Quantum footprints of symplectic rigidity

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Joint work with Laurent Charles (Paris)

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Today's story: Quantum counterpart of symplectic displacement energy, a fundamental symplectic invariant (Hofer, 1990)

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Example: For pure states $\xi, \eta \in H$, $|\xi| = |\eta| = 1$,
$\Phi(\xi, \eta) = |\langle \xi, \eta \rangle|$.

Dislocation

$F_t \in \mathcal{L}(H)$ - quantum Hamiltonian.
Schrödinger equation
$\dot{U}_t = -\frac{i}{\hbar} F_t U_t$,
$U_t : H \to H$ unitary evolution, $U_0 = \mathbb{1}$, $U_1 = U$.

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Appears e.g. in quantum computation. Margolus-Levitin (1998) address the question about *"the maximum number of distinct [i.e., non-overlapping] states that the system can pass through, per unit of time. For a classical computer, this would correspond to the maximum number of operations per second."*

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Quantum speed limit: universal bound on the energy required to a-dislocate a quantum state:

$$\Phi(\theta, U\theta U^{-1}) \leq a \;\Rightarrow\; \ell_q(F) \geq \arccos(a)\hbar$$

Mandelstam-Tamm, 1945 "time-energy uncertainty", Uhlmann 1992, Margolus-Levitin, 1998

Quantum speed limit

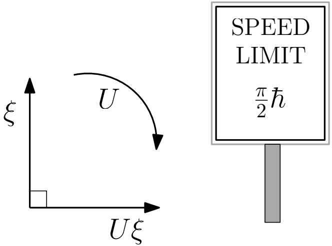

Figure: "Displacing" a pure quantum state

We explore semiclassical dislocation of semiclassical states.

Mathematical model of classical mechanics

(M^{2n}, ω)–symplectic manifold (e.g, 2-sphere)
ω– **symplectic form**. Locally $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$.

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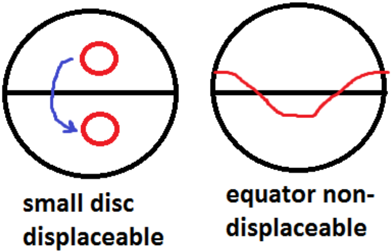
Key feature: $\varphi_t^* \omega = \omega$.

$X \subset M$ **displaceable** if $\exists\, \varphi \in Ham\colon\ \varphi X \cap X = \emptyset$ (Hofer, 1990)

Figure: (Non)-displaceability on the 2-sphere



Displacement energy

(M, ω) - closed symplectic manifold.
Let f_t, $t \in [0, 1]$ be classical Hamiltonian generating Hamiltonian diffeomorphism $\varphi \in Ham(M, \omega)$. Total energy

$\ell_c(f) = \int_0^1 ||f_t|| dt$, where $||g|| := \max |g|$-uniform norm.

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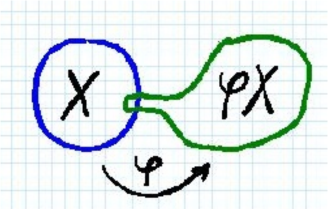
Rigidity: $e(X) > 0$ for all open X

Flexibility

Counterpoint: If $\text{Vol}(X) < \frac{1}{2} \cdot \text{Vol}(M)$, for all $\epsilon > 0, \delta > 0$ there exists f_t such that

$\text{Vol}(\varphi X \cap X) < \epsilon$, $\ell_c(f) < \delta$.



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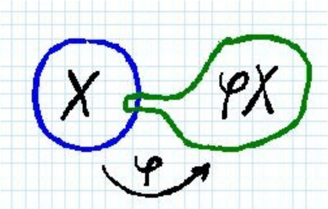

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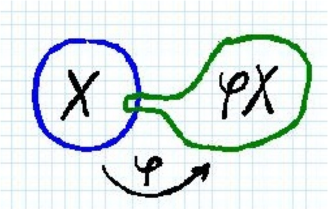

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Based on Katok's lemma, 1970, Ostrover-Wagner, 2005.

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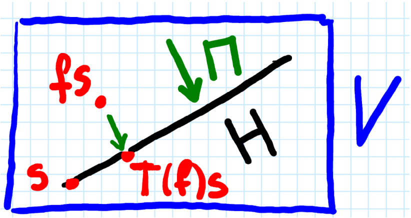
The Toeplitz operator: $T_\hbar(f)(s) := \Pi_\hbar(fs)$, $f \in C^\infty(M)$, $s \in H_\hbar$.



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Definition: For classical state τ (probability measure on M)

$$Q_\hbar(\tau) = \int_M P_{x,\hbar} d\tau(x) \in \mathcal{S}(H_\hbar)$$

"classical" quantum state, Giraud-Braun-Braun 2008

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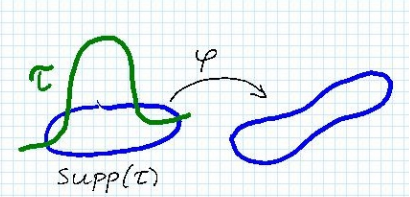
Theorem (Charles-P., 2016)

If f_t displaces supp$(\tau) \Rightarrow F_t$ $O(\hbar^\infty)$-dislocates θ.

Figure: φ-time-one-map of the flow of f_t



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Conclusion: Speed limit becomes more restrictive ~ 1 than the universal bound $\sim \hbar$.

Rigidity vs. flexibility on quantum side

Theorem (Charles-P., 2016)

Assume $Vol(supp(\tau)) < \frac{1}{2} \cdot Vol(M)$. *Then* $\forall \epsilon, \delta > 0$ *there exists* f_t *such that* F_t ϵ*-dislocates* θ *and* $\ell_q(F_t) < \delta$.

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	RIGIDITY	FLEXIBILITY
RATE OF DISLOCATION	$o(\hbar^n)$	ϵ

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(P4) **(trace correspondence)**
$\left|\text{trace}(T_\hbar(f)) - (2\pi\hbar)^{-n}\int_M f\frac{\omega^n}{n!}\right| = O(\hbar^{-(n-1)})$,

for all $f, g \in C^\infty(M)$.

Zooming into small scales

Theorems 1,2 extend to dislocation of semiclassical states which "occupy" a ball of radius $\hbar^{\varepsilon}$, $\varepsilon \in [0, 1/2)$ in the phase space. The speed limit on such a scale is $\sim \hbar^{2\varepsilon}$ which, again, is more restrictive than the universal quantum speed limit $\sim \hbar$.

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Rigidity of remainders: (Charles-P., 2016) α, β, γ cannot be small simultaneously

Quantum "proof" of $e(B) > 0$

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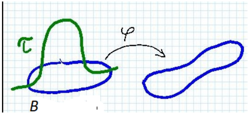

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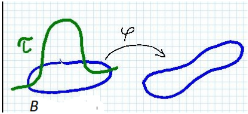

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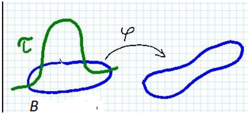

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Resolution: Remainders of quantization are large on scale $\sim \sqrt{\hbar}$

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Symplectic capacities, monotone invariants based on periodic orbits $\leftrightarrow$ **Gutzwiller type trace formula** (A. Uribe, 2016); in progress Charles, Le Floch, P., Uribe

The End

THANK YOU!