LOOP BRAIDS

AND OTHER

MOTION GROUPS

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Appear a lot in the literature...

$$\boxed{\text{LOOP BRAID GROUPS } LB_n}$$

... with many different names and formalisms!

"BASIS-CONJUGATING AUTOMORPHISMS OF THE FREE GROUP"

"MONOTONE RIBBON TUBES"

"MOTION GROUPS OF A TRIVIAL LINK"

"UNTWISTED RING GROUPS"

"GROUPS OF SYMMETRIC AUTOMORPHISMS OF THE FREE GROUP"

$\boxed{\text{Loop Braid Groups } LB_n}$

"WELDED BRAID GROUPS"

"GROUPS OF FLYING RINGS"

"BRAID-PERMUTATION GROUPS"

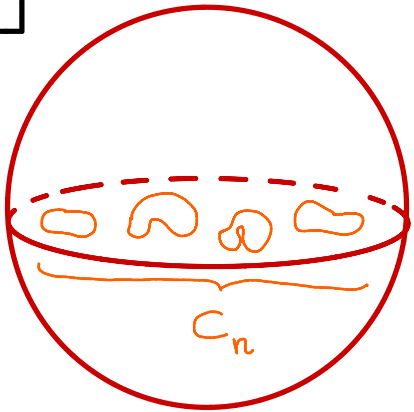
"WEAKLY VIRTUAL BRAID GROUPS"

MAPPING CLASS GROUP
(MOTION GROUP) OF THE 3-BALL B^3
W.R.T. A TRIVIAL LINK C_n
WITH n COMPONENTS
(ORIETATION PRESERVED ON B^3 AND C_n)

LOOP BRAID GROUPS LB_n

$\mathrm{Homeo}(B^3, C_n) = \{$ self-homeos of B^3 that:

- fix ∂B^3
- fix C_n setwise
- preserve orientation on B^3 and C_n $\}$



$$LB_n := \mathrm{Homeo}(B^3, C_n) \big/ \text{isotopy}$$

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AUTOMORPHISMS OF THE FREE GROUP F_n OF TYPE:

$$\begin{cases} x_i \longrightarrow a_i^{-1} x_{\pi(i)} a_i \\ x_j \longrightarrow x_j \end{cases}$$

s.t. $a_i \in F_n$, $\pi \in S_n$

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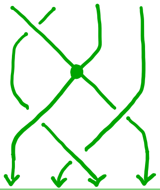
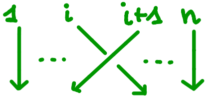
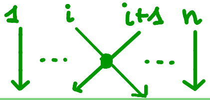
ABSTRACT PRESENTED GROUP + WELDED DIAGRAMS

$$\left\langle \sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1} \;\middle|\; \begin{array}{l} \text{braid group relations} \\ \text{symm. group relations} \\ \text{mixed relations} \end{array} \right\rangle$$



$\sigma_i \longmapsto$ 

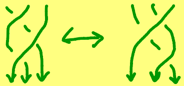
$\rho_i \longmapsto$ 

Loop Braid Groups LB_n

Braid relations

$\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j|>1$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $i=1,\dots,n-2$

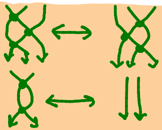


Symmetric group relations

$\rho_i \rho_j = \rho_j \rho_i$ $|i-j|>1$

$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$ $i=1,\dots,n-2$

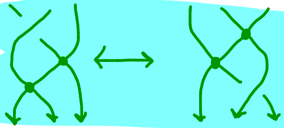
$\rho_i^2 = 1$ $i=1,\dots,n-1$



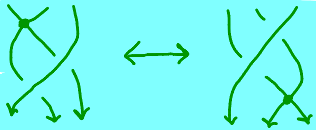
Mixed relations

$\sigma_i \rho_j = \rho_j \sigma_i$ $|i-j|>1$

$\sigma_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \sigma_i$



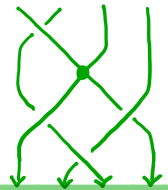
$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}$



"forbidden move of type 1"

ABSTRACT PRESENTED GROUP + WELDED DIAGRAMS

$$\left\langle \sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1} \;\middle|\; \begin{array}{l} \text{braid group relations} \\ \text{symm. group relations} \\ \text{mixed relations} \end{array} \right\rangle$$



LOOP BRAID GROUPS LB_n

MAPPING CLASS GROUP
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AUTOMORPHISMS OF
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OF TYPE:

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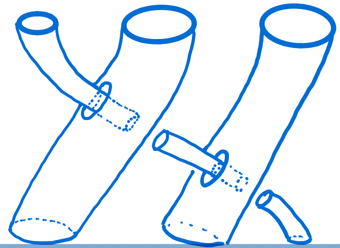
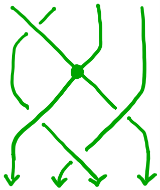
TRAJECTORIES OF
(non-flipping) LOOPS
IN A 3-SPACE
+
BROKEN SURFACES



ABSTRACT PRESENTED
GROUP

$$\left\langle \sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1} \;\middle|\; \begin{array}{l} \text{braid group relations} \\ \text{symm. group relations} \\ \text{mixed relations} \end{array} \right\rangle$$

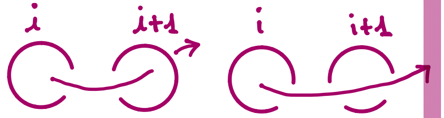
+ WELDED
DIAGRAMS



LOOP BRAID GROUPS LB_n

MAPPING CLASS GROUP (MOTION GROUP) OF THE 3-BALL B^3 W.R.T. A TRIVIAL LINK C_n WITH n COMPONENTS (ORIETATION PRESERVED ON B^3 AND C_n

π_1 (CONFIGURATION SPACE OF n RINGS ON // PLANES IN $\mathbb{R}^3$)



σ_i ρ_i

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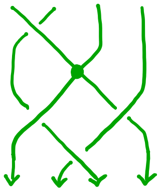

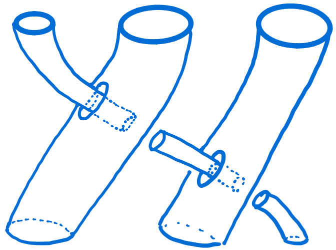

LOOP BRAID GROUPS LB_n

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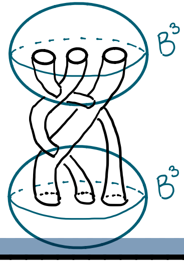
BRAIDED ANNULI IN $\mathbb{R}^4$



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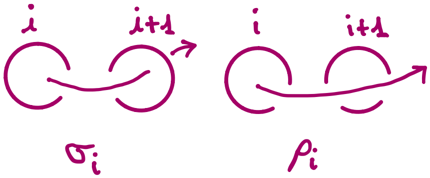

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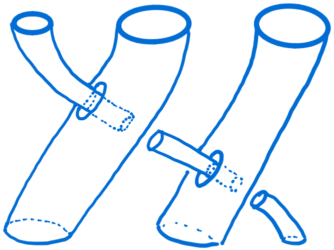

ABSTRACT PRESENTED GROUP + WELDED DIAGRAMS

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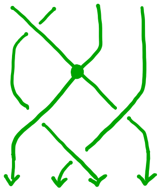


EQUIVALENCE

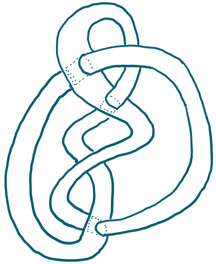
Different people, different parts!

Theorem (Savushkina, Fenn-Rimány-Rourke, Brendle-Hatcher, Baez-Wise-Crans, D.)

All these formulations are equivalent.

- MOTION GROUPS OF OTHER SPLIT LINKS?

- INVARIANT FOR RIBBON TORUS-LINKS IN $\mathbb{R}^4$?



- STABLE HOMOLOGY?

$\boxed{\text{LOOP BRAID GROUPS } LB_n}$

- QUASIMORPHISMS?

- REPRESENTATIONS OF B_n THAT EXTEND TO LB_n?

- ~~GARSIDE~~?

- LINEAR?

$\boxed{\bullet \; \ldots ?}$

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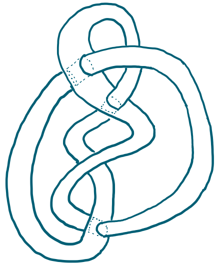

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• ~~GARSIDE~~ ?

• LINEAR?

• ...?

MOTION GROUPS

The elements

M an m-dimensional manifold

N an orientable submanifold $\subset \mathring{M}$
not necessarily connected or non-empty.

Definition A motion of N in M

is an ambient isotopy of M

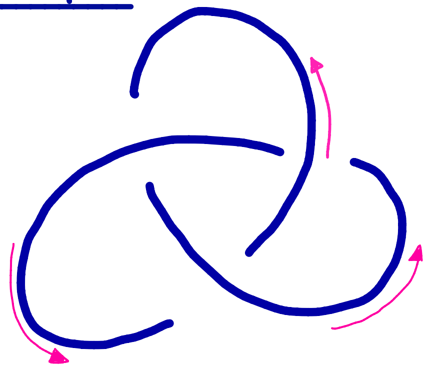
who returns N to itself.

MOTION GROUPS

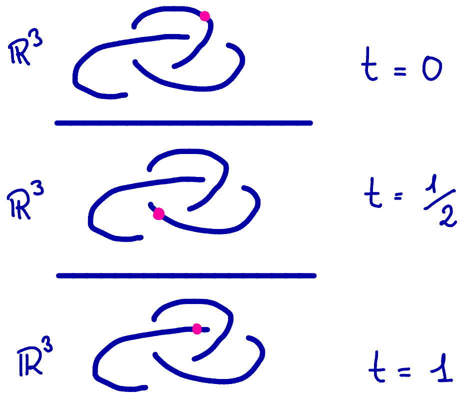
Stationary motions and group structure

Definition A motion f is stationary for N if $f_t(N) = N$, for all $t \in [0,1]$.

Example



A stationary motion of the trefoil knot in $\mathbb{R}^3$



MOTION GROUPS

Stationary motions and group structure

Can define COMPOSITION, INVERSE, EQUIVALENCE of motions

Definition Motion group $\mathcal{M}(M,N)$:

group of equivalence classes of motions of N in M.

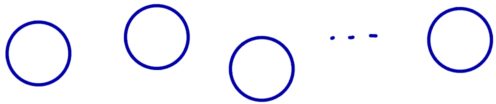
Coincides with $\pi_1\big(H_c(M), H_c(M,N); \mathrm{id}_M\big)$.

THE MOTION GROUP OF A H-TRIVIAL LINK

Joint work with SEIICHI KAMADA

H - trivial link

=



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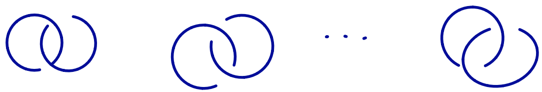


trivial link

we know a presentation for

$$\mathcal{M}(S^3, C)$$

$\cong$

LOOP BRAID GROUP

Hopf links

we know a presentation for

$$\mathcal{M}(S^3, H_i)$$

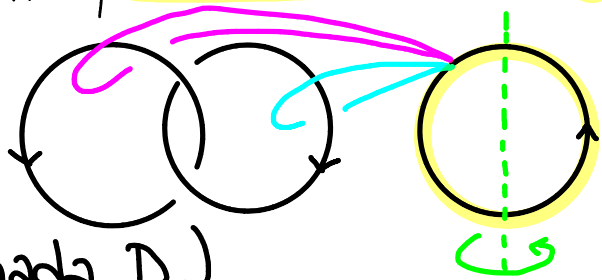
just one Hopf

(GOLDSMITH, 1982)

THE MOTION GROUP OF A H-TRIVIAL LINK

A first step

H Hopf link, C Euclidean circle $\subset S^3$



Lemma (Kamada, D.)

The motion group $\mathcal{M}(S^3 \setminus H, C)$ admits the presentation

$$\left\langle g_a, g_b, \tau_c \;\middle|\; \begin{array}{l} \tau_c^2 = 1, \\ g_a \tau_c = \tau_c g_a, \\ g_b \tau_c = \tau_c g_b, \\ g_a g_b = g_b g_a \end{array} \right\rangle$$

THE MOTION GROUP OF A H-TRIVIAL LINK

Next baby steps

we have the exact sequence:

$$1 \to \mathcal{M}(S^3 \setminus H, C) \longrightarrow \mathcal{M}(S^3, H, C) \longrightarrow \mathcal{M}(S^3, H) \longrightarrow 1$$

THE MOTION GROUP OF A H-TRIVIAL LINK

Next baby steps

Recall that we have the exact sequence:

$$1 \to \underbrace{\mathcal{M}(S^3 \setminus H, C)}_{\text{we know a presentation}} \longrightarrow \mathcal{M}(S^3, H, C) \longrightarrow \underbrace{\mathcal{M}(S^3, H)}_{\text{we know a presentation}} \longrightarrow 1$$

THE MOTION GROUP OF A H-TRIVIAL LINK

Next baby steps

Recall that we have the exact sequence:

$$1 \to \mathcal{M}(S^3 \setminus H, C) \to \boxed{\mathcal{M}(S^3, H, C)} \to \mathcal{M}(S^3, H) \to 1$$

we know a presentation

we can write a presentation!

we know a presentation