Singular reduction and quantization

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Introduction

Let M be a closed connected Riemannian manifold and K a compact Lie group acting isometrically on M. Further, let

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be an elliptic, positive, symmetric K-invariant PDO with principal symbol p and self-adjoint extension P.

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Classical mechanics		Quantum mechanics
Ω_ζ / K	$\longleftrightarrow$	$\mathrm{L}^2(M) \simeq \bigoplus_{\sigma \in \widehat{K}} \mathrm{L}^2(M)_\sigma$
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Creed: • Correspondence principle (Bohr)
• $[Q, R] = 0$ (Guillemin-Sternberg)

Questions:

- spectrum of P on $L^2(M)_\sigma \leftrightarrow$ flow of p on Ω_0/K
- shape of eigenfunctions of $P \leftrightarrow$ symmetries
- equiv. cohomology $H^*_K(T^*M) \leftrightarrow$ cohomology $H^*(\Omega_0/K)$
- ergodicity, heat asymptotics, index theory, ...

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Ultimately, one has to understand the asymptotic behavior of the Witten integral

$$I(\mu) = \int_{T^*M} \int_{\mathfrak{k}} e^{i\mu\,\mathbb{J}(x,\xi)(X)} a(x,\xi,X)\, dX\, dx\, d\xi, \quad \mu \to +\infty,$$

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via the stationary phase theorem, where $a \in \mathrm{C}^\infty_\mathrm{c}$ and

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$\hookrightarrow$ overcome this problem by using resolution of singularities.

1. Equivariant spectral geometry

The spectral function of an elliptic operator

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- a positive, symmetric, elliptic PDO $P_0 : \mathrm{C}^\infty(M) \longrightarrow \mathrm{L}^2(M)$ of order m on M with principal symbol $p(x, \xi)$,
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Define for $\mu > 0$, $\mu_j := \sqrt[m]{\lambda_j}$, and $x, y \in M$

$$e(x, y, \mu) := \sum_{\mu_j \leq \mu} \varphi_j(x)\overline{\varphi_j(y)} \qquad \text{spectral function of } Q := \sqrt[m]{P}.$$

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Proof: Via Fourier integral operators:

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$$\tilde{s}_\mu = \sum_{j=0}^{\infty} \varrho(\mu - \mu_j) E_j$$

where $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}^+)$, $\varrho(0) = 1$, $\operatorname{supp} \widehat{\varrho} \subset (-\delta, \delta)$ for some $\delta > 0$.

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The kernel of $U(t)$ is locally given by an oscillatory integral

$$\tilde{U}(t, \tilde{x}, \tilde{y}) = \int_{\mathbb{R}^n} e^{i(\psi(t,\tilde{x},\eta) - \langle \tilde{y}, \eta \rangle)} a(t, \tilde{x}, \eta) d\eta, \quad \tilde{x}, \tilde{y} \in \mathbb{R}^n,$$

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where $a \in S^0_{phg}$, and ψ solves the Hamilton-Jacobi problem

$$\frac{\partial \psi}{\partial t} + \sqrt[m]{p\Big(x, \frac{\partial \psi}{\partial \tilde{x}}\Big)} = 0, \qquad \psi(0, \tilde{x}, \eta) = \langle \tilde{x}, \eta \rangle.$$

Asymptotics for $K_{\tilde{s}_\mu}(x,y)$ then yield the assertion. □

Now, let K be a compact Lie group acting isometrically on M and P_0 be K-invariant. Define for $\sigma \in \widehat{K}$

$$e_\sigma(x,y,\mu) := \sum_{\substack{\mu_j \leq \mu, \\ \varphi_j \in \mathrm{L}^2(M)_\sigma}} \varphi_j(x)\overline{\varphi_j(y)}$$

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Approximate $e_\sigma(x, y, \mu + 1) - e_\sigma(x, y, \mu)$ by

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where

$$\Pi_\sigma = d_\sigma \int_K \overline{\chi_\sigma(k)}\pi(k)\, dk$$

is the projector onto $\mathrm{L}^2(M)_\sigma$,

- $d_\sigma = \dim \sigma$,
- χ_σ character of σ,
- $(\pi, \mathrm{L}^2(M))$ regular representation.

$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)$ is a superposition of Witten-type integrals of the form

$$I_{x,y}(\mu) = \int_{\Sigma_x} \int_K e^{i\mu \Phi_{x,y}(\omega,k)} a(x, y, \omega, k)\, dk\, d\omega,$$

where $a \in \mathrm{C^\infty_c}$, $\Sigma_x \equiv \{\omega \in \mathbb{R}^n : q(x, \omega) = 1\}$, (Y, κ) are coordinates, and

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$\hookrightarrow$ Study asymptotic behavior via stationary phase method and desingularization.

Equivariant Weyl law

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Theorem (Reduced local Weyl law, 2016)

$$e_\sigma(x,x,\mu) = \mu^{n-\kappa_x} \frac{d_\sigma [\pi_{\sigma|K_x} : \mathbf{1}]}{(2\pi)^{n-\kappa_x}} \int_{\{(x,\xi)\in\Omega_0,\, p(x,\xi)<1\}} d\xi + O_{x,\sigma}(\mu),$$

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Example. Let $M = S^2$, $G = \mathrm{SO}(2)$, $P_0 = -\Delta$ the Laplace operator, $Y_{k,m}$ the spherical harmonic functions with eigenvalues $k(k+1)$, $P_{k,m}$ the Legendre polynomials. Then ($0 \leq \varphi < 2\pi$, $0 \leq \theta < \pi$)

$$|Y_{k,0}(\varphi,\theta)|^2 = \frac{2k+1}{4\pi} |P_{k,0}(\cos\theta)|^2 \approx \begin{cases} k & \theta = 0,\pi \\ (\sin\theta)^{-1} & \theta \in (0,\pi). \end{cases}$$

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$\hookrightarrow$ resolution of singularities yields pointwise description of $e_\sigma(x,x,\mu)$ and eigenfunctions.

Consider the orbit type decomposition of M

$$M = M_1 \cup \cdots \cup M_{\aleph}.$$

Theorem (Singular reduced local Weyl law, 2016)

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Corollary (Reduced Weyl's law, 2010)

$$\int_M e_\sigma(x,x,\mu)\, dx = \frac{d_\sigma [\pi_{\sigma|H} : \mathbf{1}]}{(n-\kappa)(2\pi)^{n-\kappa}} \mathrm{vol}[(\Omega \cap S^* M)/K] \mu^{n-\kappa} + O_\mu,$$

where $O_\mu = O(\mu^{n-\kappa-1}(\log \mu)^\Lambda)$, $\Lambda \in \mathbb{N}$, and

$$S^*M = \{(x,\xi) \in T^*M : p(x,\xi) = 1\}.$$

□

Convex bounds for eigenfunctions

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Non-equivariant bounds. As a consequence of Hörmander's local Weyl law one obtains

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Corollary (2018)

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Similarly, equivariant L^p-bounds can be derived.

Subconvex bounds for Hecke–Maas forms

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Spherical bounds. Let

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Iwaniec-Sarnak proved for $\varphi_j \in \mathrm{L}^2(\Gamma \backslash G/K)$ and any $\varepsilon > 0$

$$\|\varphi_j\|_\infty \ll_\varepsilon \lambda_j^{5/24+\varepsilon}$$ spherical subconvex bound.

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Theorem (–,Wakatsuki, 2018)

For $\varphi_j \in \mathrm{L}^2(\Gamma\backslash G)$ one has uniformly in $\sigma \in \widehat{K}$

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Proof: Via asymptotics for $K_{\tilde{s}_\mu \circ \Pi_\sigma}(x,y)$ in a neighborhood of the diagonal and arithmetic amplification. $\square$

Application. Let $\mathcal{C}$ be the Casimir operator on G, so that $\Delta = -\mathcal{C} + l^2/4$ on $\mathrm{L}^2(\Gamma\backslash G)_{\sigma_l}$, $\widehat{K} \ni \sigma_l \equiv l \in \mathbb{Z}$. For fixed Casimir eigenvalue one obtains

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$\hookrightarrow$ Previously shown by Venkatesh using equidistribution methods. Best known bound by Reznikov reads $(1+|l|)^{1/3+\varepsilon}$.

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Equivariant subconvex bounds can be shown also for rather general algebraic groups, based on previous work of Marshall.

2.
Equivariant quantum ergodicity

Classical and quantum ergodicity

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As before, let M be a compact Riemannian manifold M of dimension n,

$$\Delta : \mathrm{C}^\infty(M) \longrightarrow \mathrm{L}^2(M)$$

the Laplacian, and φ_j an ONB of eigenfunctions φ_j with eigenvalues λ_j.

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Classical mechanics:

- S^*M co-shpere bundle $\equiv$ phase space of free particle
- $a \in C^\infty(S^*M)$ function $\equiv$ observable
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Quantization map. $S^l(M) \ni a \longmapsto \mathrm{Op}_\hbar(a)$, $\hbar$ Planck's constant.

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Quantum ergodicity theorem. Consider the distributions

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If Φ_t is ergodic, Shnirelman, CdV, and Zelditch showed that $\exists$ subsequence $\{\varphi_{j_k}\}$ of density 1 such that

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$\hookrightarrow$ Both the classical and quantum system behave less chaotically. By dividing out symmetries (i.e. order), ergodic properties should emerge.

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$\hookrightarrow$ Singular symplectic reduction (Sjamaar-Lerman-Bates) and desingularization yield

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Let $\widetilde{\Phi}_t$ be the reduced geodesic flow on $(\Omega_0 \cap M_{reg})/K$, $\sigma \in \widehat{K}$.

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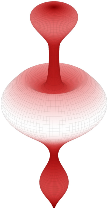
Example. Let $M \subset \mathbb{R}^3$ be a surface of revolution, $e_{l,m}(\varphi,\theta) = f_{l,m}(\theta)e^{il\varphi}$ a basis of $L^2(M)_{\sigma_l}$. Then

$$|e_{l,m}|^2\, dM \to c \cdot dM/R,$$

R distance to symmetry axis.



3.
Equivariant cohomology

Atiyah-Bott-Berline-Vergne localization

Atiyah-Bott-Berline-Vergne localization

Consider

- a manifold M with C^∞-action of a compact Lie group K,
- $\mathcal{A} := (S(\mathfrak{k}^*) \otimes \Lambda(M))^K$,
- $D\varrho(X) := d(\varrho(X)) - \iota_{\widetilde{X}}(\varrho(X)),\ \varrho \in \mathcal{A}$, $X \in \mathfrak{k}$,
- $H^*_G(M) := \mathrm{Ker}\, D/\mathrm{Im}\, D$ equivariant cohomology of M.

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Let $\varrho \in \mathcal{A}$ have compact support, $M_0 = \{m \in M : \widetilde{Y}_m = 0\}$ where $Y \in \mathfrak{k}$, χ_{NM_0} equivariant Euler form. If $D\varrho = 0$,

$$\int_M \varrho(Y) = \int_{M_0} \frac{\varrho(Y)}{\chi_{NM_0}(Y)}$$

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$$\int_M \varrho(Y) = \int_{M_0} \frac{\varrho(Y)}{\chi_{NM_0}(Y)} \quad \text{Atiyah-Berline-Bott-Vergne localization.}$$

Let (M, ω) be a compact Hamiltonian K-space, $\mathbb{J} : M \to \mathfrak{k}^*$, $J_X(p) = \mathbb{J}(p)(X)$ the moment map, $\bar{\omega} := \mathbb{J} - \omega$. As a special case, if $K = T$ is a torus, localization implies that

$$X \longmapsto \int_M e^{i\bar{\omega}(X)} \qquad \text{Duistermaat-Heckman integral}$$

is given by its exact stationary phase approximation. Its $\mathfrak{t}$-FT $\mathbb{J}_*(\omega^n/n!)$ is a **piecewise polynomial measure** on $\mathfrak{t}^*$.

Residue formulae

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More generally, consider the tempered distribution

$$\mathfrak{t} \ni X \longmapsto L_\varrho(X) := \int_M e^{i\bar{\omega}(X)} \varrho(X), \qquad \varrho \in \Lambda_K^*(M).$$

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If $0 \in \mathfrak{k}^*$ **is a regular value of** $\mathbb{J}$, Kirwan showed that

$$\mathcal{K} : H_K^*(M) \xrightarrow{\iota^*} H_K(\mathbb{J}^{-1}(0)) \xrightarrow{(\pi^*)^{-1}} H^*(\mathbb{J}^{-1}(0)/K)$$

is surjective, where $\iota : \mathbb{J}^{-1}(0) \hookrightarrow M$, $\pi : \mathbb{J}^{-1}(0) \to \mathbb{J}^{-1}(0)/K$.

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Let $T \subset G$ be a maximal torus. Using localization and Paley-Wiener theorems, Kirwan-Jeffrey showed for $\varrho \in H_K^*(M)$

$$\int_{\mathbb{J}^{-1}(0)/K} \mathcal{K}(\varrho e^{i\bar{\omega}}) = \operatorname{Res}\Big(\sum_F u_F\Big) \qquad \text{Residue formulae},$$

where F are the components of M^T and $u_F(X) \equiv \int_F \frac{\varrho(X)e^{i\omega}}{\chi_{NF}(X)}$.

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Proof. Via stationary phase asymptotics for the Witten integral

$$I_\zeta(\mu) := \int_{\mathfrak{k}} \int_M e^{i\mu\,(\mathbb{J}-\zeta)} a \frac{\omega^n}{n!}\, dX \sim \sum_{j=0}^{\infty} \mu^{-\kappa-j} L_j(\zeta),$$

where $\zeta \in \mathfrak{k}^*$, $\mu \to +\infty$, $a \in \mathrm{C}^\infty_\mathrm{c}$, $\kappa := \dim K/H_{prin}$, and

$$\mathrm{Crit}(\mathbb{J} - \zeta) = \{(p, X) : p \in \Omega_\zeta,\ X \in \mathfrak{k}_p\}$$

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Remarks.

- Full asymptotic expansion required.
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Problem. If $0 \in \mathfrak{k}^*$ is not a regular value of $\mathbb{J}$, the critical set is singular $\Longrightarrow$ stationary phase fails.

Residue formulae for circle actions

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In what follows, assume that

- (M, ω) is a (not necessarily compact) $2n$-dimensional Hamiltonian K-space with $K = \mathrm{SO}(2) \simeq S^1$.
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By the Marle-Guillemin-Sternberg normal form of the momentum map, the Witten integral reduces to integrals

$$J_\zeta(\mu) := \int_{\mathbb{R}} \int_{\mathbb{R}^{2n}} e^{ix\mu(\langle Q w, w\rangle - \zeta)} b(x, w) \, dw \, dx,$$

where $Q \in M(2n, \mathbb{R})$ is symmetric and non-degenerate.

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Proposition (Kuester,Konstantis,–, 2018)

$$J_\zeta(\mu) \sim \sum_{j=0}^{\infty} \mu^{-1-j} \sum_{l=0}^{\infty} \Theta_{l+j,l}(\zeta\mu)\zeta^l \quad \text{Singular stationary phase,}$$

where $\Theta_{j,l}$ are known coefficients with $|\Theta_{j,l}(\zeta\mu)| \ll_{j,l} 1$ given by distributions supported on the strata of $\{\langle Q w, w\rangle = 0\}$, and possibly not continuous in ζ. □

Consider

- $S^2 \subset \mathbb{R}^3 \simeq \mathfrak{so}(3)$ co-adjoint orbit of $\mathrm{SO}(3)$ with symplectic form $\omega_{S^2}(\tilde{X}, \tilde{Y})_\xi =: -\langle \xi, [X, Y] \rangle$
- Hamiltonian action of $K = \mathrm{SO}(2)$ on S^2 with moment map $\mathbb{J}_{S^2}(\xi_1, \xi_2, \xi_3) = -\xi_3$.
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Theorem (Kuester, Konstantis, –, 2018)

$$\mathrm{Res} \sum_F u_F = \int_{(\Omega_0 \cap M_{reg})/S^1} \widetilde{\mathcal{K}}_{reg}(\varrho e^{i\bar{\omega}}) + \int_{(\Omega_0 \cap \mathcal{M}_{sing})/S^1} \widetilde{\mathcal{K}}_{sing}(\varrho e^{i\bar{\omega}}).$$

□

Thank you!