Geometric quantization of toric and semitoric systems

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Geometric Quantization and applications

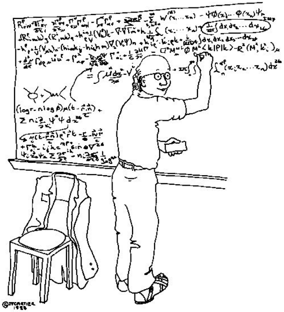
Outline

Joint work with

Classical vs. Quantum: a love & hate story

1. Classical systems
2. Observables $C^\infty(M)$
3. Bracket $\{f, g\}$

1. Quantum System
2. Operators in $\mathcal{H}$ (Hilbert)
3. Commutator
   $[A, B]_h = \frac{2\pi i}{h}(AB - BA)$



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

Quantization via real polarizations

- (M^{2n}, ω) symplectic manifold with integral $[\omega]$.
- $(\mathbb{L}, \nabla)$ a complex line bundle with a connection ∇ such that $curv(\nabla) = -i\omega$ (prequantum line bundle).
- A real polarization $\mathcal{P}$ is a Lagrangian foliation.
- Integrable systems provide natural examples of real polarizations.
- Flat sections equation: $\nabla_X s = 0$, $\forall X$ tangent to $\mathcal{P}$.

Bohr-Sommerfeld leaves

Definition

A Bohr-Sommerfeld leaf is a leaf of a polarization admitting global flat sections.

Example: Take $M = S^1 \times \mathbb{R}$ with $\omega = dt \wedge d\theta$, $\mathcal{P} =< \frac{\partial}{\partial\theta} >$, $\mathbb{L}$ the trivial bundle with connection 1-form $\Theta = td\theta \rightsquigarrow \nabla_X \sigma = X(\sigma) - i < \Theta, X > \sigma$ $\rightsquigarrow$Flat sections: $\sigma(t,\theta) = a(t).e^{it\theta} \rightsquigarrow$ Bohr-Sommerfeld leaves are given by the condition $t = 2\pi k, k \in \mathbb{Z}$.

Liouville-Mineur-Arnold $\leftrightsquigarrow$ this example is the canonical one.

Bohr-Sommerfeld leaves: continued...

Theorem (Guillemin-Sternberg)

If the polarization is a regular fibration with compact leaves over a simply connected base B, then the Bohr-Sommerfeld set is given by,

$$BS = \{p \in M, (f_1(p), \ldots, f_n(p)) \in \mathbb{Z}^n\}$$

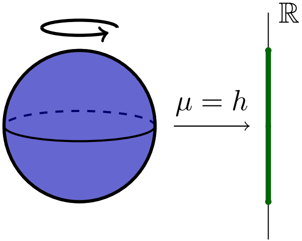
where $f_1, \ldots, f_n$ are global action coordinates on B.

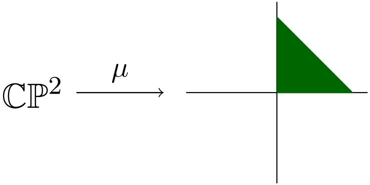
For toric manifolds the base B may be identified with the image of the moment map.

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes and the bijective correspondence is given by the image of the moment map:

$$\{\text{toric manifolds}\} \longrightarrow \{\text{Delzant polytopes}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, F) \longrightarrow F(M)$$





$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

The case of fibrations

- "Quantize" these systems counting Bohr-Sommerfeld leaves.
- For real polarization given by integrable systems Bohr-Sommerfeld leaves are just "integral" Liouville tori.

Theorem (Sniatycki)

If the leaf space B^n is Hausdorff and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibers, then quantization is given by the count of Bohr-Sommerfeld leaves.

But how exactly?

Quantization: The cohomological approach

- Following the idea of Kostant when there are no global sections we define the quantization of $(M^{2n}, \omega, \mathbb{L}, \nabla, P)$ as

$$\mathcal{Q}(M) = \bigoplus_{k\geq 0} H^k(M, \mathcal{J}).$$

- $\mathcal{J}$ is the sheaf of flat sections.

Then quantization is given by:

Theorem (Sniatycki)

$\mathcal{Q}(M^{2n}) = H^n(M^{2n}, \mathcal{J})$, *with dimension the number of Bohr-Sommerfeld leaves.*

What is this cohomology?

1. Define the sheaf: $\Omega^i_{\mathcal{P}}(U) = \Gamma(U, \wedge^i \mathcal{P})$..
2. Define $\mathcal{C}$ as the sheaf of complex-valued functions that are locally constant along $\mathcal{P}$. Consider the natural (fine) resolution

$$0 \to \mathcal{C} \xrightarrow{i} \Omega^0_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^2_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \cdots$$

The differential operator $d_{\mathcal{P}}$ is the one of foliated cohomology.

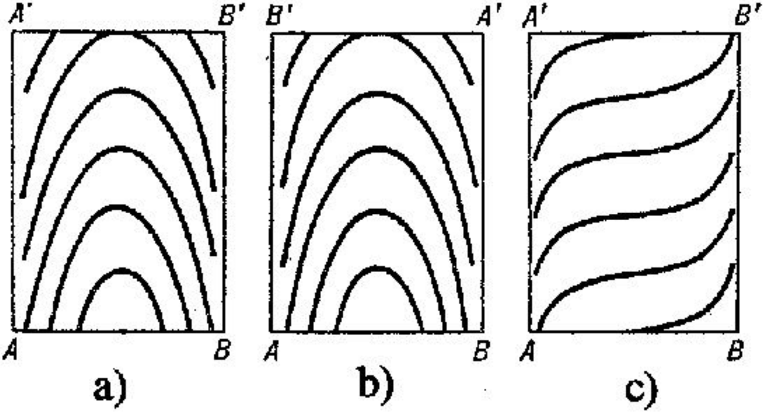
3. Use this resolution to obtain a fine resolution of $\mathcal{J}$ by twisting the previous resolution with the sheaf $\mathcal{J}$.

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{S} \xrightarrow{\nabla_{\mathcal{P}}} \mathcal{S} \otimes \Omega^1_{\mathcal{P}} \xrightarrow{\nabla_{\mathcal{P}}} \mathcal{S} \otimes \Omega^2_{\mathcal{P}} \to \cdots$$

with $\mathcal{S}$ the sheaf of sections of the line bundle $L(\otimes N^{1/2})$.

Applications to the general case of Lagrangian foliations

This fine resolution approach can be useful for polarizations given by general Lagrangian foliations.
Classification of foliations on the torus (Kneser-Denjoy-Schwartz theorem).



The case of the torus: irrational slope.

Let $(\mathbb{T}^2, \omega)$ be the 2-torus with a symplectic structure ω of integer class and consider the foliation $\mathcal{P}_\eta$ given by $X_\eta = \eta\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, with $\eta \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem (Presas-M.)

- $\mathcal{Q}(T^2, \mathcal{J})$ *is always infinite dimensional.*
- *For the limit case of foliated cohomology* $\omega = 0$ $\mathcal{Q}(\mathbb{T}^2, \mathcal{J}) = \mathbb{C} \bigoplus \mathbb{C}$ *if the irrationality measure of* η *is finite and* $\mathcal{Q}(\mathbb{T}^2, \mathcal{J})$ *is infinite dimensional if the irrationality measure of* η *is infinite.*

This generalizes a result El Kacimi for foliated cohomology.

"Quantization Computation kit" for regular foliations

Most computations rely on

1. Künneth formula: Let $(M_1, \mathcal{P}_1)$ and $(M_2, \mathcal{P}_2)$ be symplectic manifolds endowed with Lagrangian foliations and let $\mathcal{J}_{12}$ be the induced sheaf of basic sections, then under some mild conditions: $H^n(M_1 \times M_2, \mathcal{J}_{12}) = \bigoplus_{p+q=n} H^p(M_1, \mathcal{J}_1) \otimes H^q(M_2, \mathcal{J}_2)$.
2. Mayer-Vietoris: Consider $M \leftarrow U \sqcup V \stackrel{\leftarrow}{\leftarrow} U \cap V$, then the following sequence is exact,
$0 \to \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(M) \stackrel{r}{\to} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U) \oplus \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(V) \stackrel{r_0 - r_1}{\longrightarrow} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U \cap V) \to 0.$

Application II: Regular integrable system

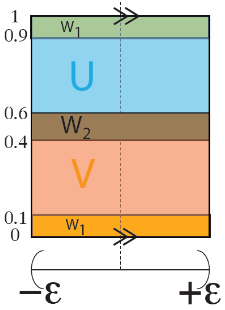
$I_j = (-\varepsilon, \varepsilon), j = 1, 2.$

Computation 1: $\mathcal{Q}(I_1 \times I_2, \omega = dx_1 \wedge dx_2; \mathcal{P} = \frac{\partial}{\partial x_2})$.

- $H^0(I_1 \times I_2; \mathcal{J}) = C^\infty(I_1, \mathbb{C})$,
- $H^1(I_1 \times I_2; \mathcal{J}) = 0$.

Computation 2: $\mathcal{Q}(I_1 \times \mathbb{S}^1_2, \omega = dx_1 \wedge d\theta_2; \mathcal{P} = \frac{\partial}{\partial \theta_1})$.

- $H^0(I_1 \times \mathbb{S}^1_2; \mathcal{J}) = 0$ since BS leaves are isolated.
- Consider $I_1 \times \mathbb{S}^1_2 = U \cup V = (I_1 \times (0.4, 1.1)) \cup (I_1 \times (-0.1, 0.6))$.



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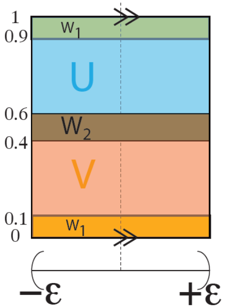

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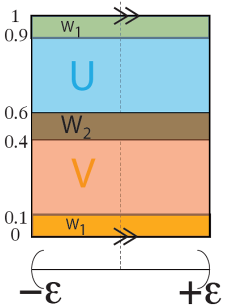

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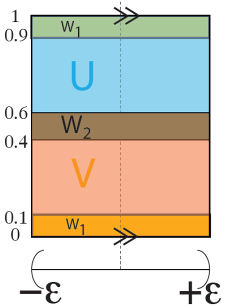

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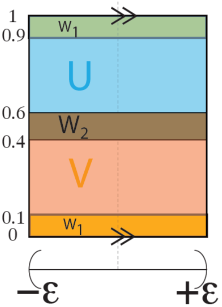

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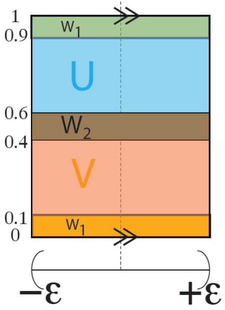

Regular integrable system

Apply Mayer-Vietoris and computation 1 to obtain

$$H^0(V) \oplus H^0(U) \hookrightarrow H^0(W_1) \oplus H^0(W_2) \twoheadrightarrow H^1(I_1 \times \mathbb{S}^1_2).$$

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$$\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ e^{i\theta x} & e^{-i\theta x} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

Thus

$$H^1(I_1 \times \mathbb{S}^1_2) = \begin{cases} 0 & \text{if non BS,} \\ \mathbb{C} & \text{if there is one BS.} \end{cases}$$

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Computation 4:

$$\mathcal{Q}(M^{2n}_{Tor,Reg}; \mathcal{P}(Torus)) = \bigoplus_{j=1}^{n} H^j(M; \mathcal{J}) = \mathbb{C}^b, \; b = \#\text{BS}.$$

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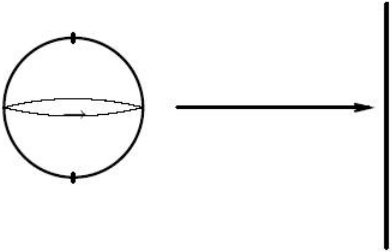
Toric Manifolds

What happens if we go to the edges and vertexes of Delzant's polytope?



Rotations of S^2 and moment map

There are two leaves of the polarization which are singular and correspond to fixed points of the action.

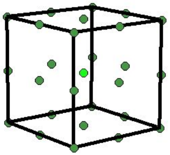
Quantization of toric manifolds

Theorem (Hamilton)

For a $2n$-dimensional compact toric manifold

$$\mathcal{Q}(M) = H^n(M; \mathcal{J}) \cong \bigoplus_{l \in BS_r} \mathbb{C}$$

with a BS_r the set of regular Bohr-Sommerfeld leaves.



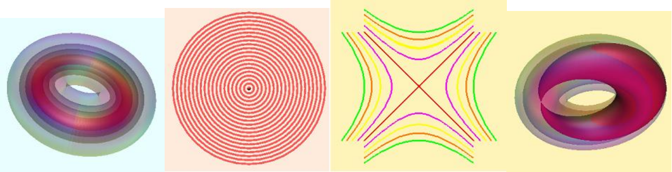
In the example of the sphere Bohr-Sommerfeld leaves are given by integer values of height (or, equivalently) leaves which divide out the manifold in integer areas.

Action-angle coordinates with singularities

The theorem of Marle-Guillemin-Sternberg for fixed points of toric actions can be generalized to non-degenerate singularities of integrable systems.

Theorem (Eliasson)

There exists symplectic Morse normal forms for integrable systems with non-degenerate singularities.



Liouville torus k_e comp. elliptic k_h hyperbolic k_f focus-focus

Description of singularities

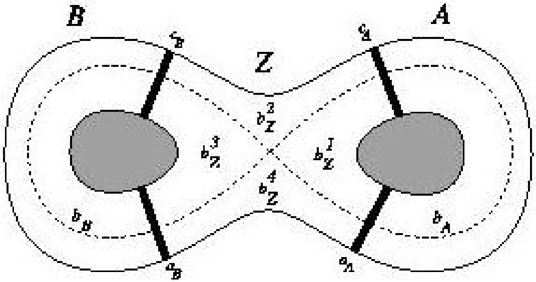
The local model is given by $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ and $\omega = \sum_{i=1}^{k} dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$. and the components of the moment map are:

1. Regular $f_i = p_i$ for $i = 1, ..., k$;
2. Elliptic $f_i = x_i^2 + y_i^2$ for $i = k+1, ..., k_e$;
3. Hyperbolic $f_i = x_i y_i$ for $i = k_e + 1, ..., k_e + k_h$;
4. focus-focus $f_i = x_i y_{i+1} - x_{i+1} y_i$, $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1,\ j = 1, ..., k_f$.

We say the system is semitoric if there are no hyperbolic components.

Hyperbolic singularities

We consider the following covering



Key point in the computation

We may choose a trivializing section of such that the potential one-form of the prequantum connection is $\Theta_0 = (xdy - ydx)$.

Theorem

Leafwise flat sections in a neighborhood of the singular point in the first quadrant are given by

$$a(xy)e^{\frac{i}{2}xy \ln \frac{x}{y}}$$

where a is a smooth complex function of one variable which is flat at the origin.

The case of surfaces

We can use Čech cohomology computation and a Mayer-Vietoris argument to prove:

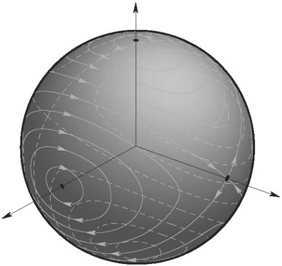
Theorem (Hamilton-M.)

The quantization of a compact surface endowed with an integrable system with non-degenerate singularities is given by,

$$\mathcal{Q}(M) = H^1(M;\mathcal{J}) \cong \bigoplus_{p\in\mathcal{H}}(\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{l\in BS_r} \mathbb{C} \ ,$$

where $\mathcal{H}$ is the set of hyperbolic singularities.

The rigid body



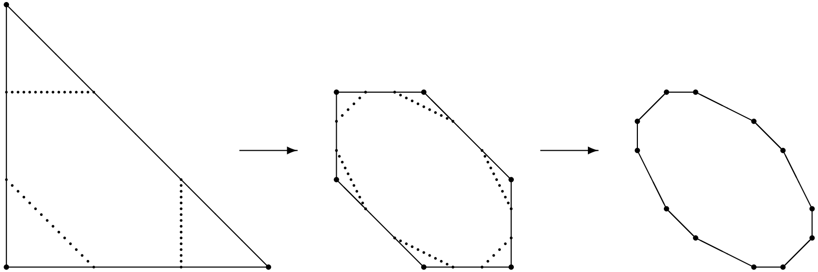
Using this recipe and the quantization of this system is

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}_p^{\mathbb{N}})^2 \oplus \bigoplus_{b \in BS} \mathbb{C}_b.$$

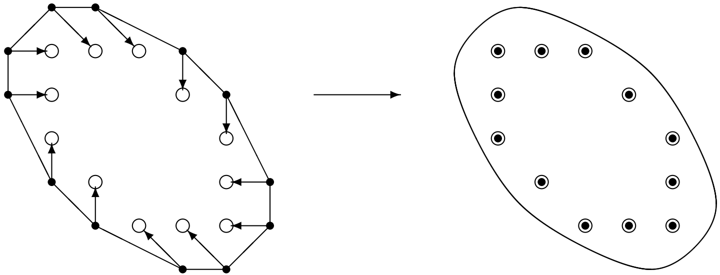
Comparing this system with the one of rotations on the sphere $\rightsquigarrow$ This quantization depends strongly on the polarization.

$\mathbb{C}P^2$, $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$, and $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$

Let us construct toric systems blowing up at 9 singular points using symplectic cutting.

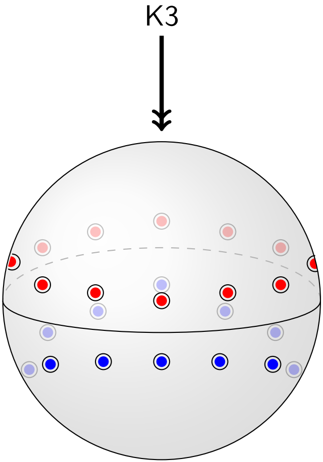


(Symington's) Nodal trades on $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$



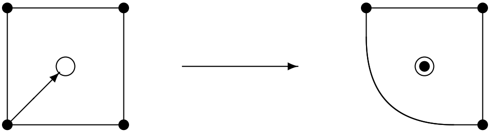
We can convert elliptic points into focus-focus points using nodal trading (Symington).

K3 surface=$(\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2) \#_{\mathbb{T}^{\not{z}}} (\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2)$



We may glue two copies to obtain a K3 surface.

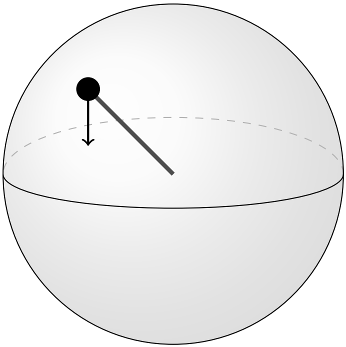
Other examples: Spin-spin system

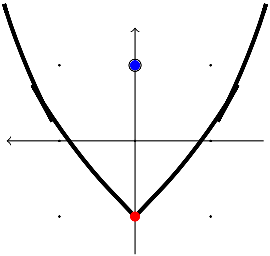


- We may perform a nodal trade on $\mathbb{C}P^1 \times \mathbb{C}P^1$ to obtain a spin-spin system.
- This is a toy model of the spin-spin system of Sadovskií and Zĥilinskií

$$\begin{cases} f_1 = \frac{z_1}{2} + \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{2} \\ f_2 = z_1 + z_2 \end{cases}$$

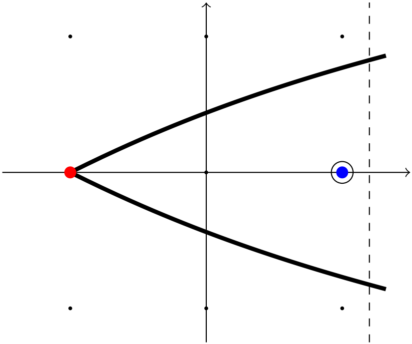
Spherical pendulum





Coupled classical spin and harmonic oscillator $\mathbb{C}P^1 \times \mathbb{C}$

$$\begin{cases} f_1 = z + \frac{1}{2}(u^2 + v^2) \\ f_2 = \frac{1}{2}(xu + yv) \end{cases}$$



Quantization of 4-dimensional almost toric manifolds

Theorem (M-Presas-Solha)

For a 4-dimensional compact almost toric manifold M,

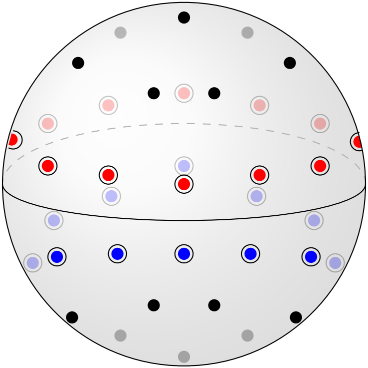
$$\mathcal{Q}(M) \cong \left(\bigoplus_{p \in BS_r} \mathbb{C} \right) \oplus \left(\bigoplus_{p \in BS_f} \oplus_{n(p)} C^{\infty}(\mathbb{R}; \mathbb{C}) \right) ,$$

where with BS_r and BS_f denotes the image of the regular and focus-focus Bohr–Sommerfeld fibers respectively on the base and $n(p)$ the number of nodes on the fiber whose image is $p \in BS_f$.

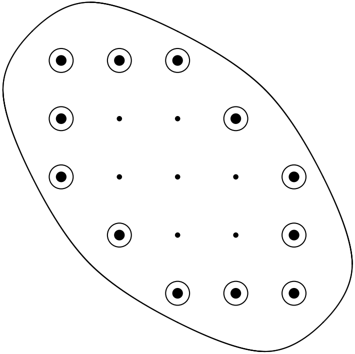
Application: Real geometric quantization of a K3 surface

For a K3 surface with **24** Bohr–Sommerfeld focus-focus fibers;

$$\mathcal{Q}(K3) \cong \mathbb{C}^{14} \oplus \bigoplus_{j\in\{1,\dots,24\}} C^{\infty}(\mathbb{R};\mathbb{C}) .$$



Bohr-Sommerfeld leaves in Gompf decomposition of K3



Kähler geometric quantization of a K3 surface

- Dimension of $H^0(K3; L)$ is $\frac{1}{2}c_1(L)^2 + 2$. and $c_1(L)^2 = \int_{K3} \omega \wedge \omega$
- The symplectic volume of a symplectic sum is the sum of the symplectic volumes $K3 = \left(\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2\right) \#_{T^2} \left(\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2\right)$.
- The symplectic volume of a toric 4-manifold is simply twice the Euclidean volume of its Delzant polytope; thus,

$$\frac{1}{2}c_1(L)^2 + 2 = \frac{1}{2}(2 \cdot 24 + 2 \cdot 24) + 2 = 50 \ .$$

- and $\mathcal{Q}(K3) \cong \mathbb{C}^{50}$.