Volume forms on the $\mathrm{SL}_N\mathbb{C}$-moduli space of surfaces with boundary

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joint work *in progress* with
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Representation Spaces, Teichmüller Theory, and their Relationship with 3-manifolds from the Classical and Quantum Viewpoints

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  for fixed *regular* elements $g_i \in G$. $\quad \partial S = \partial_1 \sqcup \cdots \sqcup \partial_n$
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- Those are smooth $\mathbb{C}$-manifolds. Want to put (compatible and natural) $\mathbb{C}$-valued volume forms on
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- For $G_{\mathbb{R}} = \mathrm{SU}(N)$ or $\mathrm{SL}_N\mathbb{R}$, get usual volume forms.

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- $\pi_1 S \cong F_k$ free group, so $R^*(S,G) \subset G \times \overset{(k)}{\cdots} \times G/G$
  Fix $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $B(X,Y) = -\mathrm{tr}(XY)$
  $B \rightsquigarrow$ vol form $\bigwedge \mathfrak{g} \to \mathbb{C}$
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- Reidemeister torsion is an invariant of a CW-complex and a representation of its fundamental group. It depends on the simple homotopy type of the CW complex (here of $\pi_1 S$)

$$\Omega : \bigwedge H^1(S, \mathfrak{g}) \to \mathbb{C}$$

Thm: (A.Weil) $H^1(S,\mathfrak{g}) \cong T_\rho R^*(S,G)$

$$0 \to T_\rho R^*(S, \partial S, G) \to \underset{\Omega = R\text{-}torsion}{T_\rho R^*(S, G)} \to \oplus_i T_\rho R^{reg}(\partial_i, G) \to 0$$

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$$\omega(a, b) = \langle \tilde{a}, b \rangle = \langle a, \tilde{b} \rangle \qquad \text{if } a = i(\tilde{a}) \text{ or } b = i(\tilde{b})$$

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- If $2m = \dim_{\mathbb{C}} R^*(S,\partial S,G))$ then
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$$A^N - \sigma_1(A)A^{N-1} + \sigma_2(A)A^{N-2} + \cdots \pm 1 = 0$$

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- We consider the form

$$d\sigma_1 \wedge \cdots \wedge d\sigma_{N-1} \colon T_\rho R^{reg}(\partial_i, G) \to \mathbb{C}$$

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Thm (Heusener-P.):

$$\Omega = \pm \frac{\omega^m}{m!} \bigwedge_{i=1}^{n} c_N d\sigma_1 \wedge \cdots \wedge d\sigma_{N-1}$$

where $c_N = \sqrt{N}(-1)^{(N-1)(N-2)/4}$.

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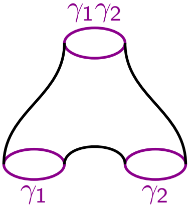
- Need to express $d\sigma_1 \wedge \cdots \wedge d\sigma_{N-1}$ in terms of Reidemeister torsion and intersection form for $H^*(S^1, \mathfrak{g})$.

Pants $P = S_{0,3}$ *and* $\mathrm{SL}_2\mathbb{C}$



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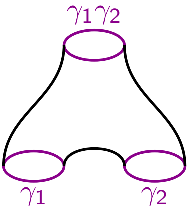

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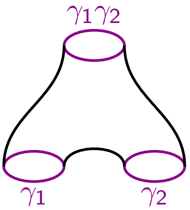

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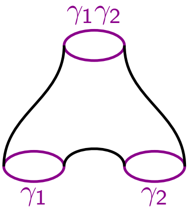

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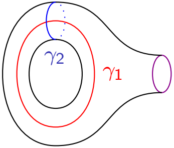
- Hence $R^*(P, \partial P, \mathrm{SL}_2\mathbb{C}) = \{*\}$ and $\omega = 1$
- As peripheral terms contribute by $\pm\sqrt{2}\, d\,\mathrm{tr}$ we get:

$$\Omega_P = \Omega_{F_2} = \pm 2\sqrt{2}\, dt_1 \wedge dt_2 \wedge dt_{12}$$

A torus minus a disk $S_{1,1}$ *and* $\mathrm{SL}_2\mathbb{C}$



- $\pi_1 S_{1,1} \cong F_2 = \langle \gamma_1, \gamma_2 \mid \rangle$ and $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ peripheral element.
  $R^*(S_{1,1}, \mathrm{SL}_2\mathbb{C}) \cong \mathbb{C}^3 - H$ with coordinates (t_1, t_2, t_{12})
  $t_{12\bar{1}\bar{2}} = \mathrm{tr}_{\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}} = t_1^2 + t_2^2 + t_{12}^2 - t_1 t_2 t_{12} - 2$
- Hence (for $c \neq 2$):

$$R^*(S_{1,1}, \partial S_{1,1}, \mathrm{SL}_2\mathbb{C}) = \{(t_1, t_2, t_{12}) \in \mathbb{C}^3 \mid t_{12\bar{1}\bar{2}} = c\}$$

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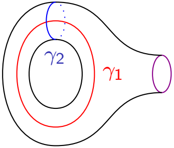

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- Using a formula of Goldman for the Poisson bracket:
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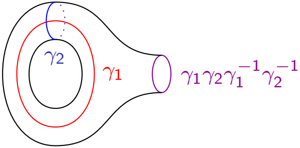

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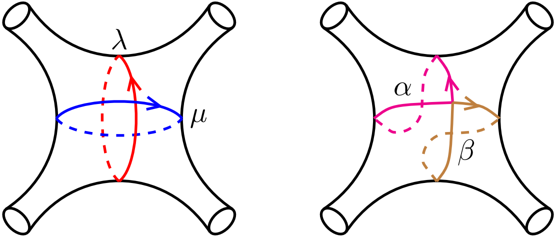
- $\Omega_{S_{1,1}} = \omega \wedge \sqrt{2}\, dt_{12\bar{1}\bar{2}} = \pm 2\sqrt{2}\, dt_1 \wedge dt_2 \wedge dt_{12}$

Planar surface $S_{0,4}$ *and* $\mathrm{SL}_2\mathbb{C}$



- Start with λ and μ, and obtain α and β by surgery:
  - $\pm(t_{\lambda\mu} - t_{\alpha\beta})$ is independent of the orientations of λ and μ,
  - (t_λ, t_μ) local parameter of $R^*(S_{0,4}, \partial S_{0,4}, \mathrm{SL}_2\mathbb{C}) - \{t_{\lambda\mu} = t_{\alpha\beta}\}$

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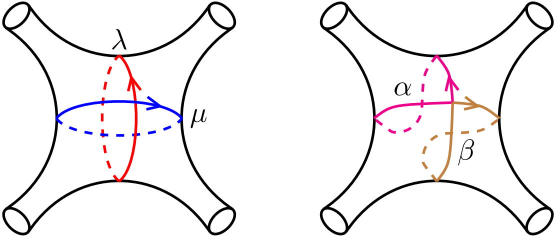

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- Using again Goldman's formula for the Poisson bracket:

$$\omega = \pm \frac{d\, t_\lambda \wedge d\, t_\mu}{t_{\lambda\mu} - t_{\alpha\beta}}$$

Free group of rank 3 and $\mathrm{SL}_2\mathbb{C}$

- $F_3 = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$,
- The map $(t_1, t_2, t_3, t_{12}, t_{13}, t_{23})\colon R^*(F_3, \mathrm{SL}_2\mathbb{C}) \to \mathbb{C}^6$ is a 2:1 branched covering. The variables t_{123} and t_{132} are solutions of

  $$z^2 + Pz + Q = 0$$

  for some $P, Q \in \mathbb{Z}[t_1, t_2, t_3, t_{12}, t_{13}, t_{23}]$. Eg:

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- From the expression for ω and the theorem:

$$\Omega_{F_3} = \Omega_{S_{4,0}} = \pm \frac{4}{t_{123} - t_{132}} dt_1 \wedge dt_2 \wedge dt_3 \wedge dt_{12} \wedge dt_{13} \wedge dt_{23}$$

Free group of rank k and $\mathrm{SL}_2\mathbb{C}$

- $F_k = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$,
- For $k \geq 3$, the $3k-3$ trace functions $t_1, t_2, t_{12}, t_3, t_{13}, t_{23}, \ldots, t_k, t_{1k}, t_{2k}$ define a local parameterization

$$R^*(F_k, \mathrm{SL}_2\mathbb{C}) - \bigcup_{i \geq 3} \{t_{12i} = t_{21i}\} \cup \{t_{12\bar{1}\bar{2}} = 2\} \to \mathbb{C}^{3k-3}$$

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- A Mayer-Vietoris argument (involving subgraphs with $\chi < 0$) yields:

$$\Omega_{F_k} = \pm 2^{(k+1)/2} dt_1 \wedge dt_2 \wedge dt_{12} \bigwedge_{i=3}^{k} \frac{dt_i \wedge dt_{1i} \wedge dt_{2i}}{t_{12i} - t_{21i}}$$

Pants $P = S_{0,3}$ *and* $\mathrm{SL}_3\mathbb{C}$

- The invariant functions of $A \in \mathrm{SL}_3\mathbb{C}$ are $\mathrm{tr}(A)$ and $\mathrm{tr}(A^{-1})$.

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- Using Lawton's computation of the Poisson bracket:

$$\omega = \frac{dt_{1\bar{2}} \wedge dt_{\bar{1}2}}{t_{21\bar{2}\bar{1}} - t_{12\bar{1}\bar{2}}}.$$

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- Using the peripheral terms $\pm\sqrt{-3}dt_i \wedge dt_{\bar{i}}$:

$$\Omega_P = \frac{\pm 3\sqrt{-3}}{t_{21\bar{2}\bar{1}} - t_{12\bar{1}\bar{2}}} dt_1 \wedge dt_{\bar{1}} \wedge dt_2 \wedge dt_{\bar{2}} \wedge dt_{12} \wedge dt_{\bar{1}\bar{2}} \wedge dt_{1\bar{2}} \wedge dt_{\bar{1}2}.$$