Advances in the understanding of parabolic subgroups of Artin-Tits groups

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Artin-Tits groups

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- $M = (m_{s,t})_{s,t \in S}$ symmetric, $m_{s,t} = 1$, $m_{s,t} \in \{2, \ldots, \infty\}$, $s \neq t$

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Artin-Tits group associated to M (of spherical type)

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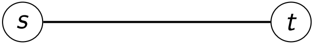

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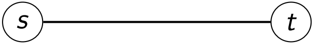

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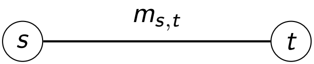

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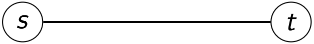

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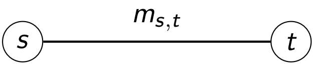

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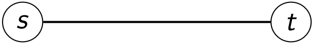

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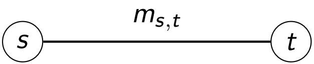

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A braid group is an irreducible Artin–Tits group of spherical type.

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Example: $\mathcal{B}_4$

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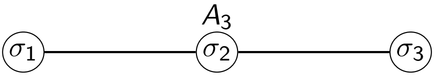
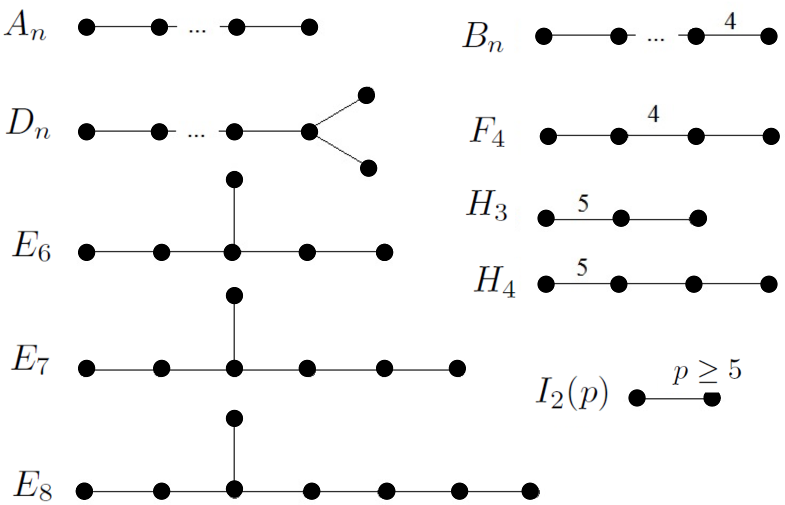


Irreducible Coxeter graphs (of finite type)



Parabolic subgroups

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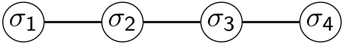
$U \subseteq S, \beta \in A$, A_U has connected Coxeter graph.

Curves in D_n and parabolic subgroups in braid groups



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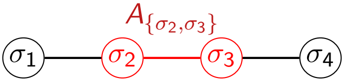

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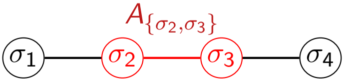

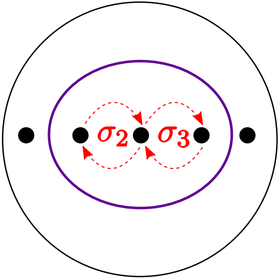

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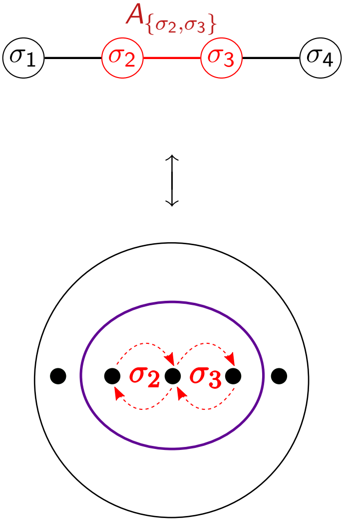

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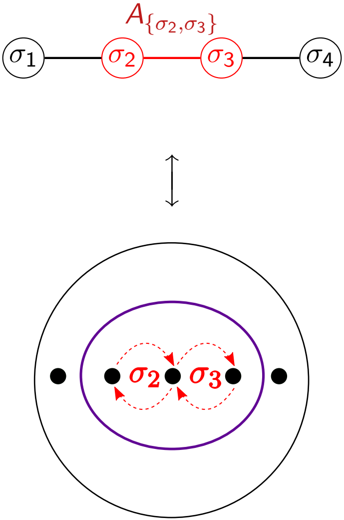

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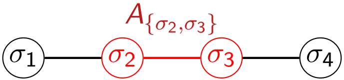

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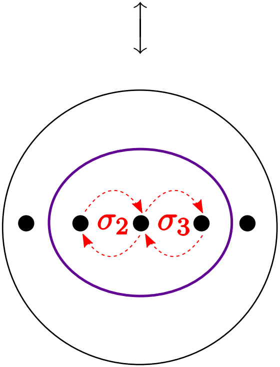

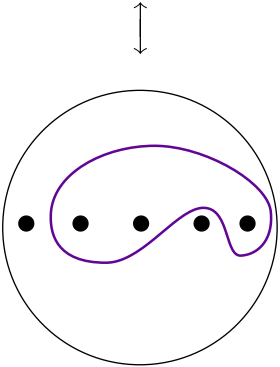

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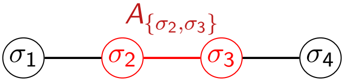

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Irreducible parabolic subgroups

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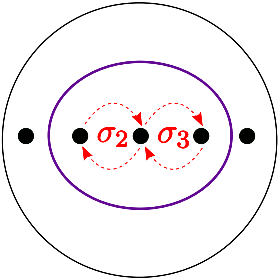

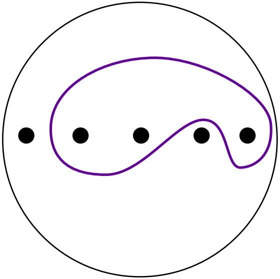

Non-degenerate simple closed curves

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We want to use irreducible parabolic subgroups as an algebraic analogue of the curve complex.

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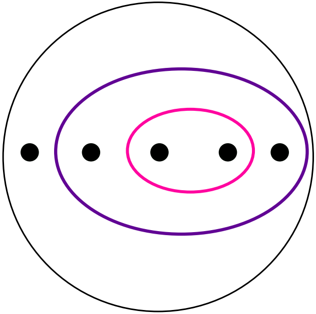

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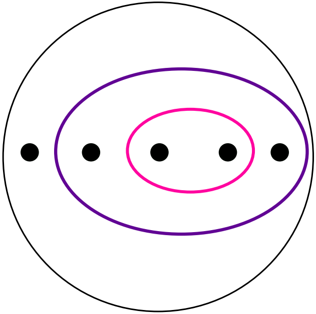

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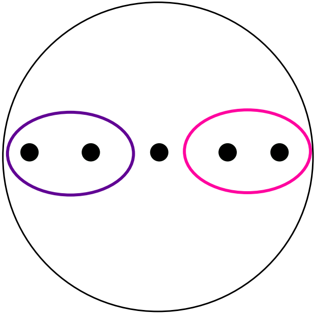

$P \cap Q = \{1\}$ and $xy = yx$, $\forall x \in P$, $y \in Q$

Central Garside element

Each Artin–Tits group of spherical type A_S has Garside structure, which allows to define the following:

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Lemma (Godelle 2003, C. 2017)

Given P, Q parabolic subgroups and $\alpha \in A_S$

$$P = \alpha^{-1} Q \alpha \iff z_P = \alpha^{-1} z_Q \alpha$$

Main results

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Theorem (Lattice of parabolic subgroups)

The set of parabolic subgroups is a lattice with respect to the order induced by the inclusion. That is, if P and Q are parabolic subgroups:

- $\exists!$ maximal parabolic subgroup contained in $P \cap Q$.
- $\exists!$ minimal parabolic subgroup containing $P \cup Q$.

Main results

Theorem ("Disjointness" of parabolic subgroups)

Let P and Q be two distinct irreducible parabolic subgroups of A_S. Then $z_P z_Q = z_Q z_P$ holds if and only if one of the following three conditions are satisfied:

1. $P \subsetneq Q$.
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3. $P \cap Q = \{1\}$ and $xy = yx$ for every $x \in P$ and $y \in Q$.

Complex of irreducible parabolic subgroups $\mathcal{P}$

- Vertices: Irreducible parabolic subgroups.
- n-simplex: $\{P(1), \cdots, P(n+1)\}$ such that $z_{P(i)} z_{P(j)} = z_{P(j)} z_{P(i)}$ for all $1 \leq i \leq j \leq n+1$.

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- The action of A_S on $\mathcal{P}$ would allow to generalize results that are known for braids.

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- The *np*-normal form of $x \in A_S$ with respect to Δ_S^p is canonical way of represent an element:

$$x = x_n^{-1} \cdots x_1^{-1} y_1 \cdots y_m,$$

  where x_i, y_j satisfies some coprimality conditions and are prefixes of Δ_S^p for every i, j and $x_1 \cdots x_n$ and $y_1 \cdots y_m$ have no prefix in common.

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- Decycling: $d(x) = y_m x_n \cdots x_1^{-1} y_1 \cdots y_{m-1}$.
- $RSSS_p(x)$: The set of conjugates of x that are in a period under twisted cycling and decycling. $RSSS_\infty(x) = \bigcap_{p \leq 1} RSSS_p(x)$.

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- Given $\alpha' \in RSSS_\infty(\alpha)$, we define $\varphi(\alpha) = |\Delta_{Supp(\alpha)}|$.

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Let P and Q be two parabolic subgroups. Then $P \cap Q$ is also a parabolic subgroup.

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- Prove $A_Z = P \cap Q$.

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The set of parabolic subgroups is a lattice with respect to the order induced by the inclusion. That is, if P and Q are parabolic subgroups:

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Theorem ("Disjointness" of parabolic subgroups)

Let P and Q be two distinct irreducible parabolic subgroups of A_S. Then $z_P z_Q = z_Q z_P$ holds if and only if one of the following three conditions are satisfied:

1. $P \subsetneq Q$.
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Thank you!

Merci!

¡Gracias!