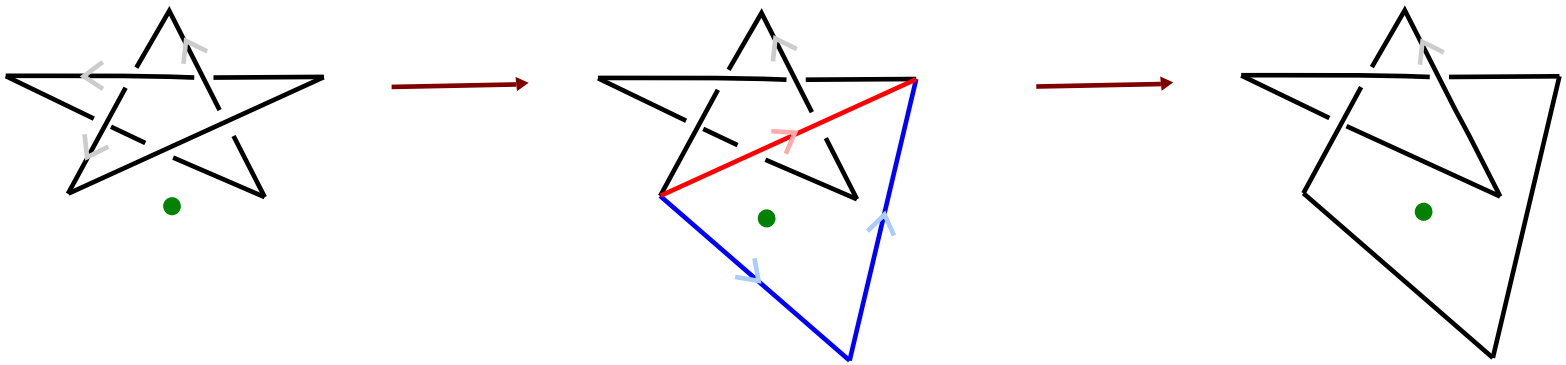

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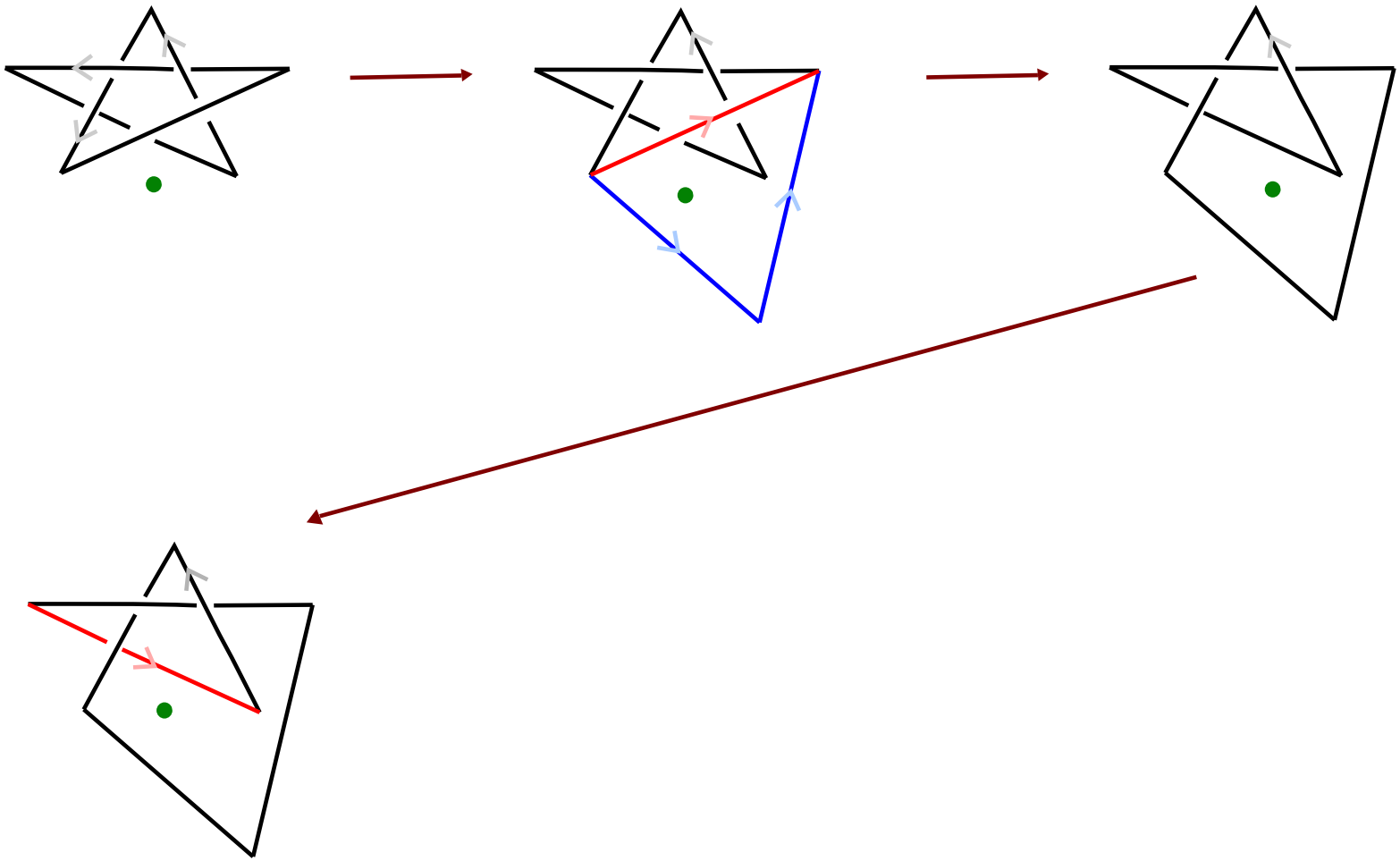

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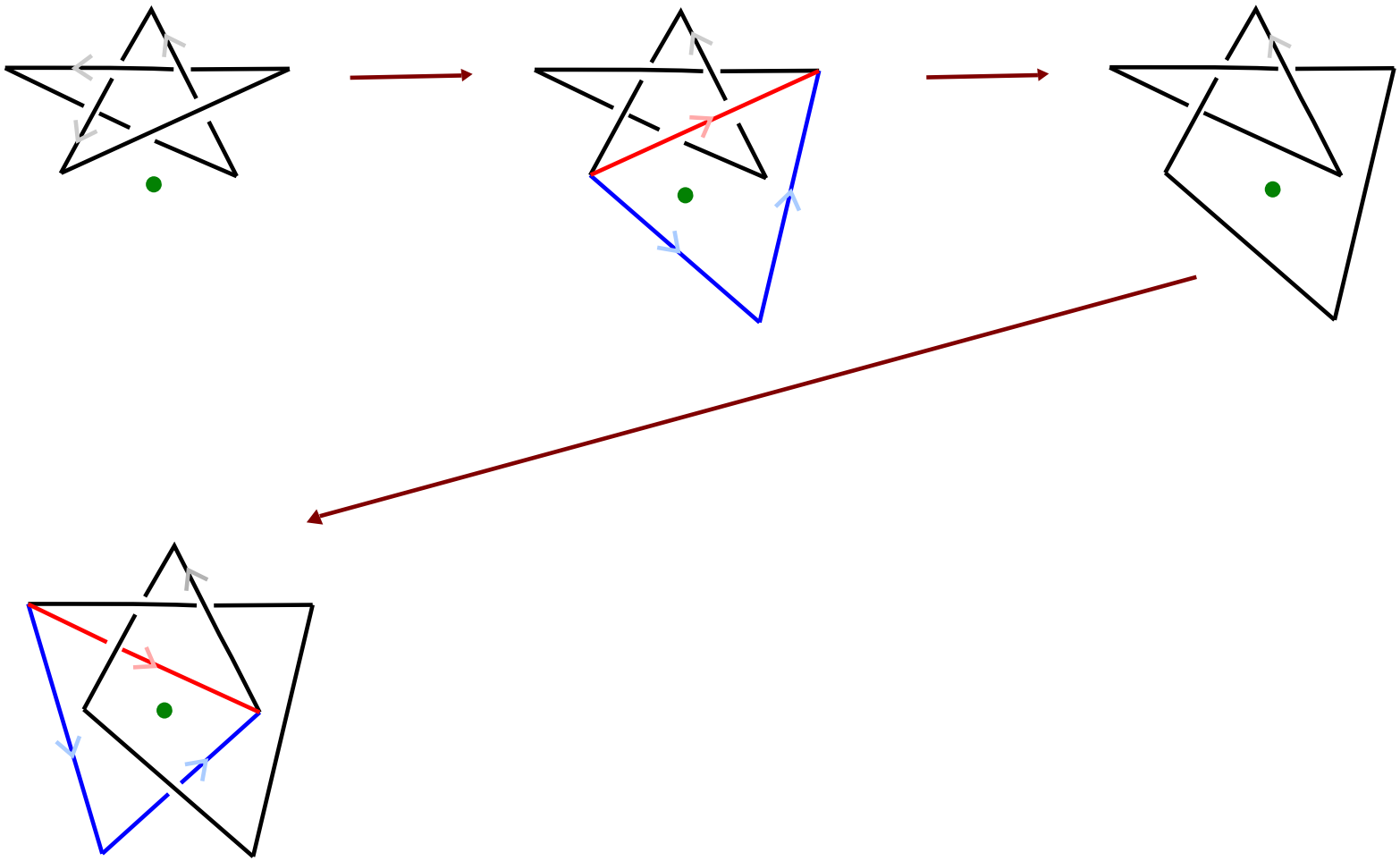

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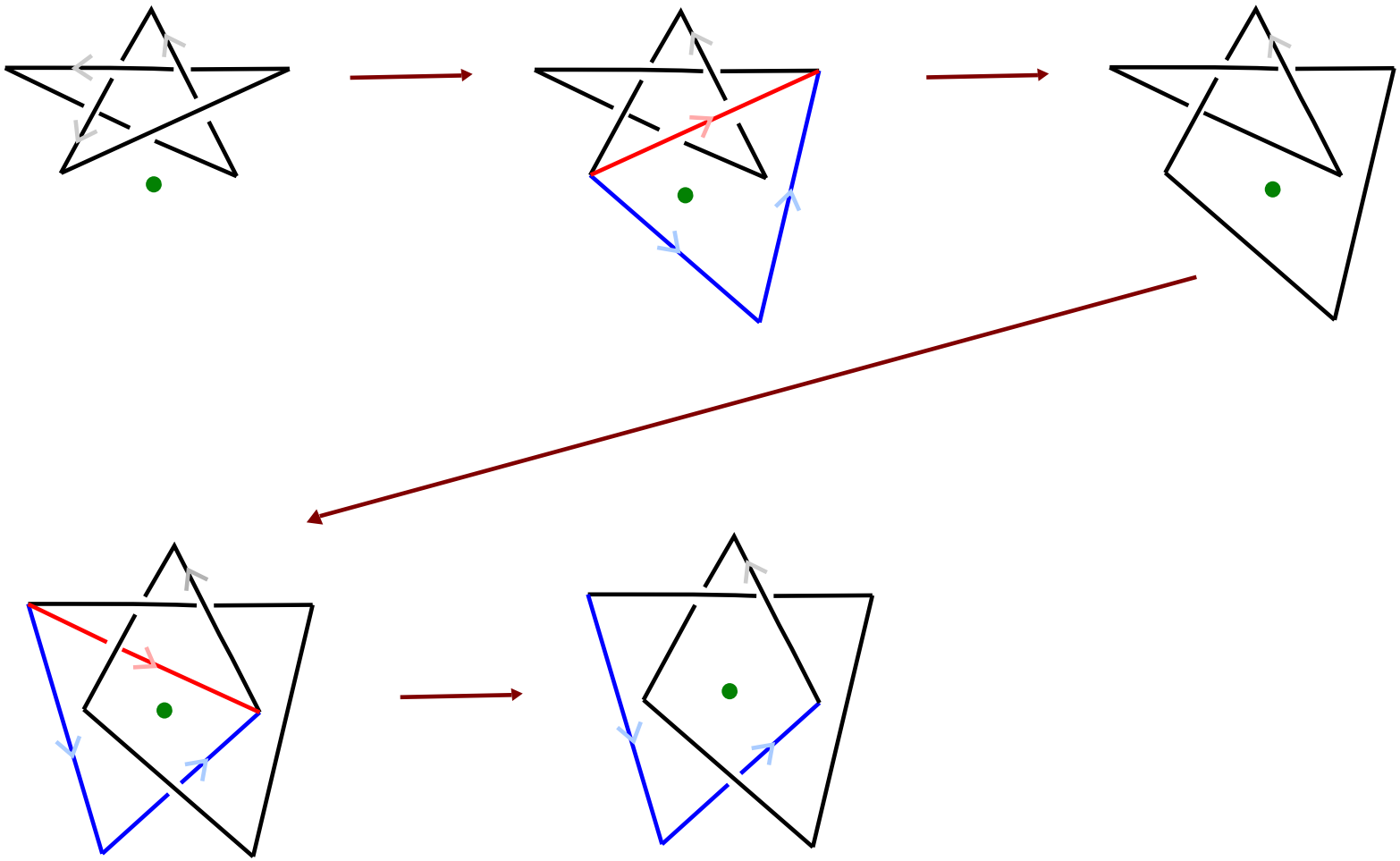

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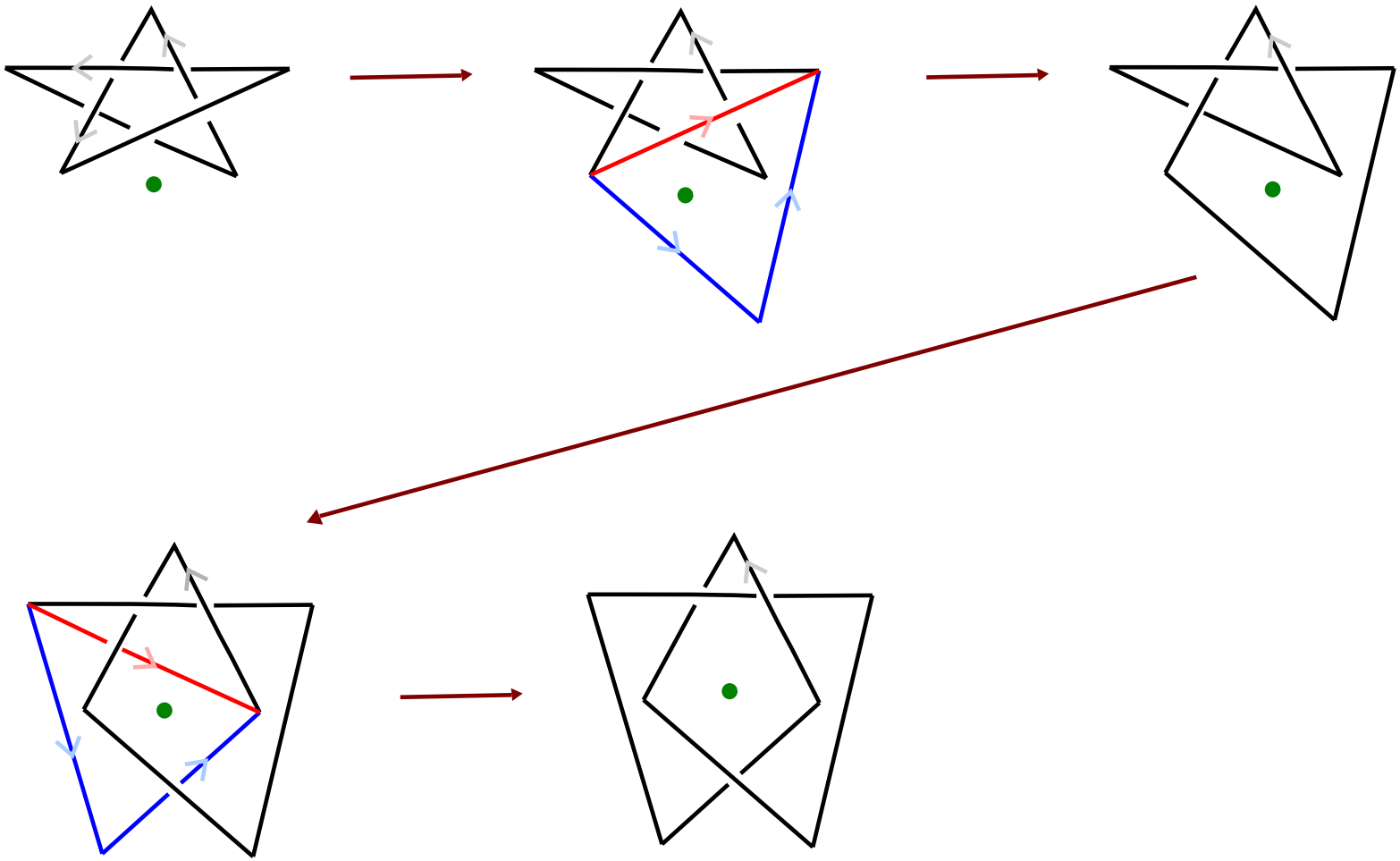

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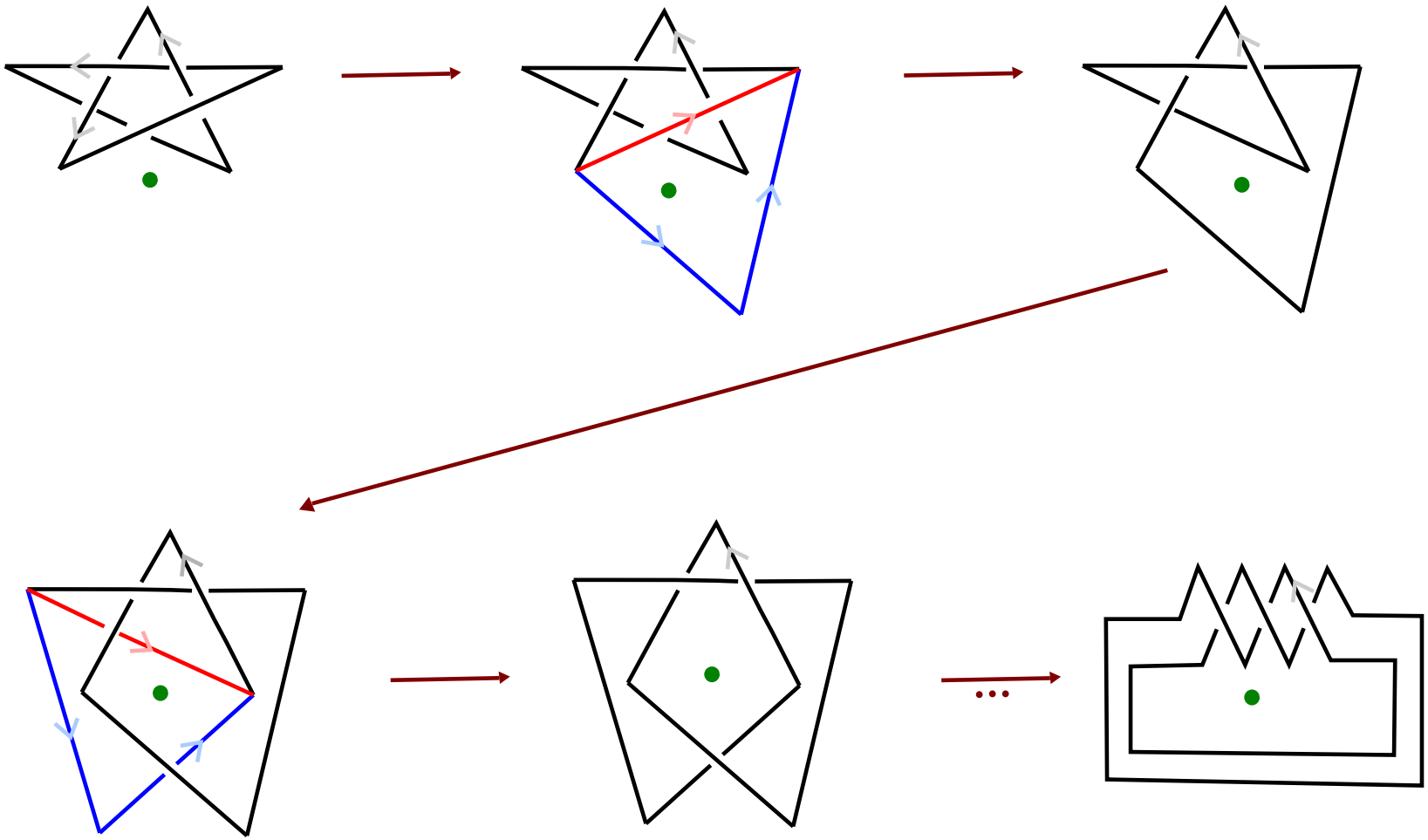

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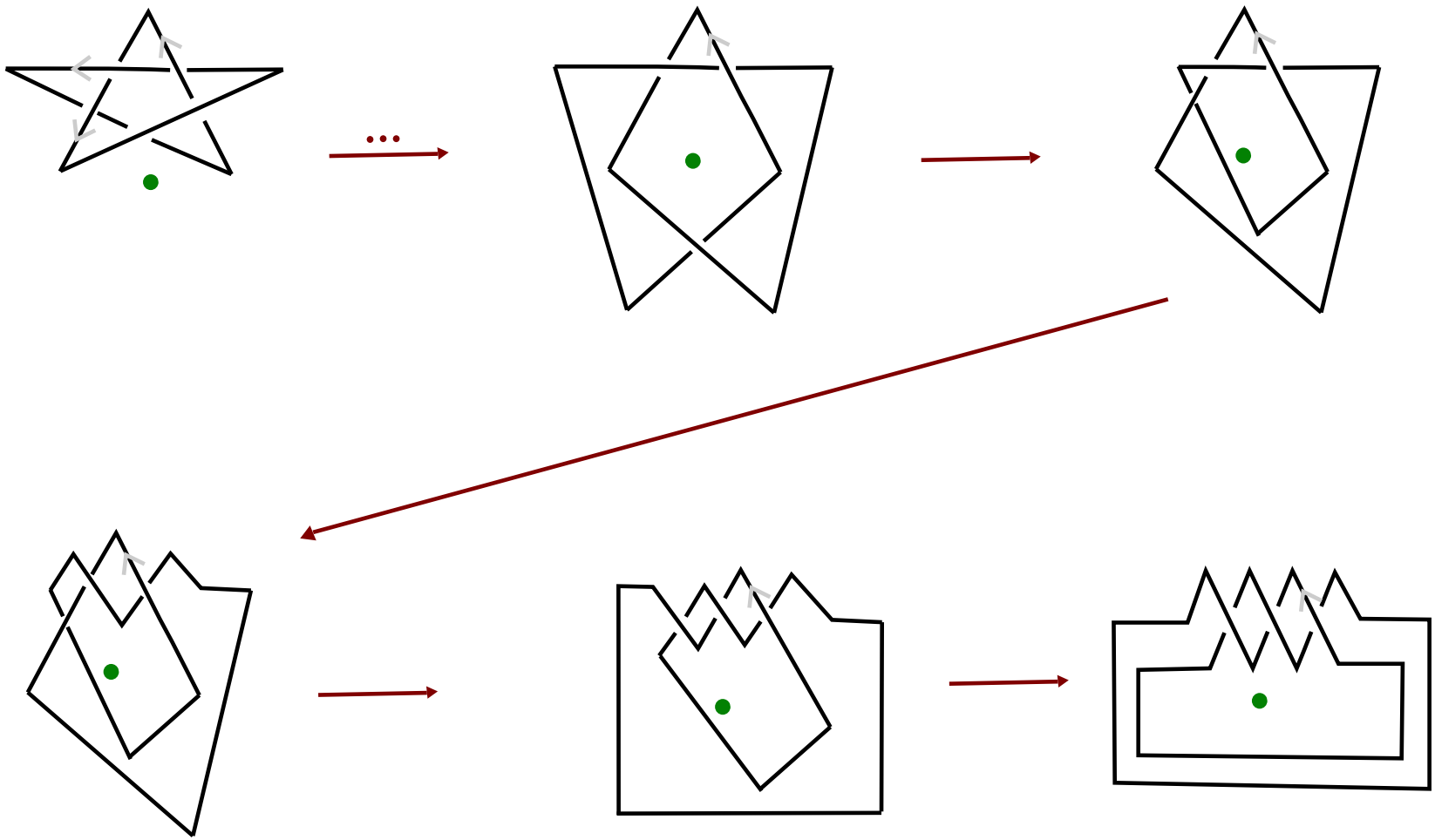
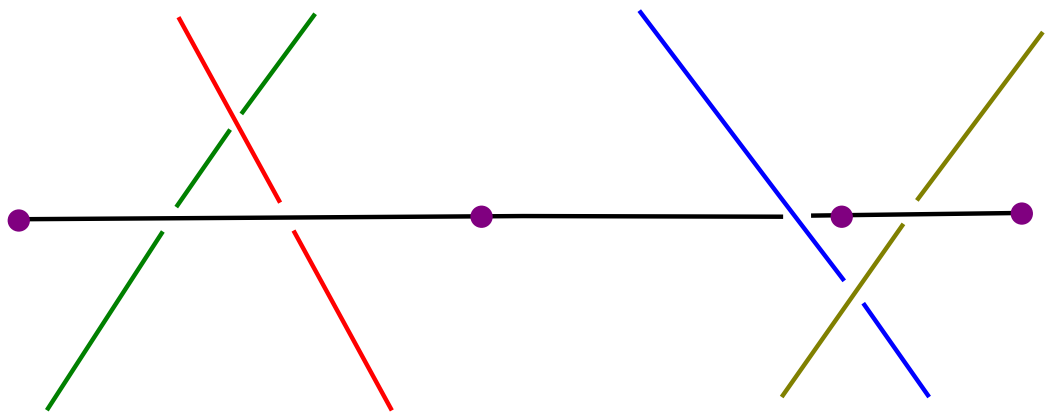


$k = 1$: Proof

Alexander's Theorem

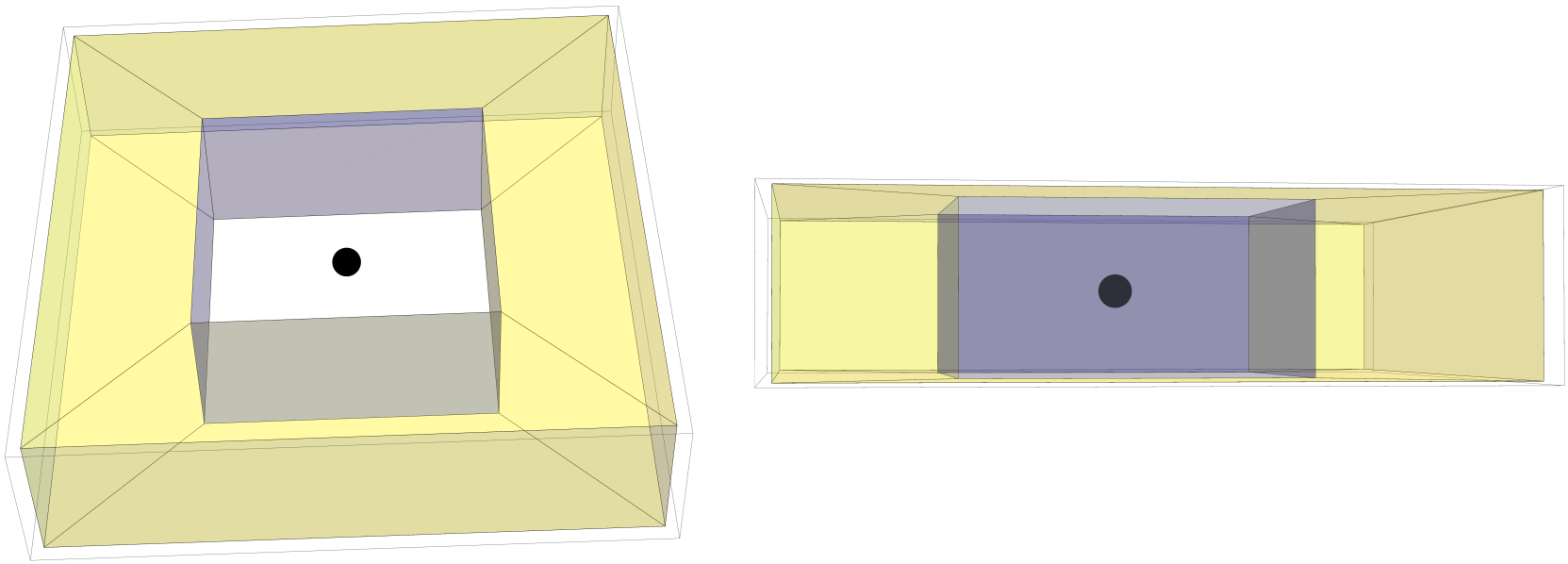
Every oriented link in $\mathbb{R}^3$ is isotopic to a closed braid.

- *Claim 1.* If a negative simplex has only over-crossings, then can find such a triangle crossing ℓ by going over.
- *Claim 2.* The result of a cellular move along such a triangle is that a negative simplex is replaced by positive simplices.



$k = 2$: Braiding the standard torus

The standard torus



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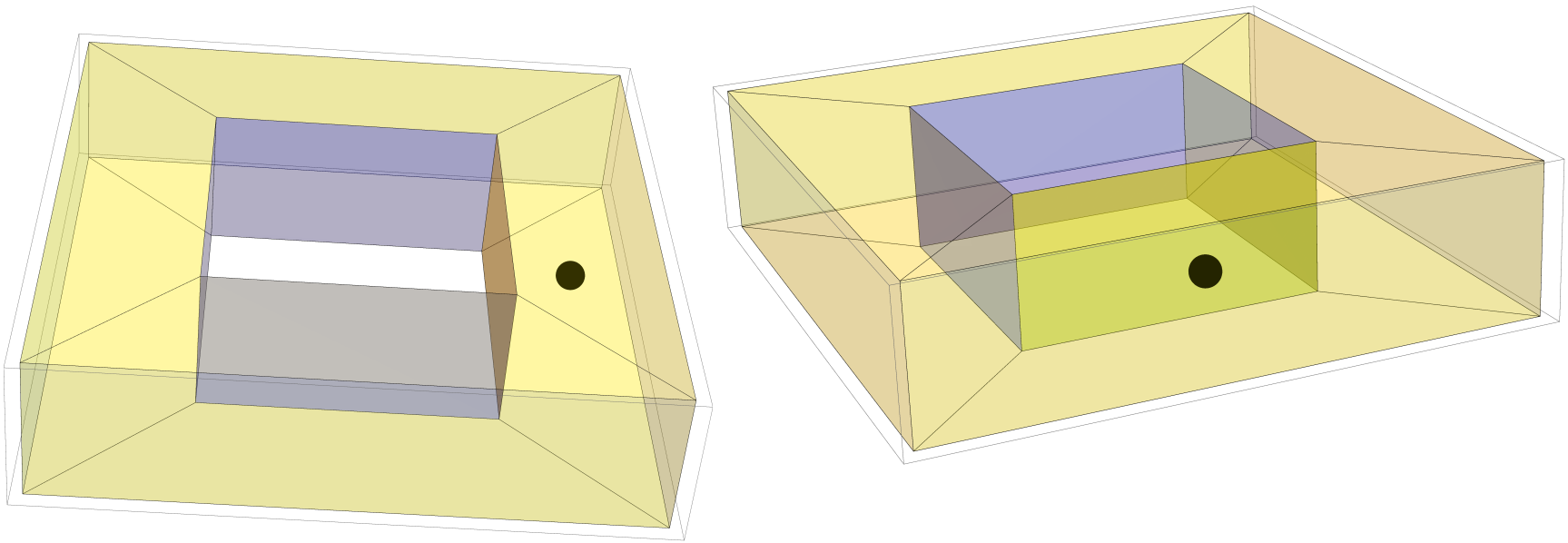
Translating the torus



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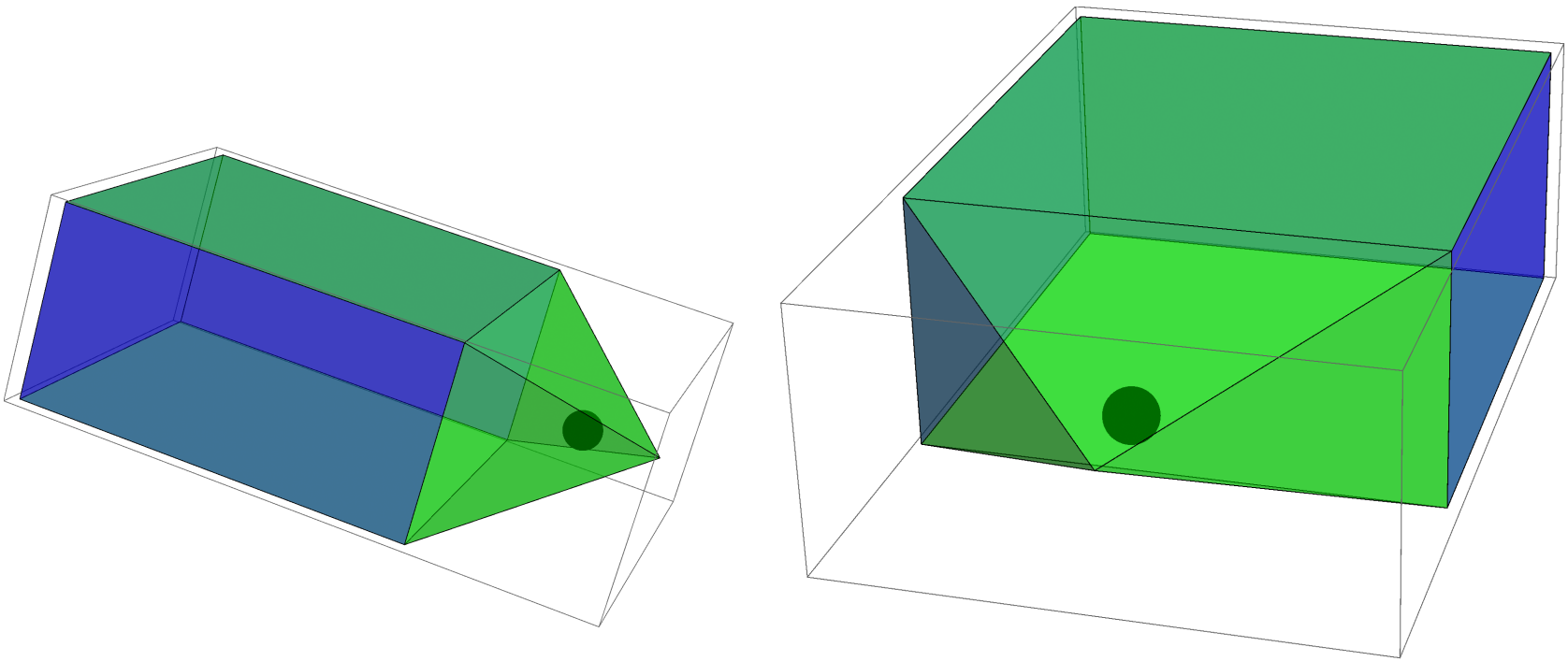
Cellular move



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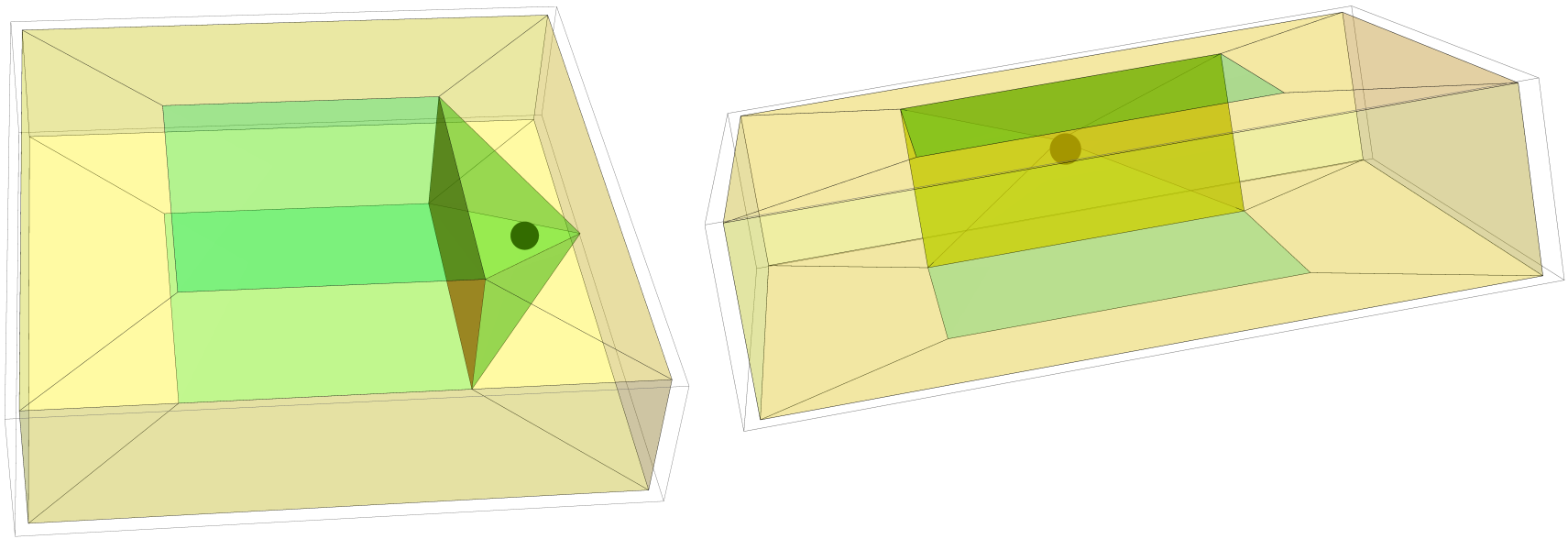

Transverse Contact Embeddings

- A *contact structure* is a maximally non-integrable hyperplane field on an odd dimensional manifold.
- Example: On $S^{2n-1} \subset \mathbb{C}^n$, $\xi_{std} = TS^{2n-1} \cap iTS^{2n-1}$.
- A smooth embedding $\iota : (M^3, \xi) \hookrightarrow (S^5, \xi_{std})$ is called a *transverse contact embedding* if $\iota \pitchfork \xi_{std}$ and $d\iota(\xi) \subset \xi_{std}$.

Theorem (Etnyre-Furukawa, 2017)

In the smooth category, if an embedding $M^3 \hookrightarrow S^5$ can be isotoped $M^3 \rightsquigarrow S^3 \times D^2 \subset S^5$ nicely, then it can be isotoped to be a transverse contact embedding.

Higher codimension

Theorem (K., *2017*)

Any closed oriented PL k-link in $\mathbb{R}^n$ can be PL isotoped to be a closed braid for $2n \geq 3k + 2$.

Corollary

Every closed oriented PL k-manifold embeds in $\mathbb{R}^{2k}$, and moreover every embedding is PL isotopic to a closed braid.

Questions

- What is the commutator length of the braid group?
- If there are two band factorizations of a braid closing to the unknot, do the band factorizations have to be slide equivalent?
- Given a braid closing to the unknot, is its centralizer always generated by itself and Δ_n^2?

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THANK YOU!