Associated varieties and geometric quantization

David Vogan

Geometric Quantization and Applications
CIRM, October 12, 2018

Outline

First introduction: classical limits and orbit method

Second introduction: solving differential eqns

Third introduction: Lie group representations

Howe's wavefront set and the size of representations

Associated varieties and the size of representations

Turning on your computer

Advertisement for Eva Miranda

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Seek guidance from **EASY** classical limit

$$\text{unitary irr repn of } G \rightsquigarrow \text{coadjt orbit}$$

This talk: define, compute classical limit(unitary rep).

What's wrong with this pic: **EASY** classical limit only computes orbit at infinity. . .

Classical limit of rep π should mean Howe's $\mathrm{WF}(\pi) \subset \mathfrak{g}^*$.

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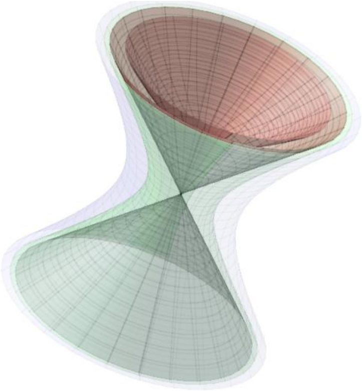
Some coadjoint orbits for $SL(2, R)$.



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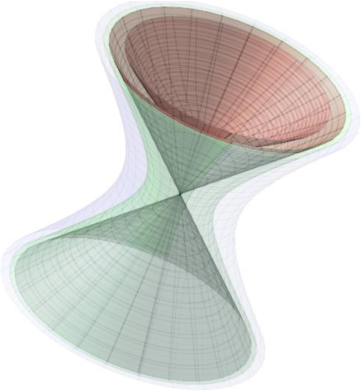

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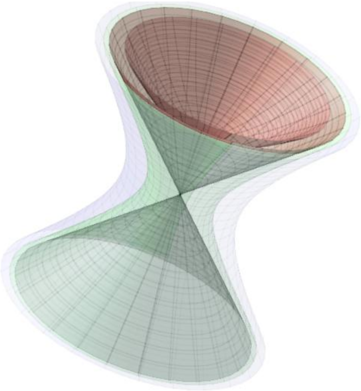

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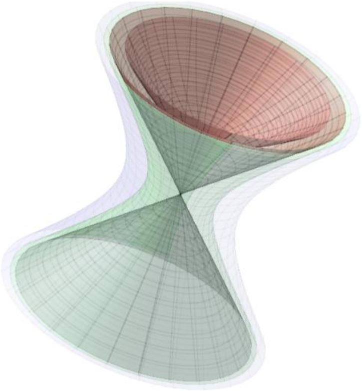

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$S \subset V$ fin diml $\rightsquigarrow$ $\mathrm{Cone}_{\mathbb{R}}(S) = \{\lim_{i\to\infty} \epsilon_i s_i\}$ $\quad(\epsilon_i \to 0^+, s_i \in S)$.

Quantization, classical limits, and cones

Here's what classical limits tell us about quantization.

$$\begin{tikzcd}
\text{coadjt orbit} \arrow[rr, "\text{quantization}"] \arrow[dr, "\text{Cone}_{\mathbb{R}}"'] & & \text{unitary irr} \arrow[dl, bend left, "\text{classical limit}"] \\
& \text{nilp coadjt orbit(s)} \arrow[ur, "\text{quantization}"] &
\end{tikzcd}$$

Desideratum for quantization: diagram commutes.

$\text{Cone}_{\mathbb{R}}(\text{nilp coadjt orbit}) = \overline{\text{nilp coadjt orbit}} \implies$

quantization of nilpotent X must be π with

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Something to do during the talk

G reductive/number field k, $\pi = \otimes_v \pi_v$ automorphic rep.

k_v local field, $G(k_v)$ reductive, $\mathfrak{g}(k_v) = \mathrm{Lie}(G(k_v))$.

Howe: $\pi_v \rightsquigarrow \mathrm{WF}(\pi_v) \subset \mathfrak{g}(k_v)^*$ nilp orbit closure[s].

Conjecture (global coherence of WF sets)

1. $\exists\ x(\pi) \in \mathfrak{g}(k)^*$, $\quad \mathrm{Cone}_{k_v}(G(k_v) \cdot x(\pi)) = \mathrm{WF}(\pi_v)$.
2. $\exists$ global version of local char expansions for π_v.

Says $G(k_v) \cdot x(\pi)$ controls asymptotics of $\pi_v|_{K_v}$.

Orbit of $x(\pi) \rightsquigarrow$ algebraic cone over $\overline{k}$

$$N(\pi) = \mathrm{Cone}_{\overline{k}}(G(\overline{k}) \cdot x(\pi)) \subset \mathcal{N}^*_{\overline{k}}$$

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$\mathrm{WF}(\pi_v) \subset N(\pi)_{k_v}$, but possibly $\mathrm{WF}(\pi_v) \cap N(\pi)^0_{k_v} = \emptyset$.

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1. $\exists\ x(\pi) \in \mathfrak{g}(k)^*, \quad \mathrm{Cone}_{k_v}(G(k_v) \cdot x(\pi)) = \mathrm{WF}(\pi_v)$.
2. $\exists$ global version of local char expansions for π_v.

Says $G(k_v) \cdot x(\pi)$ controls asymptotics of $\pi_v|_{K_v}$.

Orbit of $x(\pi) \rightsquigarrow$ algebraic cone over $\overline{k}$

$$N(\pi) = \mathrm{Cone}_{\overline{k}}(G(\overline{k}) \cdot x(\pi)) \subset \mathcal{N}^*_{\overline{k}}$$

closure of one $G(\overline{k})$ nilpotent orbit $N(\pi)^0$.

$\mathrm{WF}(\pi_v) \subset N(\pi)_{k_v}$, but possibly $\mathrm{WF}(\pi_v) \cap N(\pi)^0_{k_v} = \emptyset$.

All π_v same size EXCEPT for finite arithm set of v.

Something to do during the talk

G reductive/number field k, $\pi = \otimes_v \pi_v$ automorphic rep.

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If ϕ integrable function of $x \in \mathbb{R}^n$, Fourier transform is

$$\widehat{\phi}(\xi) = \int e^{2\pi i \langle x, \xi \rangle} \phi(x) dx.$$

Still makes sense if ϕ is compactly supported distribution on $\mathbb{R}^n$: apply ϕ to $x \mapsto e^{2\pi i \langle x, \xi \rangle}$

ϕ msre of cpt support $\implies$ $\widehat{\phi}$ bounded fn of ξ.

Take m derivs of ϕ $\rightsquigarrow$ multiply $\widehat{\phi}$ by degree m poly.

mth derivs(ϕ) = cpt supp msres $\implies$ $\widehat{\phi}(\xi) \le C_m/(1 + |\xi|)^m$.

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Function f on manifold M has *support*:

$$\text{supp}(f) = \text{closure of } \{m \in M \mid f(m) \neq 0\}.$$

Generalized fn ϕ is continuous linear fnl on test densities.

Can multiply ϕ by bump f_0 at m_0 to study "ϕ near m_0."

Singular support of ϕ is where it isn't smooth:

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Behavior of solutions of PDEs

Suppose D is order k diff op on M.

D has *symbol* $\sigma_k(D)$: fn on T^*M, hom poly on $T^*_m(M)$.

$\rightsquigarrow$ *characteristic variety* of D

$$\mathrm{Ch}(D) =_{\mathrm{def}} \{(m,\xi) \in T^*(M) \mid \sigma_k(D)(m,\xi) = 0\}$$

$$D\phi = \psi \implies \mathrm{WF}(\phi) \subset \mathrm{WF}(\psi) \cup \mathrm{Ch}(D):$$

solving D adds singularities only in $\mathrm{Ch}(D)$.

$D_1, \ldots, D_m$ diff ops on $M \rightsquigarrow$ char var of system

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Solns of systems: if $D_j\phi = 0$, all j, then

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PDE on M $\leftrightsquigarrow$ module for diff op alg $D(M)$.

Noncomm alg $D(M)$ $\approx$ comm alg Poly$(T^*(M))$.

$=$ Smooth fns that are polys along each $T_m^*(M)$.

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What groups?

$G = G(\mathbb{R}, \sigma)$ real points of complex connected reductive algebraic group G, σ_c compact real form of G commuting with σ, $K = G(\mathbb{R}, \sigma) \cap G(\mathbb{R}, \sigma_c)$ maximal compact subgroup of G.

(That's for postdocs. They should sweat a little.)

$G \subset GL(n, \mathbb{R})$ closed, transpose-stable, $K = O(n) \cap G$.

(That's for the PDE people. Thank you for showing up!)

Also keep in mind $G = GL(m, \mathbb{H})$, $G = SO(p, q)$.

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Secs of $K(\mathbb{C})$-eqvt reg holonomic $\mathcal{D}$-mod on flag variety.

Example: Normal derivs of Borel-Weil-Bott realization of $K(\mathbb{C})$-rep on $K(\mathbb{C})/B \cap K(\mathbb{C}) \subset G(\mathbb{C})/B$.

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Finite length quasisimple Fréchet rep of moderate growth.

Example: Smooth secs of eqvt vec bdle on $\mathrm{Gr}(k, n)$.

(That's for PDE people. Although demise of language requirements means only the French will know whether the accent on "Fréchet" is correct.)

Trig polys on the circle, as module for

$$\mathrm{Span}(d/d\theta, \cos(2\theta)d/d\theta, \sin(2\theta)d/d\theta) \simeq \mathfrak{sl}(2, \mathbb{R}).$$

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Want analogue of $\mathrm{Ch}(D_1, \ldots, D_m)$ for repn (π, V) of G:

$$\mathrm{WF}_{\mathrm{big}}(\pi) \subset T^*(G) \simeq G \times \mathfrak{g}^*.$$

Desideratum: $\mathrm{WF}_{\mathrm{big}}(\pi)$ closed cone, left and right G-invt.

Left invt $\implies$ $\mathrm{WF}_{\mathrm{big}}(\pi)$ determined by real closed cone

$$\mathrm{WF}(\pi) =_{\mathrm{def}} \mathrm{WF}_{\mathrm{big}}(\pi) \cap T_e^*(G) \simeq \mathfrak{g}^*$$

Right invt $\implies$ $\mathrm{WF}(\pi)$ is $\mathrm{Ad}(G)$-invt: union of orbits.

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system of PDE $D_j\phi = 0$ on $M \rightsquigarrow \mathrm{Ch}(D_1, \ldots, D_m) \subset T^*(M)$ controlling singularities of solns.

Want analogue of $\mathrm{Ch}(D_1, \ldots, D_m)$ for repn (π, V) of G:

$$\mathrm{WF}_{\mathrm{big}}(\pi) \subset T^*(G) \simeq G \times \mathfrak{g}^*.$$

Desideratum: $\mathrm{WF}_{\mathrm{big}}(\pi)$ closed cone, left and right G-invt.

Left invt $\implies$ $\mathrm{WF}_{\mathrm{big}}(\pi)$ determined by real closed cone

$$\mathrm{WF}(\pi) =_{\mathrm{def}} \mathrm{WF}_{\mathrm{big}}(\pi) \cap T^*_e(G) \simeq \mathfrak{g}^*$$

Right invt $\implies$ $\mathrm{WF}(\pi)$ is $\mathrm{Ad}(G)$-invt: union of orbits.

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Characters and sizes

Info about lin op A on n-diml V encoded by char poly:

$$\det(tI - A) = t^n - t^{n-1}\operatorname{tr}(A) + \cdots + (-1)^n \det(A).$$

Lower order coeffs are poly fns of $\operatorname{tr}(A),\ \operatorname{tr}(A^2), \cdots, \operatorname{tr}(A^n)$.

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$$\text{Size of } \pi = n = \Theta_\pi(e).$$

If V inf-diml, $\pi(g)$ isn't trace class, so Θ_π isn't function.

But Θ_π is often a generalized function: if μ is test density on G, then linear operator

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How do you control that?

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Real nilpotent cone

$\mathcal{N}^*_{\mathbb{R}} =_{\text{def}}$ zeros of Ad(G)-invt homog polys $\subset \mathfrak{g}^*$.

Proved: WF(π) $\subset \mathcal{N}^*_{\mathbb{R}}$, Ad($G$)-invt.

Howe's wavefront set defines

$$(\text{irr of } GL(n,\mathbb{R})) \overset{\text{WF}}{\rightsquigarrow} (\text{conj class of nilp mats}).$$

$$(\text{irr of } G) \overset{\text{WF}}{\rightsquigarrow} (G \text{ orbit on } \mathcal{N}^*_{\mathbb{R}}).$$

Size of π = one half real dimension of orbit.

Howe's WF(π) is the perfect classical limit:

$$\text{group representation} \overset{\text{WF}}{\rightsquigarrow} \text{symplectic manifold}$$

in a simple, natural, and meaningful way.

But after forty years, it's still a royal pain to compute.

Next: (computable) algebraic analogue of WF(π).

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$\mathcal{N}^*_{\mathbb{R}} =_{\text{def}}$ zeros of Ad(G)-invt homog polys $\subset \mathfrak{g}^*$.

Proved: WF(π) $\subset \mathcal{N}^*_{\mathbb{R}}$, Ad($G$)-invt.

Howe's wavefront set defines

(irr of $GL(n,\mathbb{R})$) $\overset{\text{WF}}{\rightsquigarrow}$ (conj class of nilp mats).

(irr of G) $\overset{\text{WF}}{\rightsquigarrow}$ (G orbit on $\mathcal{N}^*_{\mathbb{R}}$).

Size of π = one half real dimension of orbit.

Howe's WF(π) is the perfect classical limit:

group representation $\overset{\text{WF}}{\rightsquigarrow}$ symplectic manifold

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(Remember that I asked the senior professors to think about trigonometric polynomials on the circle?)

Harish-Chandra understood that this works for all reps of all reductive G, with K = max cpt subgp.

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$$V_K =_{\text{def}} \{v \in V \mid \dim\langle\pi(K)v\rangle < \infty\} \quad K\text{-finite vecs}$$
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Last slide suggested $V_K = K$-finite vectors in V as algebraic substitute for smooth G rep V.

Definition A $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-module is cplx vec space with $U(\mathfrak{g}_\mathbb{C})$ action, and alg rep of $K_\mathbb{C}$, so that

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Recall idea of WFs and PDE:

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$$\text{system of PDEs} = D\text{-module} \approx \text{Poly}(T^*(M))\text{-module}$$

$$\text{Poly}(T^*(M))\text{-module} \rightsquigarrow \text{cone in } T^*(M)$$

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$$D = \text{diff ops on } M \approx \text{Poly}(T^*(M))$$

$$\text{system of PDEs} = D\text{-module} \approx \text{Poly}(T^*(M))\text{-module}$$

$$\text{Poly}(T^*(M))\text{-module} \leftrightsquigarrow \text{cone in } T^*(M)$$

$$U(\mathfrak{g}_{\mathbb{C}}) = \text{left-invt cplx diff ops on } G \approx \text{Poly}(\mathfrak{g}_{\mathbb{C}}^*)$$

Precisely: $U(\mathfrak{g}_{\mathbb{C}})$ filtered by deg, gr $U(\mathfrak{g}_{\mathbb{C}}) \simeq \text{Poly}(\mathfrak{g}_{\mathbb{C}}^*)$.

fin length $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-mod X has $K_{\mathbb{C}}$-stable good filt,

$$\begin{aligned} \text{gr}\, X &= \text{fin. gen. graded } (\text{Poly}(\mathfrak{g}_{\mathbb{C}}^*/\mathfrak{k}_{\mathbb{C}}^*), K_{\mathbb{C}})\text{-module} \\ &= K_{\mathbb{C}}\text{-eqvt coherent sheaf on } \mathfrak{g}_{\mathbb{C}}^*/\mathfrak{k}_{\mathbb{C}}^* \end{aligned}$$

$\text{AV}(X) =_{\text{def}} \text{supp}\,\text{gr}\, X$,

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What sort of invariant is that?

$\mathcal{N}^*_\theta =_{\text{def}}$ zeros of Ad(G)-invt homog polys $\subset \mathfrak{g}^*_{\mathbb{C}}/\mathfrak{k}^*_{\mathbb{C}}$.

WF(π) proof $\rightsquigarrow$ AV(π_K) $\subset \mathcal{N}^*_\theta$, Ad($K_{\mathbb{C}}$)-invt.

Associated variety defines

$$(\text{irr } (\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})\text{-module } X) \overset{\text{AV}}{\rightsquigarrow} K_{\mathbb{C}}\text{-orbits on } \mathcal{N}^*_\theta).$$

Size of X = complex dim of orbit.

AV(X) is the perfect algebraic classical limit:

$$(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})\text{-module} \overset{\text{AV}}{\rightsquigarrow} \text{algebraic cone}$$

in a simple, natural, and meaningful way.

One way to understand the meaning:

$$\begin{aligned} X|_{K_{\mathbb{C}}} &\simeq (\text{gr } X)|_{K_{\mathbb{C}}} \\ &= (\text{coherent sheaf on AV}(X))|_{K_{\mathbb{C}}}. \end{aligned}$$

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KNOW how to calculate $X|_{K_{\mathbb{C}}}$. So…

FIND eqvt sheaf M on $\mathcal{N}^*_\theta$ such that $X|_{K_{\mathbb{C}}} = M|_{K_{\mathbb{C}}}$.

KNOW how to do that as well. Pet computers are awesome.

CONCLUDE AV(X) = supp(M).

Restate: $AV(X)$ = what can carry the K-types of X.

Such thms $\rightsquigarrow$ Kashiwara & Vergne (Luminy 1978).

Connect 1978$\leftrightsquigarrow$2018 needs $(\mathcal{N}^*_{\mathbb{R}})/G \leftrightsquigarrow (\mathcal{N}^*_\theta)/K_{\mathbb{C}}$.

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How that looks for $SL(2,\mathbb{R})$

$G = SL(2,\mathbb{R})$, $K = SO(2)$, $\widehat{K} = \mathbb{Z}$.

Standard representations are

1. holomorphic (lims of) disc series $I^+(m)$ $(m \geq 0)$,
   $I^+(m)|_K = \{m+1, m+3, m+5, \ldots\}$
2. antihol (lims of) disc series $I^-(m)$ $(m \geq 0)$,
   $I^-(m)|_K = \{-m-1, -m-3, -m-5, \ldots\}$
3. spher princ series $I^{\text{even}}(\nu)$,
   $I^{\text{odd}}(\nu)|_K = \{0, \pm 2, \pm 4 \ldots\}$
4. nonspher princ series $I^{\text{odd}}(\nu)$, $\nu \neq 0$,
   $I^{\text{odd}}(\nu)|_K = \{\pm 1, \pm 3, \pm 5 \ldots\}$

N.B. $I^{\text{odd}}(0) = I^+(0) + I^-(0)$.

Three nilp $SO(2,\mathbb{C})$ orbits on $\mathcal{N}^*_\theta$: $\mathcal{O}^+$, $\mathcal{O}^-$, $\{0\}$.

$\underbrace{\text{Coherent sheaves}}_{\text{restriction to } K}$ on $\overline{\mathcal{O}^+}$: $\underbrace{[I^{\text{even}}(0)] - [I^-(1)]}_{\{0,2,4,\ldots\}}$, $\underbrace{[I^+(0)]}_{\{1,3,5,\ldots\}}$.

Coherent sheaves on $\overline{\mathcal{O}^-}$: $\underbrace{[I^{\text{even}}(0)] - [I^+(1)]}_{\{0,-2,-4,\ldots\}}$, $\underbrace{[I^-(0)]}_{\{-1,-3,\ldots\}}$.

Coh on $\{0\}$: $\underbrace{[I^{\text{even}}(0) - I^+(1) - I^-(1)]}_{\{0\}}$, $\underbrace{[I^+(m) - I^+(m+2)]}_{\{m+1\} \quad (m \geq 0)}$, $\underbrace{[I^-(m) - I^-(m+2)]}_{\{-m-1\} \quad (m \geq 0)}$.

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$G = SL(2,\mathbb{R})$, $K = SO(2)$, $\widehat{K} = \mathbb{Z}$.

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Algorithm for AV(X)

0. Make formulas (Achar theory, `atlas` practice)

$$\mathcal{S}_j = \text{coh shf on } \mathcal{O}_j) = \sum_i s_j^k[I_k] \qquad (I_k \text{ standard rep})$$

1. Write (KL theory, `atlas` practice) char formula

$$X = \sum_i m_i I_i \qquad (I_i \text{ standard rep}).$$

2. Restrict to K: set cont parameters equal to zero.
3. Write (linear algebra)

$$\sum_i m_i I_i|_K = \sum n_j \mathcal{S}_j$$

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Here's how that looks for $SL(2,\mathbb{R})$ (reprise)

Library of coherent sheaves on orbit closures:

$\underbrace{\text{Coherent sheaves}}_{\text{restriction to } K}$ on $\overline{\mathcal{O}^+}$: $\underbrace{[I^{\text{even}}(0)] - [I^-(1)]}_{\{0,2,4,\dots\}}$, $\underbrace{[I^+(0)]}_{\{1,3,5,\dots\}}$.

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Here $[I^+(0)]$ means class in Groth grp of gr $I^+(0)$.

Try X = three-diml adjoint rep, character formula

$$X = I^{\text{even}}(3) - I^+(3) - I^-(3)$$

$$X|_K = I^{\text{even}}(0) - I^+(3) - I^-(3)$$

$$= (I^{\text{even}}(0) - I^+(1) - I^-(1))$$

$$+ (I^+(1) - I^+(3)) + (I^-(1) - I^-(3)).$$

Three terms from orbit $\{0\}$, so $\text{AV}(X) = \overline{\{0\}}$.

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$$X = I^{\text{even}}(3) - I^+(3) - I^-(3)$$

$$\begin{aligned} X|_K &= I^{\text{even}}(0) - I^+(3) - I^-(3) \\ &= (I^{\text{even}}(0) - I^+(1) - I^-(1)) \\ &\qquad + (I^+(1) - I^+(3)) + (I^-(1) - I^-(3)). \end{aligned}$$

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Here $[I^+(0)]$ means class in Groth grp of gr $I^+(0)$.

Try X = three-diml adjoint rep, character formula

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Three terms from orbit $\{0\}$, so $\mathrm{AV}(X) = \overline{\{0\}}$.

Here's how that looks for $SL(2,\mathbb{R})$ (reprise)

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