Selberg's twisted trace formula and asymptotics of equivariant analytic torsion

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Max Planck Institute for Mathematics

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Contents

In this talk

Two parts:

1. Evaluate twisted orbital integrals on symmetric spaces by an explicit geometric formula.

2. Apply computations to equivariant analytic torsion of locally symmetric spaces.

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General setting

- $X = G/K$ symmetric space, σ automorphism of (G, K, B).
- Δ^X Laplace-Beltrami on X.
- Twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[\exp(t\Delta^X/2)]$: integration on orbit of γ in G under σ-twisted conjugation.
- Such integrals appear when evaluating σ-equivariant traces on $Z = \Gamma\backslash X$, compact locally symmetric space.

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Twisted orbital integrals

When $\gamma\sigma$ is semisimple, we have the explicit local formula:

Theorem A (Bismut 2011 for $\sigma = 1$, L. 2018)

For $t > 0$, we have

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(t(\Delta^X/2 - c/2)] = \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \int_{\mathfrak{k}_\sigma(\gamma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) e^{-|Y_0^{\mathfrak{k}}|^2/2t} \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

$J_{\gamma\sigma}$ explicitly defined analytic function on $\mathfrak{k}_\sigma(\gamma)$.

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- λ highest weight of compact form U of G, E_d rep. of G with $d\lambda$.
- $P \xrightarrow{G} M$ with flat connection, $F_d = P \times_G E_d$ flat vector bundles on M.
- $\mathcal{T}_d$ real analytic torsions for F_d.

Theorem(Bismut-Ma-Zhang, 2017)

There is a locally computable diff. form W on M associated with λ such that as $d \to +\infty$, the leading term of $\mathcal{T}_d$ is given by

$$d^{n+1} \int_M W.$$

Also true for asymptotic analytic torsion forms.

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A *fixed point formula* : following BMZ

Equivariant case: $F_d|_{d\in\mathbb{N}}$ sequence of flat vector bundles on $Z=\Gamma\backslash X$ equipped with action of σ.

- Bismut-Zhang(1994): equivariant analytic torsion can be evaluated on the fixed point set.
- $W^j_{\gamma\sigma}$ locally computable forms on fixed point set of σ in Z: W-invariants by BMZ.

Theorem B (L. 2018)

As $d\to+\infty$,

$$d^{-m(\sigma)-1}\mathcal{T}_\sigma(F_d) = \sum_{\substack{\text{certain elliptic}\\ \text{classes } [\gamma]_\sigma}} \mathrm{Vol}(\underbrace{\Gamma\cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)}_{\text{fixed pts of } \sigma \text{ in Z}}) \Big(\sum_j \underbrace{s_j(\gamma\sigma)^d}_{\text{oscillating}}[W^j_{\gamma\sigma}]^{\max}\Big)+o(1).$$

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Our tools

1. Hypoelliptic Laplacian of Bismut interpolates between the two sides of our formula for twisted orbital integrals.

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Geometric setting

- (G, K, θ, B) connected real reductive Lie group.
- $\mathfrak{g}$, $\mathfrak{k}$ Lie algebras of G, K.
- Cartan decomposition: $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$.
- $X = G/K$ symmetric space, $p : G \to X$.
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- $X \simeq \mathfrak{p}$ is of nonpositive sectional curvature.

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The twist σ

- $\sigma \in \mathrm{Aut}(G, \theta, B)$, acts on X isometrically.
- $\Sigma^\sigma \subset \mathrm{Aut}(G)$ compact subgroup generated by σ:

$$G^\sigma = G \rtimes \Sigma^\sigma,\ K^\sigma = K \rtimes \Sigma^\sigma,\ X = G^\sigma / K^\sigma.$$

- If $h \in G$, twisted conjugation,

$$C^\sigma(h)\gamma = h\gamma\sigma(h^{-1}) \in G.$$

- Twisted centralizer of $\gamma \in G$:

$$Z_\sigma(\gamma) = \{h \in G : C^\sigma(h)\gamma = \gamma\}.$$

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- $\rho^E : K^\sigma \to \mathrm{Aut}(E)$ unitary representation.
- $F = G \times_K E = G^\sigma \times_{K^\sigma} E$ Hermitian vector bundle on X.
- Q a G^σ-invariant integral operator acting on $C^b(X, F)$ with kernel $q(g) \in \mathrm{End}(E)$,

$$|q(g^{-1}g')| \leq C \exp(-C' d^2(pg, pg')).$$

- If $\gamma\sigma$ semisimple, twisted orbital integral

$$\mathrm{Tr}^{[\gamma\sigma]}[Q] = \int_{Z_\sigma(\gamma)\backslash G} \mathrm{Tr}^E[\rho^E(\sigma) q(g^{-1}\gamma\sigma(g))] dg.$$

- We need a geometric interpretation for $\mathrm{Tr}^{[\gamma\sigma]}[Q]$.

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Semisimple isometries

- If $\phi \in \mathrm{Isom}(X)$, displacement function

$$d_\phi(x) = d(x, \phi(x)).$$

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- $X(\phi) = d_\phi^{-1}(m_\phi)$ closed convex subset of X.

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The minimizing set for $\gamma\sigma$

- Take $\gamma \in G$ such that $\gamma\sigma \in \mathrm{Isom}(X)$ semisimple.
- After conjugation, one has

$$\gamma = e^a k^{-1},\ a \in \mathfrak{p},\ k \in K,\ \mathrm{Ad}(k^{-1})\sigma(a) = a.$$

- $m_{\gamma\sigma} = |a|$.
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Normal bundle to $X(\gamma\sigma)$:

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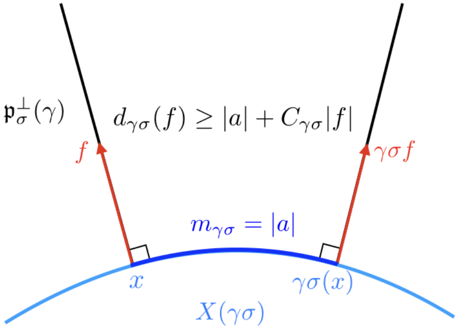
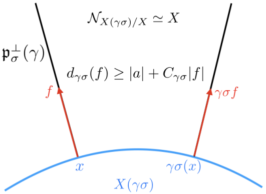


Geometric twisted orbital integrals

- $q(x, x')$ kernel of Q, $|q(x, x')| \leq Ce^{-Cd^2(x,x')}$.
- Geometric formula for twisted orbital integral:

$$\mathrm{Tr}^{[\gamma\sigma]}[Q] = \int_{\mathfrak{p}^{\perp}_{\sigma}(\gamma)} \mathrm{Tr}^F[\gamma\sigma q(f, \gamma\sigma f)] \underbrace{r(f)}_{\text{Jacobian}} df.$$



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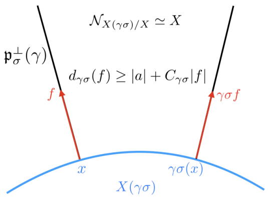

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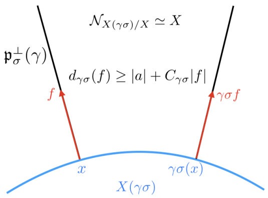

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Casimir operator

- $U\mathfrak{g}$ enveloping algebra $\simeq$ left-inv. diff. operators on G.
- $C^{\mathfrak{g}} \in U\mathfrak{g}$ Casimir operator associated with $(\mathfrak{g}, B)$:

$$C^{\mathfrak{g}} = -\sum e_i^* e_i = -\sum_{\mathfrak{p}\text{ part}} e_i^2 + \underbrace{\sum_{\mathfrak{k}\text{ part}} e_i^2}_{C^{\mathfrak{k}}}.$$

- $C^{\mathfrak{g}}$ descends to $C^{\mathfrak{g},X}$ acting on $C^\infty(X, F)$:

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Elliptic operator $\mathcal{L}^X$ on X

- $\mathcal{L}^X = \frac{1}{2}C^{\mathfrak{g},X} + \frac{1}{2}c$ with c a fixed constant.
- $\mathcal{L}^X$ self-adjoint, Bochner-like Laplacian.
- It commutes with action of G^σ on $C^\infty(X, F)$.
- Goal: an *explicit* formula for $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)]$.
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Recall: Hypoelliptic deformation

- $\mathcal{L}^X$ can be deformed to a family of hypoelliptic operators $\mathcal{L}^X_b|_{b>0}$ on an enlarged space $\widehat{\mathcal{X}}$, connecting $\mathcal{L}^X$ on the base (as $b \to 0$) and the geodesic flow on tangent bundle (as $b \to +\infty$).
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The hypoelliptic Laplacian

- $N = G \times_K \mathfrak{k}$ vector bundle on X.
- $\widehat{\mathcal{X}}$ the total space of $\widehat{\pi} : TX \oplus N = G \times_K \mathfrak{g} \to X$.
- Hypoelliptic Laplacian of Bismut $\mathcal{L}_b^X|_{b>0}$ acts on

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Twisted orbital supertrace

- (B. 2011) For $b > 0$, $t > 0$, $\exp(-t\mathcal{L}_b^X)$ has smooth kernel $q_{b,t}^X((x,Y),(x',Y'))$.
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A fundamental identity

Theorem (Bismut 2011 for $\sigma = 1$, L. 2018)

For $b > 0, t > 0$,

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_b^X)].$$

- $\mathrm{Tr_s}^{[\gamma\sigma]}[\cdot]$ supertrace on algebra of G^σ-invariant kernels.
- Right-hand side does not depend on $b > 0$ like McKean-Singer index theorem.
- (B. 2011) As $b \to 0$,

$$q_{b,t}^X((x,Y),(x',Y')) \to \mathbf{P}\frac{1}{\pi^{(m+n)/2}} p_t^X(x,x') e^{-\frac{1}{2}\left(|Y|^2+|Y'|^2\right)}\mathbf{P}.$$

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Making $b \to +\infty$

- As $b \to +\infty$, generator of geodesic flow $\nabla_{Y^{TX}}$ dominates.
- For $t > 0$: $\varphi_t(x, Y^{TX}) = \gamma\sigma(x, Y^{TX}) \Rightarrow x \in X(\gamma\sigma)$.
- Evaluation of $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_b^X)]$ can be localized near $X(\gamma\sigma)$.
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Depends on the action of $\gamma\sigma$ on $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$.

Formula of $J_{\gamma\sigma}(Y_0^{\mathfrak{k}}), Y_0^{\mathfrak{k}} \in \mathfrak{k}_\sigma(\gamma)$

$$J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) = \frac{1}{|\det(1-\mathrm{Ad}(\gamma\sigma))|_{\mathfrak{z}_0^\perp}|^{1/2}} \frac{\widehat{A}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}_\sigma(\gamma)})}{\widehat{A}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}_\sigma(\gamma)})}$$
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An explicit geometric formula

Put $p = \dim \mathfrak{p}_\sigma(\gamma),\ q = \dim \mathfrak{k}_\sigma(\gamma),\ r = p + q.$

Theorem A

For $t > 0$, we have

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \times$$
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Geometric setting

- $\rho^E : G^\sigma \to E$, $F = G^\sigma \times_{K^\sigma} E \simeq X \times E$ with flat connection ∇^F.
- Σ^σ acts on $F \to X$ preserving ∇^F.
- $\Gamma \subset G$ discrete, cocompact, torsion-free subgroup such that $\sigma(\Gamma) = \Gamma$.
- $Z = \Gamma \backslash X$ compact smooth manifold.
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- $\mathbf{D}^{Z,F}$ Hodge-de Rham operator:

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Equivariant analytic torsion

- If $s \in \mathbb{C}$, $\mathrm{Re}\, s$ large enough, put

$$\vartheta_\sigma(s) = -\mathrm{Tr_s}\left[N\sigma[\mathbf{D}^{Z,F,2}]^{-s} P^{\perp}_{\ker \mathbf{D}^{Z,F}}\right].$$

- Equivariant Ray-Singer torsion

$$\mathcal{T}_\sigma(F) = \frac{1}{2}\frac{\partial \vartheta_\sigma}{\partial s}(0).$$

- We have

$$\frac{1}{2}\mathbf{D}^{Z,F,2} = \mathcal{L}^Z + \mathcal{A}$$

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A family of representations of G^σ

- U compact form of G with Lie algebra $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$.
- Unitary trick: representations of G can be constructed from representations of U.
- $U^\sigma = U \rtimes \Sigma^\sigma$.
- Take λ fixed by σ, irreducible representation of U extends to representation of U^σ.
- This gives representation of G^σ.
- For $d \in \mathbb{N}^*$, representation E_d of G^σ with highest weight $d\lambda$:

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- (E_d, ρ^{E_d}) defines a family of flat vector bundles F_d on Z equipped with action of σ.
- Nondegeneracy condition (BMZ):

$$\mu(M_\lambda) \cap \mathfrak{k}^* = \emptyset \longleftrightarrow W_U \lambda \cap \mathfrak{t}_K^* = \emptyset.$$

- By BMZ, Müller-Pfaff, if λ is non-degenerate, one has spectral gap

$$\mathbf{D}^{Z,F_d,2} \geq cd^2 - C.$$

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A fixed point formula:

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Under the non-deg. condition, as $d \to +\infty$,

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$$d^{-m(\sigma)-1}\mathcal{T}_\sigma(F_d) = \sum_{\substack{\text{certain elliptic} \\ \text{classes } [\gamma]_\sigma}} \mathrm{Vol}(\underbrace{\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)}_{\text{fixed pts of } \sigma \text{ in Z}}) \Big(\sum_j \underbrace{s_j(\gamma\sigma)^d}_{\text{oscillating}} [W^j_{\gamma\sigma}]^{\max} \Big) + o(1).$$

Non-elliptic elements do not contribute

- Mellin transform: $\underbrace{\int_0^{1/d}\cdots}_{\text{main term}}+\underbrace{\int_{1/d}^{+\infty}\cdots}_{\mathcal{O}(e^{-cd})}.$
- Selberg's twisted trace formula:

$$\mathrm{Tr_s}[(N-\frac{m}{2})\sigma\exp(-t\mathbf{D}^{Z,F_d,2}/2)]=$$
$$\sum_{[\gamma]_\sigma\in C_\sigma(\Gamma)}\mathrm{Vol}(\Gamma\cap Z_\sigma(\gamma)\backslash X(\gamma\sigma))\mathrm{Tr}^{[\gamma\sigma]}[(N-\frac{m}{2})\exp(-t\mathbf{D}^{X,F_d,2})].$$

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- If $r_\sigma(\gamma) = \mathrm{rk}(Z_\sigma(\gamma)^0) - \mathrm{rk}(K_\sigma(\gamma)^0) \neq 1$, the corresponding twisted orbital integral vanishes (extension of a result of Moscovici-Stanton (1991)).
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When $r_\sigma(\gamma) = 1$

If $r_\sigma(\gamma) = 1$, as $d \to +\infty$

$$d^{-n(\gamma\sigma)-1}(1 + 2t\frac{\partial}{\partial t})\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\Big(N - \frac{m}{2}\Big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big] = 2\sum_j \underbrace{s_j(\gamma\sigma)^d}_{\text{oscillating terms}} [d_t^j]^{\max} + \mathcal{O}(d^{-1}).$$

Where d_t^j are locally computable real diff. forms on $X(\gamma\sigma)$ used to define the W-invariants.

- May assume $\gamma = k^{-1} \in K$.
- Apply Theorem A to $\mathrm{Tr_s}^{[\gamma\sigma]}[\cdot]$:

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\cdot] = \frac{d^p}{(2\pi t)^{p/2}}\exp(\frac{t}{d^2}\ldots)$$

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- ${}^{\gamma\sigma}M_\lambda$ fixed point set of $\gamma\sigma$ on M_λ:

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- For λ non-degenerate, W-invariants associated with $Z_\sigma(\gamma)$ are

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