Analyticity of the Bargman projection and semiclassical asymptotics

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Joint work with Ophélie Rouby and Johannes Sjöstrand

Contents

Contents

Classical-quantum correspondence

Quantum mechanics:

- Quantum state: $\psi \in \mathcal{H}$ (Hilbert). eg: $\mathcal{H} = L^2(X)$.
- Quantum observable (Q. Hamiltonian): selfadjoint operator $\mathcal{L}$.
- evolution: (Schrödinger equation)

$$\frac{\hbar}{i}\partial_t \psi = \mathcal{L}\psi. \qquad \psi(t) = e^{\frac{it}{\hbar}\mathcal{L}}\psi(0).$$

- Stationary states $\psi(t,q) = e^{\frac{i\lambda t}{\hbar}} u(q)$: hence $\mathcal{L}u = \lambda u$.

Classical mechanics:

- Phase space: (M, ω) symplectic manifold. eg: $M = \mathbb{R}^{2n}$.
- Hamiltonian $H \in C^\infty(M)$.
- evolution: Hamiltonian flow

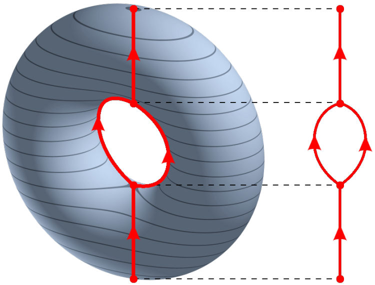
Semiclassical regime: $\hbar \to 0$

Morse Hamiltonian and Reeb graph

Georges Reeb

Morse Hamiltonian and Reeb graph

(M, ω) a 2-dimensional symplectic manifold (surface). $p : M \to \mathbb{R}$ a Morse function. We are interested in the (singular) foliation of M by level sets of p. The **Reeb graph** is the set of leaves.



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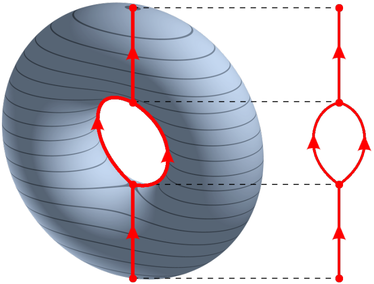

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Question

What is the quantum analogue/footprint[1] of the singular foliation?

[1] Cf Polterovich's talk

Inverse spectral theory for 1D pseudo-differential operators

Let $P := \mathsf{Op}_\hbar^w(p_\hbar)$ where $p_h := p + \hbar p_1 + \hbar^2 p_2 + \cdots$ is a symbol on $\mathbb{R}^2$, elliptic at infinity. Let $I \subset \mathbb{R}$ be an interval. We assume $M = p^{-1}(I)$ is compact ($\Rightarrow \sigma(P) \cap I$ is discrete).

Theorem (*[VN,2011]*)

Suppose that $p_{\restriction M}$ is a simple Morse function. Assume that the graphs of the periods of all trajectories of the hamiltonian flow defined by $p_{\restriction M}$, as functions of the energy, intersect generically. Then the knowledge of the spectrum $\sigma(P) \cap I + \mathcal{O}(\hbar^2)$ determines the symplectic type of (M, ω, p).

Strategy:

- detect singularities
- detect topology (Reeb graph)
- detect symplectic invariants
- use the *[Dufour-Molino-Toulet, 1992]* classification

Berezin-Toeplitz quantization on Kähler manifolds

Let M be a compact Kähler manifold with a prequantum line bundle $L \to M$. Let $H^0(X; L^k)$ be the space of holomorphic sections of L^k, for $k \geq 1$.
A family of linear operators $(A_k : H^0(X; L^k) \to H^0(X; L^k))_{k \geq 1}$ is a **Berezin-Toeplitz** operator if there is a smooth symbol $a : M \to \mathbb{R}$ with

$$a \sim a_0 + k^{-1} a_1 + k^{-2} a_2 + \cdots$$

(in the C^∞ topology) such that

$$A_k = BT(a) := u \mapsto \Pi_k(au),$$

where $\quad \Pi_k : L^2(X; L^k) \to H^0(X; L^k)$
is the orthogonal projection
(Bergman projection).

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[Berezin, 1975], *[Boutet de Monvel-Guillemin, 1981]*, *[Bordemann-Meinrenken-Schlichenmaier, 1994]*, *[Borthwick-Paul-Uribe, 1998]*, *[Charles, 2003]*, etc…
and extension to more general symplectic manifolds: *[Ma-Marinescu, 2008]*, *[Charles, 2014]*, etc.

Inverse spectral theorem for Berezin-Toeplitz operators

Theorem ([Le Floch, 2014])

Let M be a compact, connected Kähler manifold of real dimension 2, with a prequantum line bundle $L \to M$. Let $(A_k)_{k\in\mathbb{N}^*}$ be a self-adjoint Berezin-Toeplitz operator on M whose symbol a is a simple Morse function (...) Then the knowledge of the spectrum $\sigma(A_k) \cap I + \mathcal{O}(k^{-2})$ determines the symplectic type of (M, ω, a).

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In particular, we can recover from the spectrum the Reeb graph and its (singular) integral affine structure.

Remark: recovering the Reeb graph is *not obvious* (spectrum is 1D).

Non selfadjoint operators

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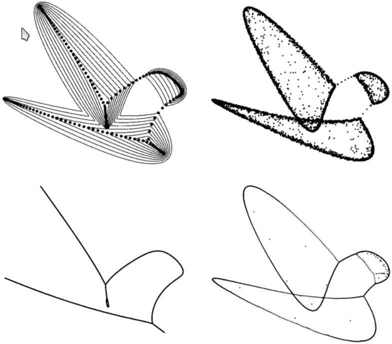
Non selfadjoint operators

An idea to "reveal" the topology in the spectrum is to make the spectrum complex. (Non-selfadjoint operators).
But there is a (huge) price to pay: general non-selfadjoint operators have very unstable spectrum:
Eg: limit of large Toeplitz matrices, from *[Trefethen-Embree, 2005]*:



$$\mathbf{A} = \begin{pmatrix} 0 & 2\mathrm{i} & -1 & 2 & & \\ 0 & 0 & 2\mathrm{i} & -1 & 2 & \\ -4 & 0 & 0 & 2\mathrm{i} & -1 & 2 \\ -2\mathrm{i} & -4 & 0 & 0 & 2\mathrm{i} & -1 \\ & -2\mathrm{i} & -4 & 0 & 0 & 2\mathrm{i} \\ & & -2\mathrm{i} & -4 & 0 & 0 \end{pmatrix};$$

Bohr-Sommerfeld for analytic non-selfadjoint operators

In order to fight non-selfadjoint instability, we need strong "rigidity" or "integrability" conditions.

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Theorem ([Rouby, 2018])

Let P_ϵ be an analytic pseudodifferential operator on $\mathbb{R}$ or S^1 of the form $P_\epsilon = P_0 + i\epsilon Q$, where P_0 is selfadjoint with discrete spectrum, and Q is P_0-bounded.
Then, near any regular value of the symbol p_0, the spectrum of P_ϵ is given by $\{g(\hbar m;\epsilon); m \in \mathbb{Z}\}$, where $g : \mathbb{C} \to \mathbb{C}$ is holomorphic and

$$g \sim g_0 + \hbar g_1 + \hbar^2 g_2 + \cdots$$

Moreover, g_0 is the inverse of the action variable, and

$$g_0 \sim p_0 + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2)$$

.

[Melin-Sjöstrand, 2003], [Hitrik-Sjöstrand 2004, 2016], [Hitrik-Sjöstrand-VN, 2007]

Towards Analytic Berezin-Toeplitz calculus

We would like to use this idea for Morse functions on compact manifolds.

Question

Can you extend Rouby's theorem to Berezin-Toeplitz quantization?

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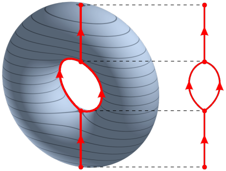
Can you extend Rouby's theorem to Berezin-Toeplitz quantization? Then, can you see the Reeb graph?



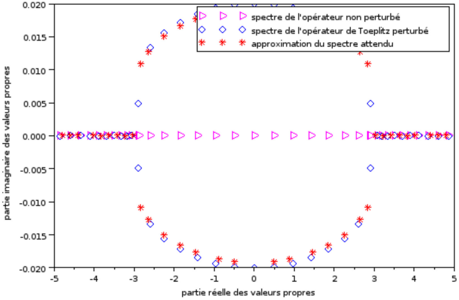


FIGURE 8.4 – $D_k^\epsilon = D_k + i\epsilon A_k$,
$d^\epsilon(p,q) = d_0(p,q) + i\epsilon 2\cos(2\pi q)$.

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Zelditch's question

Recall: $BT(a) := u \mapsto \Pi_k(au)$.

All advances in the microlocal calculus of Berezin-Toeplitz operators rely on the nice semiclassical C^∞ asymptotics of the **Bergman kernel** (Schwartz kernel of Π_k), as $k \to \infty$ *[Fefferman, 1974]*, *[Boutet-Sjöstrand, 1976]*, *[Kashiwara, 1977]*,...

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[Rouby-Sjöstrand-VN, preprint 2018].
Recent simultaneous works by *[Hezari-Xu, preprint 2018]* and *[Deleporte, in preparation]*.

Analytic symbols

[Boutet de Monvel-Kree, 1967], *[Sjöstrand, 1982]*

Definition

A **formal classical analytic symbol** $\hat{a}_\hbar$ in $\Omega \subset \mathbb{C}^n$ is a formal series $\hat{a}_\hbar = \sum_{j=0}^{\infty} a_j \hbar^j$, where $a_j \in \mathrm{Hol}(\Omega)$ satisfies:

$$\forall K \Subset \Omega, \exists C > 0, \forall j \geq 0, \quad \sup_K |a_j| \leq C^{j+1} j^j.$$

Definition

We say that $a_\hbar \in S^0(\Omega)$ is a **classical analytic symbol** if there exists a formal *cas* $\hat{a}_\hbar = \sum_{j=0}^{\infty} a_j \hbar^j$, with the asymptotic expansion: $\forall K \Subset \Omega, \exists C > 0, \forall N \geq 0,$

$$\sup_K \left| a_\hbar - \sum_{j=0}^{N-1} a_j \hbar^j \right| \leq \hbar^N C^{N+1} N^N.$$

Holomorphic Hermitian Line Bundles

Let X be a compact complex manifold of complex dimension n, let L be a holomorphic line bundle over X, equipped with a Hermitian metric g, giving rise to a metric g^k on the tensor product L^k, $k \in \mathbb{N}^*$.

Assume that g has strictly positive curvature: near any point $|s(x)|_L = e^{-\Phi(x)}$, where s is a trivializing section and such that $\omega := i\partial\overline{\partial}\Phi$ is a Kähler form.

ω induces a volume form ω_n on X, and hence a scalar product $\langle\cdot,\cdot\rangle_k$ on the space of sections of L^k. The orthogonal projection

$$\Pi_k : L^2(X, L^k) \to \mathcal{H}^0(X, L^k)$$

is called the Bergman projection. Its distribution kernel is a smooth section $K(\cdot,\cdot;k)$ of $F_k \boxtimes F_k^*$:

$$\Pi_k u(x) = \int_X K(x,y;k)u(y)\omega_n(\mathrm{d}y).$$

Bergman kernels on compact Kähler manifolds

Using trivializing sections s, t near x_0 and y_0, we can write

$$K(x, y; k) = b(x, \bar{y}; k) s_k(x) \otimes t_k(y)^*.$$

Theorem (first part: off diagonal)

Assume that the Hermitian metric g is real-analytic.

1. *If $x_0 \neq y_0$ then there exists $C > 0$ such that, uniformly in a neighborhood $\Omega_0 \times V_0$ of (x_0, y_0),*

$$|K(x,y;k)|_{F_{k,x} \otimes F^*_{k,y}} \leq Ce^{-\frac{k}{C}}.$$

Equivalently, in local trivializations:

$$e^{-k(\Phi_{x_0}(x) + \Phi_{y_0}(y))} |b(x, \bar{y}; k)| = \mathcal{O}(e^{-\frac{k}{C}}).$$

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Mentioned by *[Hezari-Lu-Xu, 2018]* and *[Christ, 2018]*. There are many related results in the literature; see also *[Ma-Marinescu, 2015]*.

Bergman kernels on compact Kähler manifolds

Using a trivializing section s near x_0, we can write, for x, y near x_0:

$$K(x, y; k) = b(x, \bar{y}; k) s_k(x) \otimes s_k(y)^*.$$

Theorem (second part: near diagonal)

2. *For any $x_0 \in X$, there exists a neighborhood Ω_0 of x_0, and a classical analytic symbol a on $\Omega_0 \times \overline{\Omega_0}$, such that, for all $(x, y) \in \Omega_0 \times \Omega_0$, for all $k \geq 1$,*

$$e^{-k(\Phi(x)+\Phi(y))} \left| b_k(x, \bar{y}) - \frac{(2k)^n}{\pi^n} a(x, \bar{y}; k^{-1}) e^{2k\psi(x,\bar{y})} \right| \leq C e^{-\frac{k}{C}},$$

for some constant $C > 0$, where ψ is the polarized form of Ψ.

Cf also *[Hezari-Lu-Xu, 2018]*, *[Hezari-Xu, preprint 2018]*.

Proof strategy

1. Construct an approximate Bergman projection that has the required properties.
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- And then prove that this quantization is equivalent (by an **elliptic analytic FIO**) to the complex Weyl quantization.
- Finally, in the Weyl quantization, it is obvious to construct an approximate Bergman projection.

Brg quantization

Mimicking the formula:

$$\Pi_k u(x) = \int_X K(x,y;k)u(y)\omega_n(\mathrm{d}y),$$

and working locally in $\mathbb{C}^n$ (Φ is a spsh real-analytic function: $\partial\overline{\partial}\Phi \geq m\mathsf{Id}$, $m > 0$, and $\hbar = 1/k$), we define $\mathsf{Op}_r^{\mathrm{Brg}}(a_\hbar)$:

Definition

Let $a_\hbar$ be defined near the anti-diagonal $\{(x, \bar{x})\} \subset \mathbb{C}^n \times \mathbb{C}^n$.

$$[\mathsf{Op}_r^{\mathrm{Brg}}(a_\hbar)u](x) := \int_{B(x,r)} k_\hbar(x,y)u(y)L(\mathrm{d}y),$$

where $r > 0$ and

$$k_\hbar(x,y) := \frac{2^n}{(\pi\hbar)^n} e^{\frac{2}{\hbar}(\psi(x,\overline{y})-\Phi(y))} a_\hbar(x,\overline{y}) \det\left(\partial_{\tilde{w}}\partial_x\psi\right)(x,\overline{y})$$

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Remark: When Φ is quadratic, we have formally $\Pi_\Phi = \mathsf{Op}_\infty^{\mathrm{Brg}}(1)$.

H_Φ spaces

In local coordinates, the holomorphic sections $\mathcal{H}^0(X, L^k)$ are of the form $\tilde{u}(x) = u(x)s_k(x)$ for a holomorphic function u such that

$$\|\tilde{u}\|_k^2 = \int |u|^2 \, e^{-2k\Phi}\omega_n < +\infty,$$

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- If $\Phi = \frac{1}{2}|z|^2$, then $H_\Phi(\mathbb{C}^n)$ is the usual **Bargmann space**.
- Let $u \in H_\Phi(\mathbb{C}^n)$, and $\hbar := 1/k$. We have $u = e^{\Phi/\hbar}v$, with $v \in L^2(\mathbb{C}^n, |\mathrm{d}z|)$, and the condition $\overline{\partial}u = 0$ writes:

$$(\hbar\partial_{\bar{z}} + \partial_{\bar{z}}\Phi)v = 0.$$

It is a semiclassical equation with characteristic set

$$\tilde{\Lambda}_\Phi := \{(x, y; \xi, \eta); (\xi + i\eta) = 2i\partial_{\bar{z}}\Phi(x, y)\} \subset T^*(\mathbb{R}^{2n})$$

FBI transforms

Assume here that Φ is quadratic. Then $\tilde{\Lambda}_\Phi$ is a linear subspace of $T^*\mathbb{R}^{2n}$. If we consider $T^*\mathbb{R}^{2n} \simeq (T^*\mathbb{R}^n) \otimes \mathbb{C}$, then $\tilde{\Lambda}_\Phi \simeq \Lambda_\Phi$ becomes the image of $T^*\mathbb{R}^n$ under a complex symplectic map.

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- Λ_Φ is the "real phase space" associated with functions in $H_\Phi(\mathbb{C}^n)$. (*i.e.*, $H_\Phi(\mathbb{C}^n)$ is the quantization of Λ_Φ.)

FBI transforms

Assume here that Φ is quadratic. Then $\tilde{\Lambda}_\Phi$ is a linear subspace of $T^*\mathbb{R}^{2n}$. If we consider $T^*\mathbb{R}^{2n} \simeq (T^*\mathbb{R}^n) \otimes \mathbb{C}$, then $\tilde{\Lambda}_\Phi \simeq \Lambda_\Phi$ becomes the image of $T^*\mathbb{R}^n$ under a complex symplectic map.

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- *[Sjöstrand, 1982]* showed how to extend this to the non-linear setting (Φ not necessarily quadratic), up to *exponentially small errors* $\mathcal{O}(e^{-\epsilon/\hbar})$.

Complex Weyl quantization *[Sjöstrand, 1982]*

Let $b_\hbar$ be a classical analytic symbol defined in a neighborhood of $(x_0, \theta_0) \in \Lambda_\Phi$ (*i.e.* $\theta_0 := \frac{2}{i}\frac{\partial \Phi}{\partial x}(x_0)$).

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Definition (Complex pseudodifferential operator)

$$\mathsf{Op}_\hbar^{\mathrm{w}}(b_\hbar)u(x) = \frac{1}{(2\pi\hbar)^n}\iint_{\Gamma(x)} e^{\frac{i}{\hbar}(x-y)\cdot\theta} b_\hbar\left(\frac{x+y}{2}, \theta\right) u(y)\mathrm{d}y\mathrm{d}\theta,$$

where $\Gamma(x)$ is a "good contour":

$$\Gamma(x) := \left\{(y,\theta) \in \Omega \times \mathbb{C}^n; \quad \theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x) + iR\overline{(x-y)}; \quad |x-y| \le r\right\}.$$

Theorem

- *If b is bounded then $\mathsf{Op}_\hbar^{\mathrm{w}}(b_\hbar)$ is bounded: $H_\Phi \to H_\Phi$ $\mod \mathcal{O}(e^{-\epsilon/\hbar})$.*
- *If $b \equiv 1$ then $\mathsf{Op}_\hbar^{\mathrm{w}}(b_\hbar) = \mathsf{Id} + \mathcal{O}(e^{-\epsilon/\hbar})$.*

A microlocal equivalence

Theorem (*[Rouby-Sjöstrand-VN]*)

Up to exponentially small errors,

1. *Any* Brg*-operator* $\mathsf{Op}_r^{\mathrm{Brg}}(a_\hbar)$ *can be written as a complex pseudodifferential* $\mathsf{Op}_\hbar^{\mathrm{w}}(b_\hbar)$.
2. *Any complex pseudodifferential* $\mathsf{Op}_\hbar^{\mathrm{w}}(b_\hbar)$ *can be written as a* Brg*-operator* $\mathsf{Op}_r^{\mathrm{Brg}}(a_\hbar)$;

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We shall use item 2 to construct an approximate Bergman projection.
Moreover, the map $b_\hbar \to a_\hbar$ is an elliptic, analytic FIO.
Therefore, if $b_\hbar$ is an analytic symbol, then $a_\hbar$ is also an analytic symbol.

Approximate Bergman projections

Corollary

There exists a classical analytic symbol $a_\hbar$ *such that* $P := \mathsf{Op}_r^{\mathrm{Brg}}(a_\hbar)$ *is an approximate Bergman projection:*

1. P *is selfadjoint;*
2. *The range of* P *is* H_Φ *(modulo* $\mathcal{O}(e^{-\epsilon}/\hbar)$*);*
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Finally, since the true Bergman projection Π_Φ is another Brg-operator with the same properties, one can show that its kernel must be exponentially close to the kernel of $P := \mathsf{Op}_r^{\mathrm{Brg}}(a_\hbar)$. *This proves the main theorem.*

What's next

A large number of potential applications

- Develop symbolic calculus modulo exponentially small remainders for analytic Berezin-Toeplitz operators (Deleporte?)
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Thank you for your attention!