Int'l Workshop on Geometric Quantization and Applications

Note Title

Deformation of the Prequantum Action

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- geometric quantisation

  background + joint work with W. Kirwin (2006)

- deformation quantisation

  background + joint work with A. Yoshioka (to appear)

- fermionic systems

  background + W. (2015 + to appear)

Classical mechanics

(M^{2n}, ω) symplectic manifold $\omega \in \Omega^2(M)$ non-degenerate, $d\omega = 0$.

Locally $\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$, contains position and momentum variables

Poisson structure $\pi = \omega^{-1} \in \Gamma(M, \Lambda^2 TM)$

Hamiltonian vector field $\iota_{H_f} \omega = -df$ or $H_f = \pi(df, \cdot)$

Poisson bracket $\{f, g\} = \pi(df \wedge dg) = -\omega(H_f, H_g)$.

Geometric quantisation

Kostant (1970) Souriau (1970)

classical phase space (M,ω) $\rightsquigarrow$ Hilbert space $\mathcal{H}$ of quantum states

prequantisation. $\begin{matrix} \ell \\ \downarrow \\ M \end{matrix}$ Hermitian line bundle with connection ∇,

$\mathrm{curv}(\nabla) = \frac{\omega}{\sqrt{-1}\hbar}$. $\hbar > 0$ is a constant.

the prequantum action of $C^\infty(M)$ on $\Gamma(M,\ell)$

$$f \mapsto \hat{f} := -\sqrt{-1}\hbar\, \nabla_{H_f} + f$$

$$\widehat{\{f,g\}} = \frac{1}{\sqrt{-1}\hbar}[\hat{f},\hat{g}]. \quad \forall f,g \in C^\infty(M)$$

In physics, $\Gamma(M,\ell)$ is two big as the quantum Hilbert space

(the representation of $C^\infty(M)$ on $\Gamma(M,\ell)$ is highly reducible)

Choose a complex polarisation. an almost complex structure J on M compatible with ω. Let $\mathcal{J}$ = the set of such J.

Then $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, $\nabla = \nabla_J^{1,0} + \nabla_J^{0,1}$

$$\Gamma_J(M,\ell) = \{\psi \in \Gamma(M,\ell) : \nabla_J^{0,1}\psi = 0\}$$

$$\mathcal{H}_J = L^2\text{- completion of } \Gamma_J(M,\ell).$$

<u>Problem</u> How does $\mathcal{H}_J$ depend on $J \in \mathcal{J}$?

Axelrod - Della Pietra - Witten (JDG, 1991)

Hilbert space bundle

$$\begin{tikzcd} \mathcal{H}_J \arrow[d] & \mathcal{H} \arrow[d] & \subset \mathcal{J} \times \Gamma(M,L) \arrow[d] \\ J \in & \mathcal{J} & = \mathcal{J} \end{tikzcd} \quad \rightsquigarrow \quad \text{connection } \nabla^{\mathcal{H}} \text{ on } \mathcal{H} \to \mathcal{J}.$$

If $\nabla^{\mathcal{H}}$ is projectively flat, then the concept of quantum states does not depend on the choice of J.

This is so if (M,ω) is a symplectic vector space (V,ω) and $\mathcal{J}$ is the set of compatible linear complex structures.

Kirwin - W. (CMP, 2006)

$\mathcal{J}$ is identified as Siegel's upper-half space $Sp(2n,\mathbb{R})/U(n)$

which is a non-compact Hermitian symmetric space,

or a classical domain in $\mathbb{C}^{\frac{1}{2}n(n+1)}$ (Siegel, 1943)

If $n=1$, $\mathcal{J}$ is the upper-half plane or the unit disc in $\mathbb{C}$.

$\mathrm{curv}(\nabla^{\mathcal{H}}) = -\frac{1}{4}\mathrm{tr}_{V_J^{1,0}}(\delta J \wedge \delta J)$ ~ standard Kähler form

Kirwin-W. (2006) continued

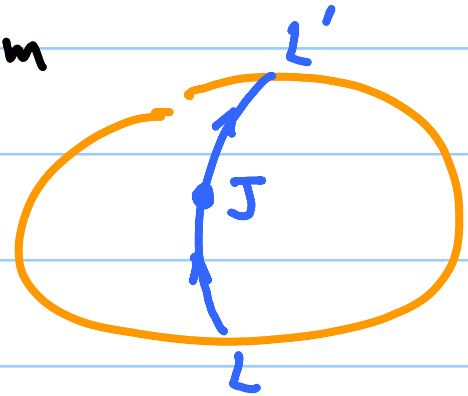
the Shilov boundary of J is $\{L:$ real Lagrangian subspace $\}$

$\mathcal{H}_L = \{L^2$-functions on $V/L\}$ in the real polarisation L.

parallel transport along the geodesics in $\mathcal{J}$

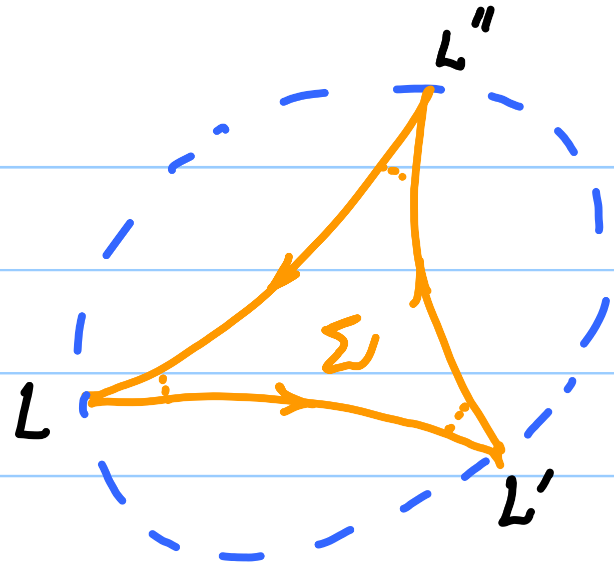
from L to J : Segal-Bargmann transform

from L to L' : Fourier transform



Generalisations: toric manifold Kirwin-Mourão-Nunes (2013)

Given $L, L', L'' \in \mathcal{L}$, mutually transverse.



$$\mathcal{F}_{LL''} \circ \mathcal{F}_{L''L'} \circ \mathcal{F}_{L'L} = e^{\frac{\sqrt{-1}}{2}\int_\Sigma \sigma} \, \mathrm{id}_{\mathcal{H}_L}$$

(holonomy of proj flat conn.)

Compare Lion-Vergne (1980)

$$\mathcal{F}_{LL''} \circ \mathcal{F}_{L''L'} \circ \mathcal{F}_{L'L} = e^{\frac{\sqrt{-1}}{4}\pi\tau(L,L',L'')} \, \mathrm{id}_{\mathcal{H}_L}$$

$$\tau(L, L', L'') = \frac{1}{2\pi}\int_\Sigma \sigma$$

W. (2011)

Deformation quantisation Bayen-Flato-Frønsdal-Lichnerowicz - Sternheimer (1978)

non-commutative but associative star product

$$f * g = fg + c_1(f,g)\hbar + c_2(f,g)\hbar^2 + \cdots$$

each $c_k(f,g)$ is a differential polynomial of f, g. $c_1(f,g) - c_1(g,f) = \{f,g\}$.

$1 * f = f * 1 = f$ and $(f*g)*h = f*(g*h)$.

existence of formal star product on symplectic manifolds (de Wilde-Lecomte, 1983)

Poisson manifolds (Kontsevich, 2003)

Problem Is the formal power series in $\hbar$ convergent ?

Yes if (M,ω) is a symplectic vector space (V,ω)

Moyal product (1949) $\quad f *_0 g = f\, e^{\frac{\sqrt{-1}\hbar}{2}\pi^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}\, g$

converges for "reasonable" functions f, g.

e.g. Omori-Maeda-Miyazaki-Yoshioka (1999)

$\mathcal{E}_p(V) = \{ f : \|f\|_{p,s} = \sup_{x\in V} |f(x)| e^{-s|x|^p} < \infty \}$ Fréchet space

If $0 < p \leq 2$, $*_0 : \mathcal{E}_p(V) \times \mathcal{E}_p(V) \to \mathcal{E}_p(V)$ is non-formal

They also introduced $\quad f *_K g = f\, e^{\frac{\sqrt{-1}\hbar}{2}(\pi + K)^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}\, g$, $\forall K \in S^2 V^{\mathbb{C}}$.

associative, non-formal

$$\begin{tikzcd} \mathcal{O} = S^2V^{\mathbb{C}} \times \mathcal{E}_p(V) \arrow[d] \\ S^2V^{\mathbb{C}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_p(V) \arrow[r, "I_K^{K'}"] \arrow[d] & \mathcal{E}_p(V) \arrow[d] \\ K & K' \end{tikzcd}$$

intertwining operator $\quad I_K^{K'} f = e^{\frac{\varepsilon\hbar}{4}(K'-K)^{\mu\nu}\partial_\mu\partial_\nu} f.$

$$I_K^{K'}(f *_K g) = (I_K^{K'} f) *_{K'} (I_K^{K'} g).$$

$\rightsquigarrow$ $\exists$ flat connection $\nabla^{\mathcal{O}}$ on $\mathcal{O} \to S^2V^{\mathbb{C}}$ such that the parallel transport from K to K' is $I_K^{K'}$.

W.- Yoshioka (to appear)

$J \hookrightarrow S^2 V^a$ $\qquad J \mapsto K_J := \frac{\sqrt{-1}}{2}(J\otimes 1 - 1\otimes J)\Pi$

$*_J = *_{K_J}$ is the normal order star product w.r.t. J.

$\frac{1}{2}(\Pi + K_J) = \left(\frac{1+\sqrt{-1}J}{2} \otimes \frac{1-\sqrt{-1}J}{2}\right)\Pi$

$f *_J g = f\, e^{\sqrt{-1}\hbar\, \Pi^{\bar{i}j} \overleftarrow{\partial}_{\bar{i}} \overrightarrow{\partial}_j} g$, where i, j are indices in $V_J^{1,0}$.

Now $f \in \mathcal{O}_J$, $\psi \in \mathcal{H}_J$, define

$$f *_J \psi = f\, e^{\frac{\sqrt{-1}\hbar}{2}(\Pi + K_J)^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\nabla}_\nu} \psi = f\, e^{\sqrt{-1}\hbar\, \Pi^{\bar{i}j} \overleftarrow{\partial}_{\bar{i}} \overrightarrow{\nabla}_j} \psi$$

Note: $\nabla_{\bar{i}} \psi = 0$, $[\nabla_j, \nabla_k] = 0$.

Properties

1. non-formal, $f *_J \psi = \overbrace{f\psi - \sqrt{-1}\hbar\, \nabla_{H_f}\psi}^{\text{prequantum action}} + O(\hbar^2)$
2. If $\nabla_J^{0,1}\psi = 0$, then $\nabla_J^{0,1}(f *_J \psi) = 0$
3. associativity: $(f *_J g) *_J \psi = f *_J (g *_J \psi)$.
4. for any path $\{J_t\}_{t\in[0,1]}$ from J_0 to J_1 in $\mathcal{J}$,

$$U_{\{J_t\}}(f *_{J_0} \psi) = (I_{J_0}^{J_1} f) *_{J_1} (U_{\{J_t\}}\psi),$$

where $U_{\{J_t\}}$ is the parallel transport under $\nabla^{\mathcal{H}}$.

proof of 2: $\nabla_{\bar i}(f *_J \psi) = \cdots$

$$= f *_J \Big(\cancel{\overleftarrow{\partial}_{\bar i}} + \sqrt{-1}\hbar\, \pi^{jk} \cancel{\overleftarrow{\partial}_j \underbrace{\overrightarrow{[\nabla_{\bar i}, \nabla_k]}}_{\omega_{\bar i k}/\sqrt{-1}\hbar}} + \cancel{\overrightarrow{\nabla}_{\bar i}} \Big)\psi$$

$$= 0$$

Fermionic systems

phase space : Euclidean space V

$C^\infty(V)$ or $S(V^*)$ $\rightsquigarrow$ $\Lambda^\bullet V^*$

Weyl algebra $\rightsquigarrow$ Clifford algebra

Geometric quantisation $\begin{cases} \text{prequantisation} & \text{Kostant (1975)} \\ \text{quantisation} & \text{Woodhouse (1981)} \\ \text{projectively flat connection} & \text{W. (2015)} \end{cases}$

$$\begin{array}{c} \mathcal{H} \\ \downarrow \\ \mathcal{J} \end{array} \cong SO(m)/U([\tfrac{m}{2}])$$

compact Hermitian symmetric space

L-K Hua (1944)

$\forall J \in \mathcal{J}$, (pre)quantisation $\Rightarrow$

$\mathcal{H}_J$ = (spinor repr.) fermionic gaussian $\subset \Lambda^\bullet V_{\mathbb{C}}^*$.

They form a vect bundle $\mathcal{H} \to \mathcal{J}$ with a projectively flat connection W. (2011, 2012)

- holonomy ~ fermionic analog of triple Maslov index introduced by Magneron (1984, 1988)
- the connection becomes flat upon metaplectic correction

Deformation quantisation

- fermionic star product on $\Lambda^\bullet V_{\mathbb{C}}^*$

  Hirshfeld, Henselder (2003, 2005)

- star product of $\Lambda^\bullet V_{\mathbb{C}}^*$ on $\mathcal{H}_J$

- flat and proj'ly flat connections over $\mathcal{J}$

  W. (to appear)