# Hypoelliptic Laplacian, index theory and the trace formula

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- Two connections gauge equivalent $\Leftrightarrow$ they have conjugate holonomies.
- If $g \in G$, $O_g \subset G$ adjoint orbit in $G$.
- Coadjoint orbit $\mathcal{O}_g \subset \widehat{L}G$ = smooth paths in $G$ connecting 1 and $O_g$: $\frac{d}{dt} + A \leftrightarrow \dot{g} + A_r = 0$.

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- $\widetilde{L}G$ central extension of $\widehat{LG}$.
- Full coadjoint orbit $= \left\{\frac{d}{ds} + A_s, E\left(A\right)\right\}$.

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- By DH, BV, should localize on 1-parameter semigroups $\Longrightarrow$ correct formula in terms coroot lattice (affine Weyl group).

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- 'Explains' fantastic cancellations in local index theory.

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- $\ldots b \to +\infty$ localizes integral on $\ddot{g} = 0$.
- $b^2 \ddot{g} + \dot{g} = \dot{w}$ Gaussian suggests appearance of hypoelliptic operator.

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- We will delete $\Lambda^\cdot(\mathfrak{g}^*)$ by tensoring with $S^\cdot(\mathfrak{g}^*)$, and use Bargmann isomorphism.

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- $\mathcal{L}_b = \frac{1}{2b^2}\left(-\Delta^{\mathfrak{g}} + |Y|^2 - n\right) + \frac{1}{b}\left(\nabla_Y + \widehat{c}(\mathrm{ad}(Y))\right)$.

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- $\mathcal{L}_b$ hypoelliptic by Hörmander.

# A deformed Dirac operator

- Operators will act on $C^{\infty}(G) \otimes \mathcal{A}(\mathfrak{g}^*) \simeq C^{\infty}(G \times \mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
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- $\mathcal{L}_b$ hypoelliptic by Hörmander.
- As $b \to 0$, by collapsing, $\mathcal{L}_b$ deforms $\frac{1}{2}\left(-\Delta^G + c\right)$.

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**Theorem (B08)**

For $t > 0, g_1, g_2 \in G$,

$$\operatorname{Tr}\left[L_{g_1} R_{g_2}^{-1} \exp\left(t\left(\Delta^G - c\right)/2\right)\right] = \operatorname{Tr_s}\left[L_{g_1} R_{g_2}^{-1} \exp\left(-t\mathcal{L}_b\right)\right].$$

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Independence of $b$ like in index theorem, and convergence as $b \to 0$.

**Remark**

Hamiltonian counterpart to Lagrangian deformation for DH, BV formulas.

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- We get the required formulas for the above trace.

# Reductive groups and symmetric spaces

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- Symbol of $C^{\mathfrak{g}} = B(\xi, \xi)$ positive on $\mathfrak{p}$, negative on $\mathfrak{k}$.

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- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ splits as direct sums of Euclidean vector spaces.
- Introduce algebraic de Rham on $\mathfrak{g}$, and use Bargmann on $\mathfrak{p}, \mathfrak{k}$ separately.

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$$\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \sqrt{-1}c\left(\left[Y^{\mathfrak{p}}, Y^{\mathfrak{k}}\right]\right) + \frac{1}{b}\left(d^{\mathfrak{p}} + Y^{\mathfrak{p}} + d^{\mathfrak{p}*} + i_{Y^{\mathfrak{p}}}\right) + \frac{\sqrt{-1}}{b}\left(d^{\mathfrak{k}*} + i_{Y^{\mathfrak{k}}} - d^{\mathfrak{k}} - Y^{\mathfrak{k}}\right).$$

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- $\widehat{\mathcal{X}} \to X$ total space of $TX \oplus N = G \times_K (\mathfrak{p} \oplus \mathfrak{k})$.
- $\mathfrak{D}_b^X, \mathcal{L}_b^X$ act on $C^\infty\left(\widehat{\mathcal{X}}, \Lambda^{\cdot}(T^*X \oplus N^*) \otimes F\right)$.

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- $b \to +\infty$, geodesic flow $\nabla_{Y^{TX}}$ dominant $\Rightarrow$ closed geodesics.

# The case of locally symmetric spaces

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- $Z = \Gamma \setminus X$ compact locally symmetric.

# A fundamental identity

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**Theorem B11**

For $t > 0, b > 0$,

$$\mathrm{Tr}^{C^{\infty}(Z,F)}\left[\exp\left(-t\left(C^{Z}-c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right].$$

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## Proof

Limit as $b \to 0$,

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## Proof

Limit as $b \to 0$, Bianchi identity

$$\left[\mathfrak{D}_b^{Z}, \mathcal{L}_b^{Z}\right] = \left[\mathfrak{D}_b^{Z}, \left(\mathfrak{D}_b^{Z,2} + C^{Z}\right)/2\right] = 0.$$

# Splitting the identity

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2. The identity splits as identity of orbital integrals.

# Semisimple orbital integrals

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- $\gamma \in G$ semisimple, $[\gamma]$ conjugacy class.
- For $t > 0$, $\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^X - c\right)/2\right)\right]$ orbital integral of heat kernel on orbit of $\gamma$:

$$I\left([\gamma]\right) = \int_{Z(\gamma)\backslash G} \mathrm{Tr}^E\left[p_t^X\left(g^{-1}\gamma g\right)\right] dg.$$

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• $X(\gamma) \subset X$ totally geodesic symmetric space for the centralizer $Z(\gamma)$.

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$$I(\gamma) = \int_{N_{X(\gamma)/X}} \mathrm{Tr}\left[\gamma p_t^X(Y, \gamma Y)\right] \underbrace{r(Y)}_{\text{Jacobian}} dY.$$

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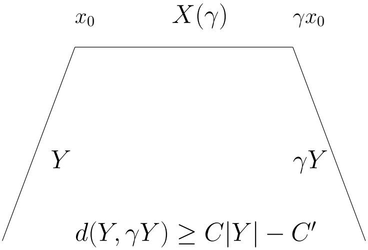

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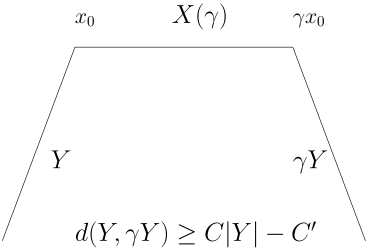


$$p_t^X(x, x') \leq C \exp(-C' d^2(x, x')).$$

# The heat kernel for $\mathcal{L}_b^X$

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## Theorem (B11)

- For $b \in ]0, M]$, $t > 0$ fixed,

$$\left| q_{b,t}^X \left( (x, Y), (x', Y') \right) \right| \leq C \exp \left( -C' \left( d^2 (x, x') + |Y|^2 + |Y'|^2 \right) \right).$$

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- As $b \to 0$,

$$q_{b,t}^X\left(\left(x, Y\right), \left(x', Y'\right)\right) \to \mathbf{P} p_t^X\left(x, x'\right) \pi^{-\dim \mathfrak{g}/2} \exp\left(\frac{1}{2}\left(\left|Y\right|^2 + \left|Y'\right|^2\right)\right) \mathbf{P}.$$

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**Remark**

The proof uses the fact that $\mathrm{Tr}^{[\gamma]}$ is a trace on the algebra of $G$-invariants smooth kernels on $X$ with Gaussian decay.

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- $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k)\, a = a$.
- $Z(\gamma)$ centralizer of $\gamma$, $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$ Lie algebra of $Z(\gamma)$.

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$$\int_{\mathfrak{k}(\gamma)} J_\gamma\left(Y_0^{\mathfrak{k}}\right) \operatorname{Tr}^E\left[\rho^E\left(k^{-1}\right)\exp\left(-i\rho^E\left(Y_0^{\mathfrak{k}}\right)\right)\right]$$
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The formula extends to any kernel.

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## Definition

$$J_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)_{\mathfrak{k}(\gamma)}\right)}$$

$$\left[\frac{1}{\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}}\right.$$

$$\left.\frac{\det\left(1-\exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

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- Here $TX$ replaced by $TX \ominus N$.

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- Clifford algebras $\widehat{c}\left(\mathfrak{g}\right)$ for $D^{\mathrm{Ko}}$, $c\left(\mathfrak{p}\right)$ for $D^X$.
- $\mathfrak{g} \oplus \overline{\mathfrak{p}} = \overline{\mathfrak{p}} \oplus \mathfrak{p} \oplus \mathfrak{k}$.
- Use action of $\mathrm{SO}\left(2m\right)$ on $\overline{\mathfrak{p}} \oplus \mathfrak{p}$ to extract $D^X$ from $\widehat{D}^{\mathrm{Ko}}$.

# Eta invariants and the trace formula

- If $D^X$ classical Dirac operator on $X$, $D^{X,2} = C^{\mathfrak{g}} + c$.
- Methods do not apply to $\mathrm{Tr}^{[\gamma]}\left[D^X \exp\left(-tD^{X,2}\right)\right]$.
- Dirac operators $\widehat{D}^{\mathrm{Ko}}, D^X$ on $G, X$ closely related.
- Clifford algebras $\widehat{c}\left(\mathfrak{g}\right)$ for $D^{\mathrm{Ko}}$, $c\left(\mathfrak{p}\right)$ for $D^X$.
- $\mathfrak{g} \oplus \overline{\mathfrak{p}} = \overline{\mathfrak{p}} \oplus \mathfrak{p} \oplus \mathfrak{k}$.
- Use action of $\mathrm{SO}\left(2m\right)$ on $\overline{\mathfrak{p}} \oplus \mathfrak{p}$ to extract $D^X$ from $\widehat{D}^{\mathrm{Ko}}$.
- Recover results by Moscovici-Stanton on eta invariants of locally symmetric spaces.

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