On a characterization of strongly quasipositive links

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Representation Spaces, TeichmÏler Theory, and their Relationship with 3-manifolds from the Classical and Quantum Viewpoints

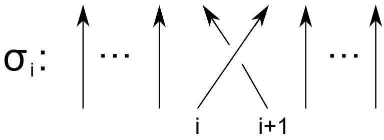
I. Generalized SQP conjecture

Braid group

The n-strand braid group

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1 \end{array} \right\rangle.$$

σ_i is called the standard Artin's (or the standard) generator.



As usual we view $\beta \in B_n$ as a geometric object.

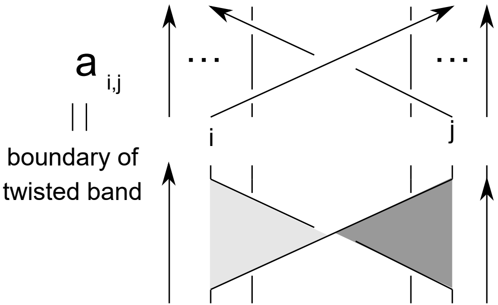
Band generators

Definition: band generators

For $1 \leq i < j \leq n$, let

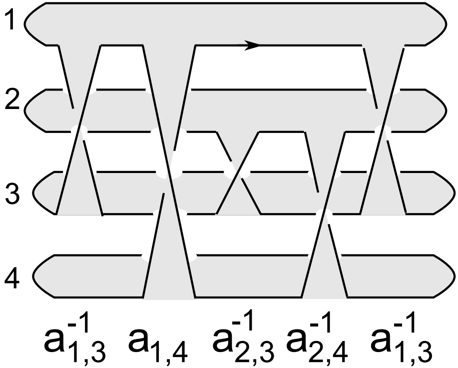
$$a_{i,j} = (\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1})^{-1}\sigma_i(\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1}) \in B_n.$$

$\{a_{i,j}\}_{1 \leq i < j \leq n}$ are called band generators (or, Birman-Ko-Lee generators).



Band generator and Bennequin surface

When $\beta \in B_n$ is written by a product of band generators, we have a canonical Seifert surface consisting of disks and twisted bands. We call such a Seifert surface a Bennequin Surface.



Strongly quasipositive braids

Definition

1. An n-braid β is strongly quasipositive if β is written by a product of positive band generators $\{a_{i,j}\}$ (no $\{a_{i,j}^{-1}\}$ appears).
2. An oriented knot/link K in S^3 is strongly quasipositive if K is represented by a closure of a strongly quasipositive braid.

Motivating question

Give an intrinsic (non-diagrammatic) characterization of strongly quasipositive knot/link.

(c.f (Fox's question): A non-diagrammatic characterization alternating of knots/links is recently given by Howie and Greene ('17))

Properties of strongly quasipositive braids/knot

Facts

1. Algebraically, the band generators $\{a_{i,j}\}$ have a nice property similar to the standard generators $\{\sigma_i\}$: They give rise to a (dual) Garside structure. Accordingly, one can
   - solve the word problem/conjugacy problem using band generators.
   - check a braid it is (conjugate to) a strongly quasipositive braid or not.

   Of course, it does not say that one can check whether a knot is strongly quasipositive or not.
2. If K is positive (i.e. represented by a diagram with only positive corssings), then K is strongly quasipositive (Rudolph,Nakamura)
3. If K is strongly quasipositive then $g(K) = g_4(K)$. (Rudolph)

Background: Bennequin's inequality

Theorem (Bennequin's inequality)

If a knot K is a closure of an braid β,

$$-n(\beta) + e(\beta) \leq 2g(K) - 1$$

where $n(\beta)$ is the number of strands and $e(\beta)$ is the exponent sum of β.

If β is strongly quasipositive then

$$\text{the genus of the Bennequin surface from } \beta = \frac{1 - n(\beta) + e(\beta)}{2} \geq g(K).$$

Corollary

If a knot K is a closure of a strongly quasipositive braid β,

$$-n(\beta) + e(\beta) = 2g(K) - 1$$

Background: Contact structure and transverse link

The quantity $-n(\beta) + e(\beta)$ in the Bennequin's inequality plays an important role in contact geometry.

Definition

A contact structure on an oriented, closed 3-manifold M is a plane field $\xi \subset TM$ of the form

$$\xi = \mathrm{Ker}\,\alpha, \quad \alpha \wedge d\alpha > 0 \quad (\alpha \text{ is a 1-form on } M).$$

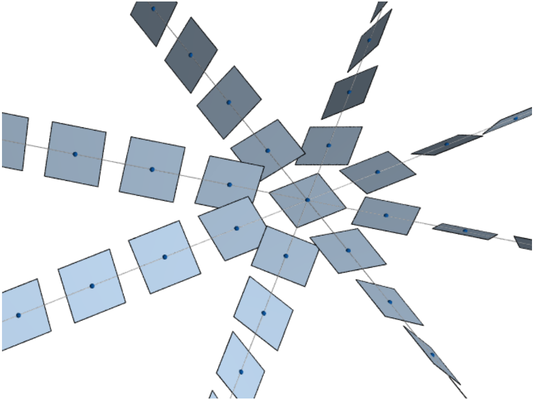
A 3-manifold M with a contact structure (M, ξ) is called a contact 3-manifold.

c.f. Foliation

ξ is a (tangent plane field of) a foliation $\iff$ $\alpha \wedge d\alpha = 0$.

Example: Standard contact structure of S^3

(r, θ, z): Cylindrical coordinate of $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$
$\xi_{std} = Ker(dz + r^2 d\theta)$



(Picture bollowed from P. Massot's web page)

Transverse knot

Definition

An oriented knot K in contact 3-manifold (M, ξ) is a transverse knot if K positively transverse to ξ_p at every $p \in K \subset M$.

Observation

A closed braid $\widehat{\beta} \in \mathbb{R}^3 \subset S^3$ (with z-axis as its axis) is a transverse knot in the standard contact structure $(S^3, \xi_{std} = Ker(dz + r^2 d\theta))$.
(When r is large, $Ker(dz + r^2 d\theta) \sim Ker(d\theta)$!)

- We will view a closed braid as a transverse knot if necessary.
- We use $\mathcal{K}$ to represent transverse knot, and use K to represent underlying topological knot type of $\mathcal{K}$.

Transverse Markov Theorem

Transverse Markov Theorem (Wrinkle,Orevkov-Shecvhishin '03)

Two braids α, β represents the same transverse link (in the standard contact S^3)

$$\Updownarrow$$

They are related by

- Conjugation: $\alpha \leftrightarrow \gamma\alpha\gamma^{-1}$ $(\alpha, \gamma \in B_n)$
- Positive (de)stabilization: $\alpha \leftrightarrow \alpha\sigma_n$ $(\alpha \in B_n)$.

c.f. Markov Theorem

Two braids α, β represents the same link

$$\Updownarrow$$

They are related by

- Conjugation: $\alpha \leftrightarrow \gamma\alpha\gamma^{-1}$ $(\alpha, \gamma \in B_n)$
- (de)stabilization: $\alpha \leftrightarrow \alpha\sigma_n^{\pm 1}$ $(\alpha \in B_n)$.

Self-linking number

Σ: a Seifert surface of a transverse knot $\mathcal{K}$.
v: non-zero section of $\xi|_\Sigma$
$K' = \mathcal{K}$: a parallel copy of K, pushed along v.

Definition

The self-linking number of a transverse knot $\mathcal{K}$

$$sl(\mathcal{K}) = lk(\mathcal{K}, K')$$

This is an invariant of transverse knot – it does not depend on Σ and v.

Remark

In a general case, $sl(\mathcal{K})$ depends on a homology class of a Seifert surface $[\Sigma] \in H_2(M, \mathcal{K})$.

Self-linking number

Theorem (Bennequin's formula '83)

When a transverse knot $\mathcal{K}$ is represented as a closure of a braid β,

$$sl(\mathcal{K}) = -n(\beta) + e(\beta).$$

Thus, Bennequin's inequality can be regarded as

$$sl(\mathcal{K}) \leq 2g(K) - 1$$

Example

If $\mathcal{K} = \widehat{\beta}$ and $\mathcal{K}' = \widehat{\beta\sigma_n^{-1}}$ Then $\mathcal{K}$ and $\mathcal{K}'$ are topologically isotopic, but they are not transverse isotopic, since $sl(\mathcal{K}) \neq sl(\mathcal{K}')$.

Remark:

$\exists$ topologically isotopic transverse knots $\mathcal{K}, \mathcal{K}'$ with $sl(\mathcal{K}) = sl(\mathcal{K}')$, which are not transversely isotopic (Etnyre-Honda '05, Birman-Menasco '06).

SQP conjecture

Definition

The maximal self-linking number

$$SL(K) = \max\{sl(\mathcal{K}) \mid \mathcal{K} \text{ is topologically isotopic to } K\}$$

(so Bennequin inequality says that $SL(K) \leq 2g(K) - 1$).

SQP Conjecture (Folklore ???)

K is strongly quasipositive $\iff$ $SL(K) = 2g(K) - 1$

Remark

In general, Bennequin's inequality is far from the equality. We have other stronger inequalities, such as, slice Bennequin inequality

$$SL(K) \leq 2g_4(K) - 1$$

SQP conjecture for fibered case

The SQP conjecture is true if K is fibered:

Theorem (Hedden '07, Etnyre, van-Horn-Morris '11)

Assume that K is fibered. Then

$$K \text{ is strongly quasipositive}$$
$$\Updownarrow$$
$$SL(K) = 2g(K) - 1$$

(The proof uses contact geometry – we view K as a binding of an open book and use a contact structure supported by K).

Defect of the Bennequin inequality

The SQP conjecture asks a characterization of equality for the Bennequin inequality. We ask more general question.

Definition

The defect of Bennequin's inequality

$$\delta(K) := \frac{1}{2}((2g(K)-1) - SL(K)) \in \mathbb{Z}_{\geq 0}$$

$$(\text{or, } \delta(\mathcal{K}) := \frac{1}{2}((2g(K)-1) - sl(\mathcal{K})) \in \mathbb{Z}_{\geq 0}$$

Motivating question, more general form

Characterize knots with $\delta(K) = n$.

(Remark: for any $n \geq 0$ there exists a knot K with $\delta(K) = n$).

Defect of the Bennequin inequality

Generalizing the SQP conjecture, we propose:

Generalized SQP Conjecture (I-Kawamuro)

$\delta(K) \leq n \iff$ K is represented as a closure of a braid β which is a product of positive band generators and n negative band generators.

i.e.

$$\delta(K) = \min\{\#\text{negatively twisted bands of Bennequin surface}\}$$

Actually we expect slightly stronger assertion

Generalized SQP Conjecture Conjecture, (strong version)

$\delta(K) \leq n \iff$ K is represented as a closure of a $(b(K)+n)$ braid β which is a product of positive band generators and n negative band generators.

Here $b(K)$ =braid index of K.

Remark on strong version

In a strong version of generalized SQP conjecture, $b(K)+n$ cannot be replaced with $b(K)$.

Example (Hirasawa and Stoimenow '03)

Let K be the closure of a 4-braid

$$a_{1,2}a_{2,4}^{2}a_{1,2}^{-1}a_{1,3}a_{1,2}a_{2,4}^{-1}a_{1,2}^{-2}a_{1,3}^{-2}.$$

This knot satisfies $b(K)=4$, $\delta(K)=1$. Any Bennequin surface S of $4(=b(K))$-braid representative of K, the number of negative bands $>1=\delta(K)$. On the other hand, K has a $5(=b(K)+\delta(K))$-braid representative with exactly $1=\delta(K)$ negative band.

Main results I: 3-braids

When $b(K) = 3$, (more strong version) of a generalized SQP Conjecture is true:

Theorem (I-Kawamuro)

If $b(K) = 3$,

$\delta(K) \leq n \iff$ K is represented as a closure of a 3-braid β which is a product of positive band generators and n negative band generators.

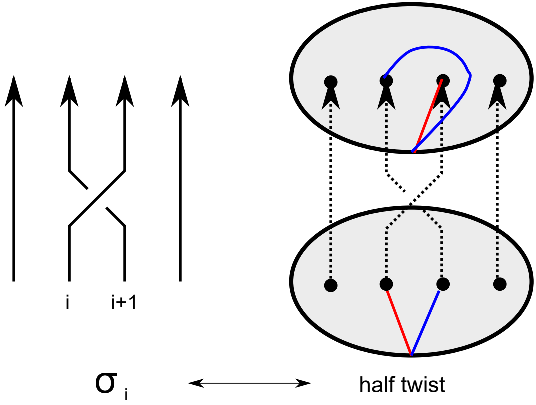
Corollary: SQP conjecture for closed 3-braid

If $b(K) = 3$, K is strongly quasipositive if and only if the Bennequin's inequality is sharp, i.e, $SL(K) = 2g(K) - 1$.

Braid as MCG

Let $D_n = D^2 \setminus \{n-\text{points}\}$. Recall that

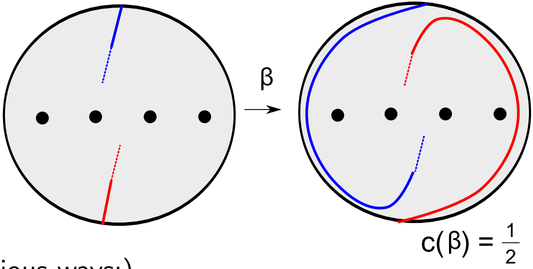
$$B_n \cong MCG(D_n) = \{f : D_n \to D_n \mid f|_{\partial D_n} = id\}/\text{isotopy}$$



Fractional Dehn twist coefficient (FDTC)

The fractional Dehn twist coefficient (FTDC, in short)

$$c(\beta) := \text{ the amount of rotations near } \partial D_n$$



(Defined by various ways:)

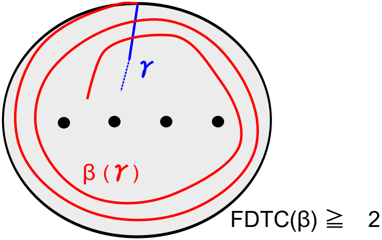
- Nielsen-Thurston type of MCG (Honda-Kazez-Matic '08)
- Using $MCG(D_n) \curvearrowright \partial\widetilde{D_n} \subset \overline{\mathbb{H}^2}$) (Malyutin '05, I-Kawamuro '17.)
- Dehornoy's ordering (Malyutin'05)
- Homogenization of the Upsilon invariant Υ (Feller-Hubbard'17)

Estimates of FDTC

It is easy to estimate FDTC : for any arc γ

$$c(\beta) \geq \#\{\text{Intersection of } \gamma \text{ and } \beta(\gamma) \text{ near the boundary}\}$$
$$= \#(\gamma \cap \beta(\gamma)) - \#\left(p(\gamma) \cap p(\beta(\gamma))\right)$$

$(p : D_n \to S^2 \setminus \{n+1 \text{ points}\})$: contraction of ∂S.



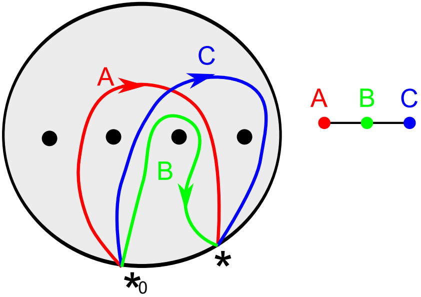
Proposition (I-Kawamuro)

FDTC is easy to compute – computed in polynomial time.

Translation distance

$\mathcal{A}$: arc complex of D_n

- vertex:= $\left\{ \begin{array}{l} \text{isotopy class of oriented, non-}\partial \text{ parallel} \\ \text{properly embedded arc } \gamma \text{ from } *_0 \text{ to } * \end{array} \right\}$
- edge: $[\gamma]$ and $[\delta]$ are connected by an edge of length 1
  $\iff$ Interior of $[\gamma]$ and $[\delta]$ are disjoint



Translation distance

B_n acts on $\mathcal{A}$ as an isometry.

Definition

The translation number $t(\beta)$ of $\beta \in B_n$

$$t(\beta) = \min\{d_{\mathcal{A}}([\gamma], \beta([\gamma]) \mid [\gamma] \in \mathcal{A}\}.$$

Simple observation

FDTC can be estimated by the number of intersections of γ and $\beta(\gamma)$ near the boundary: In particular,

$$t(\beta) \geq |c(\beta)|$$

(remark: $|t(\beta) - |c(\beta)||$ may be arbitrary large).

Main results II: sufficiently complicated braids

Assume that K is represented by a closure of a braid β.

Theorem (I.Kawamuro)

If $c(\beta) > 1 + \frac{N}{2}$, then for any $n \leq N$,

$\delta(K) \leq n \iff$ K is represented as a closure of a braid β which is a product of positive band generators and n negative band generators.

Corollary: SQP conjecture for FDTC>1 closed braids

If $c(\beta) > 1$, K is strongly quasipositive if and only if the Bennequin's inequality is sharp.

Main results II: sufficiently complicated braids

Assume that K is represented by a closure of a braid β.

Theorem (I.Kawamuro)

If $t(\beta) \geq 2(N+2)$, then for any $n \leq N$,

$\delta(K) \leq n \iff$ K is represented as a closure of a braid β which is a product of positive band generators and n negative band generators.

Corollary: SQP conjecture for translation number ≥ 4 closed braids

If $t(\beta) \geq 4$, K is strongly quasipositive $\iff$ the Bennequin's inequality is sharp.

Remark:SQP conjecture is “almost surely” true

Thus, our (generalized SQP) conjecture is true if

K is represented by “sufficiently complicated” braid.

This assumption is generic (unlike fiberedness assumption) in some sense, although it is not true for many knots with small crossing number.

Remark: random knots – conjecture is “almost surely” true

For a given constant C and a random braid β, the probability that $|c(\beta)| \leq C$ (or, $|t(\beta)| \leq C$) is zero.

$\Rightarrow$ when we consider a random knot which arise as a closure of random braid, (generalized) SQP conjecture is true with probability 1.

Geometric background: existence of Bennequin surface

(Generalized) SQP conjectures is actually viewed as a geometric problem of transverse knots.

Bennequin surface Conjecture

Every transverse knot admits a minimum genus Bennequin surface

Theorem (I.-Kawamuro)

If the Bennequin conjecture is true, the generalized SQP conjecture is true.

An existence of minimum genus Bennequin surface is a subtle problem:

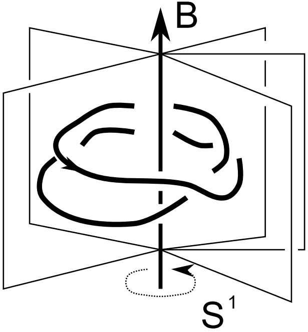
Theorem (Birman-Menasco, I-Kawamuro)

Every topological knot admits a minimum genus Bennequin surface.

Actually we study Bennequin surface Conjecture to prove main theorems.

Generalization

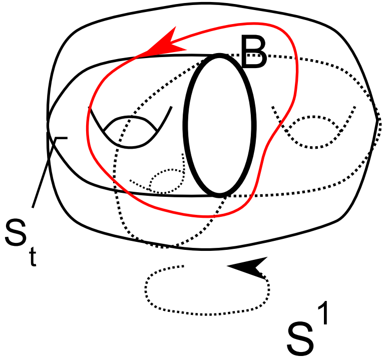
A closed braid in S^3 can be seen as a link transverse to fiber of the disk fibration $S^3 \setminus B \to S^1$ for unknot (=axis B).



We can generalize a theory for a general 3-manifold M.

Generalization

- Take a fibered link $B \subset M$ of a 3-manifold M and fibration $\pi : M \setminus B \to S^1$ (:=open book decomposition of M): one can construct a contact structure of M from (B, π) (Giroux correspondence).
- An oriented link $L \in M \setminus B$ is a closed braid if it transverse fiber (page $S_t = \pi^{-1}(t)$.)



General case

We can define a notion of strongly quasipositive closed braid and Bennequin surface. Assuming suitable conditions (the tightness of the contact structure, null-homologous property of knots,...) we can ask the same conjectures:

SQP Conjecture

K is strongly quasipositive $\iff SL(K) = 2g(K) - 1$

Generalized SQP Conjecture

$\delta(K) \leq n \iff K$ is represented as a closed braid β bounding a Bennequin surface with n negative bands.

Bennequin surface Conjecture

Every transverse knot admits a minimum genus Bennequin surface

Theorem:general case

Actually we will work in general open book case, and the S^3 case is a corollary of the general case argument.

Theorem (I.-Kawamuro)

- For general open books, if the Bennequin surface conjecure is true, then our generalized SQP conjecture is true.
- Every (null-homologous) transverse link in genera open book bounds a minimum genus Bennequin surface.

Theorem:general case

Theorem (I.-Kawamuro (work in progress))

For general case, Generalized SQP conjecture is true assuming that:

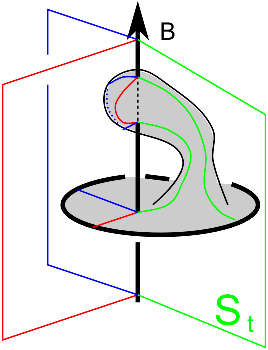
- a knot is represented by a closed braid with large FDTC/translation number.

In particular, under this assumption, a knot K is strongly quasipositive if and only if the Bennequin's inequality is sharp.

II. Idea and sketch of proofs

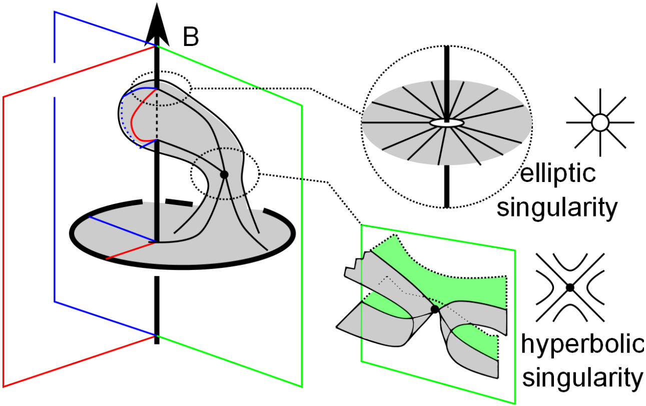
Idea of proof: braid/open book foliation

The proof of theorem uses a method of braid/open book foliation (Birman-Menasco, I-Kawamuro)
For a Seifert surface F, we consider the foliation $\mathcal{F} = \{S_t \cap F\}$, where $S_t = \pi^{-1}(\{t\})$, $\pi : S^3 \setminus B \to S^1$.



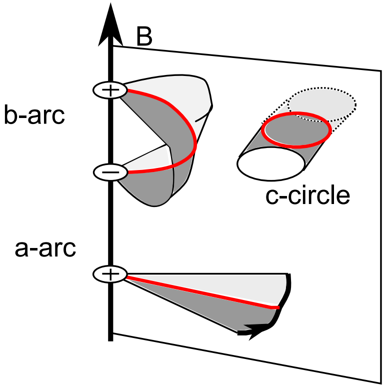
Idea of proof: braid/open book foliation

- F transverse to B, and $F \cap B$ is an elliptic singular point of $\mathcal{F}$.
- S_t is transverse to F or has saddle tangency with F. The saddle tangency is a hyperbolic singular point of $\mathcal{F}$
- According to (co)orientation ellip./hyp. singular point has a sign.



Idea of proof: braid/open book foliation

Regular leaf of $\mathcal{F}$ is classified into 3-types: a-arc, b-arc, and c-circle.



By sign reasons, every regular leaf from negative elliptic points is b-arc.

Some technical remarks

There are several important (sometimes technical, and sometimes critical) differences between the case braid foliation (S^3, disk open book) and general open book foliation.

Remark

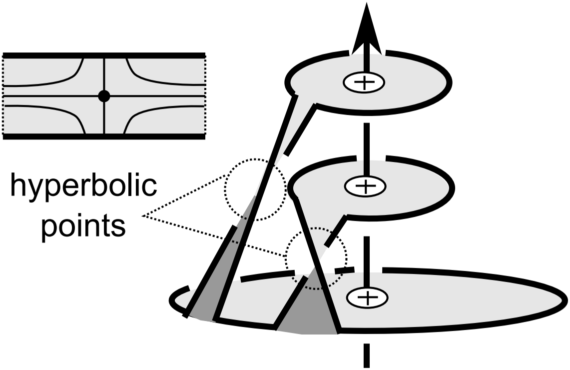
- For the braid foliation case, (when F is incompressible) we may assume that there are no c-circles – this greatly simplifies the arguments.
- For the braid foliation case, b-arc is separating arc – this is technically important, since it means we can always apply certain operation of braid/open book foliation, called foliation change.

Braid foliation view of Bennequin surface

Observation

F is a Bennequin surface $\iff$ every leaf is an a-arc.

(actually we adopt this as a definition of Bennequin surface in general open books)



Braid foliation and self-linking number

Proposition

Let $e_\pm$ and $h_\pm$ be the number of positive/negative elliptic/hyperbolic points of the braid(open book) foliation $\mathcal{F}$. Then

$$sl(\mathcal{K}) = (-e_+ + e_-) + (h_+ - h_-), \quad \chi(F) = (e_+ + e_-) - (h_+ + h_-).$$

Corollary

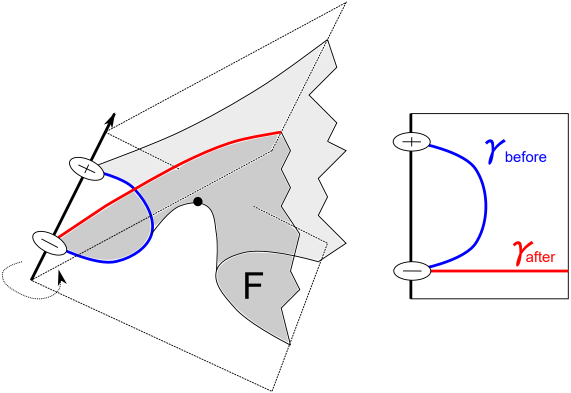
If F is a minimum genus Bennequin surface of a transverse knot $\mathcal{K}$, then $e_- = 0$ hence

$$\delta(K) = h_- = \#\{\text{negative bands in a Bennequin surface}\}.$$

Key observation: FDTC (translation number) bound

Key observation

When we pass hyperbolic point, b-arcs from a negative hyperbolic point changes. Furthermore $\gamma_{before} \cap \gamma_{after} = \emptyset$.
$\Rightarrow$ number of hyperbolic point gives an upper bound of FDTC/translation number !!!



Outline of the proof

- Put a minimum genus Seifert surface F having braid foliation.
- Assume that F is not a Bennequin surface – then it has b-arc, and negative elliptic points.
- Put F in a better position – for example, the number of singular point is small, etc) – (this part uses several techniques in braid/open book foliation theory)
- Assumption that $\delta(K) = n \Rightarrow$ upper bound on the number of the (negative) hyperbolic points (near some negative elliptic points)
- bounds on the number of hyperbolic points $\Rightarrow$ bounds on FDTC/translation number
- Since we assume FDTC/translation number is large enough, it is a contradiction !!
  $\Rightarrow F$ is a Bennequin surface.

Characterization for knots admitting minimum genus Bennequin surface

Theorem (I-Kawamuro, in preparation:)

Let M be a 3-manifold with a fixed open book decomposition (B, π) (i.e. we fix a fibered link and fibration $M \setminus B \to S^1$).
A transverse knot $\mathcal{K}$ has a minimum genus Bennequin surface (w.r.t the fixed open book), if and only if there minimum genus Seifert surface F of $\mathcal{K}$ such that

- all the intersections of F and B are positive, and
- $\partial\mathcal{K}$ is a closed braid.

Reference

Main part of this talk is based on the paper

- T. Ito and K. Kawamuro,
  The defect of Bennequin-Eliashberg inequality and Bennequin surfaces,
  Indiana Univ. Math. J. to appear.