Lifting braids

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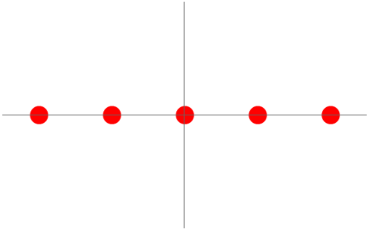
Winterbraids VIII
CIRM, 6th February 2018

SPOCK Scientific Properties Of Complex Knots

Braids as loops in the space of polynomials

Let $\tilde{V}_d$ be the set of monic complex polynomials with d disjoint simple roots.
Let $p \in \tilde{V}_d$.

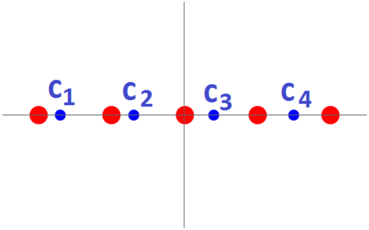


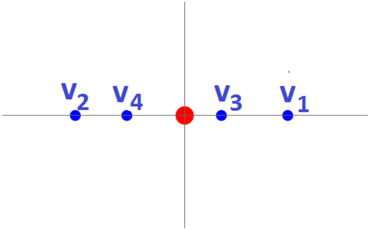
$p = 0$

We can think of loops in $\tilde{V}_d$ as braids on d strands (formed by the roots of the polynomials).

Critical braids

Instead of taking the roots of p, we could have focused on the critical points of p, i.e. $p'(c_j) = 0$ or the critical values $p(c_j)$.





$p'(c_j) = 0$ $\quad$ $p(c_j) = v_j$

The covering map

Let

$$V_d = \{\text{monic polynomials } p \in \mathbb{C}[z] \text{ of degree } d \text{ and with } d-1 \text{ disjoint, non-zero critical values and constant term equal to } 0\}$$

and

$$W_d = \{(v_1, v_2, \ldots, v_{d-1}) : v_j \in \mathbb{C}\backslash\{0\} \text{ and } v_i \neq v_j \text{ for all } i \neq j\}/S_{d-1}.$$

Theorem (Beardon, Carne, Ng)

The map $\phi_d : V_d \to W_d$ that sends a polynomial $p \in V_d$ to its set of critical values is a covering map of degree d^{d-1}.

This means that we can lift loops γ in W_d to paths in V_d and these lifts only depend on the homotopy class of γ.

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Easiest example: $d = 2$.

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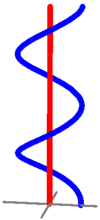
(For $d = 2$: $AB^{-1} \sim \sigma^k$ for some $k \in \mathbb{Z}$. Is k even?)
If AB^{-1} is a pure braid, we can find a parametrisation of AB^{-1}, where one strand is stationary (at zero). AB^{-1} is therefore a loop γ in W_d.
(For $d = 2$: $v_1(t) = e^{2\pi ikt/2}$, $t \in [0,1]$.)



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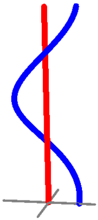
(For $d = 2$: The 2 lifts are $\sigma^{k/2}$.)



How can this be used for braid invariants

Keep lifting!

(For $d = 2$: Every lift halves the exponent of σ. Thus either $k = 0$ or we obtain a nontrivial permutation after at most $\log_2 |k|$ steps.)