Bergman kernels on punctured Riemann surfaces

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Content

I-Bergman kernels on complete manifolds

a) Landscape

- Start with an hermitian holomorphic line bundle (L, h) over a complete Kähler manifold (X^n, ω_X)
  (h **does not** necessarily polarize ω_X).
- Consider, for $p \geq 1$, the Hilbert space

$$H^0_{(2)}(X, L^p) = \left\{\sigma \in \boldsymbol{L}^2(X, L^p) \,\middle|\, \overline{\partial}^{L^p} \sigma = 0\right\}$$

  (here and below, L^p is a shortcut for $(L^{\otimes p}, h^p)$).
  It might be of infinite dimension when X is non-compact.
- To these data, associate the **Bergman kernels**

$$\mathcal{B}_p : (x, y) \longmapsto \sum_{\ell \geq 0} s_\ell^{(p)}(x) \otimes s_\ell^{(p)}(y)^* \in L^p_x \otimes (L^p_y)^*$$

  for some (any) orthonormal basis $(s_\ell^{(p)})_{\ell \geq 0}$ of $H^0_{(2)}(X, L^p)$. More particularly, look at the **density functions** $B_p(x) = \mathcal{B}_p(x, x) = \sum_{\ell \geq 0} |s_\ell^{(p)}(x)|^2_{h^p} \geq 0$.
- Alternatively: $B_p(x) = \sup\limits_{\sigma \in H^0_{(2),p}, \sigma \neq 0} \dfrac{|\sigma(x)|^2_{h^p}}{\|\sigma\|^2_{\boldsymbol{L}^2}}$.

I-Bergman kernels on complete manifolds

b) B_p asymptotics: general results

Theorem 0 (Ma-Marinescu, 2007)

With previous notations, assume that:

i) *(**"uniform ampleness"**) there exists $\varepsilon > 0$ such that:*
$iR^h \underset{\text{loc}}{=} -i\partial\overline{\partial}\log(|\sigma|_h^2) \geq \varepsilon\omega_X$ *on* X*;*

ii) *(**"bounded geometry"**)* $\mathrm{Ric}(\omega_X) \geq -C\omega_X$ *on* X*, for some* $C \geq 0$*.*

Then: for all $j \geq 0$, there exists $\boldsymbol{b}_j \in C^\infty(X)$ such that:

$$\forall K \Subset X, \forall k, m \geq 0, \exists Q = Q(K, k, m, \varepsilon, C, n), \forall p \geq 1,$$

$$\left\| p^{-n} B_p(x) - \sum_{j=0}^{k} \boldsymbol{b}_j p^{-j} \right\|_{C^m(K)} \leq Q p^{-k-1}.$$

More precisely, $\boldsymbol{b}_0 = \frac{\omega_h^n}{\omega_X^n}$ *(with* $\omega_h = \frac{i}{2\pi} R^h$*) and*
$\boldsymbol{b}_1 = \frac{\boldsymbol{b}_0}{8\pi}\left(\mathrm{scal}(\omega_h) - 2\Delta_{\omega_h}\log \boldsymbol{b}_0\right)$*.*

I-b) B_p asymptotics: general results

A few remarks:

- Long history; many names associated to this result: Tian (1990, $k = 0$, $m = 2$), Bouche (1990), Catlin-Zelditch (1999-98, compact X), ...
- Quantization of Kodaira embedding theorem / scalar curvature in Kähler geometry.
- The proof requires two steps:
  1- localization on $\mathcal{B}_p$;
  2- computations of the asymptotics with geometric data brought to $\mathbb{C}^n$ (scaling techniques).
- This statement does not say what happens to the Bergman density functions on neighbourhoods of infinity...

II-Punctured Riemann surfaces
a) Setting

"The most elementary class of complete non-compact Kähler manifolds."

- Take:
  - $\Sigma = \bar{\Sigma} \setminus D$, where $D = \{a_1, \ldots, a_N\}$ is the puncture divisor inside a compact Riemann surface $\bar{\Sigma}$, and ω_Σ a smooth Kähler form on Σ;
  - an hermitian line bundle $(L_{|\Sigma}, h)$, with L holomorphic on $\bar{\Sigma}$.
- Suppose moreover that there are trivializations

$$L_{|V_j} \xrightarrow{\sim} \mathbb{C}_{z_j} \times \mathbb{D}_r$$

$(0 < r < 1)$ around the a_j's, such that:
(α) $|1|_h^2(z_j) = \big|\log(|z_j|^2)\big|$;
(β) $i(R^h)_{|V_j^*} = \omega_{\Sigma|V_j^*}$.
In particular,

$$\omega_\Sigma = \omega_{\mathbb{D}^*}(z_j) \text{ on } V_j^*,$$

where $\omega_{\mathbb{D}^*} = \frac{idz\wedge d\bar{z}}{|z|^2\log^2(|z|^2)}$ (Poincaré metric on $\mathbb{D}^*$).

II-a) Setting

An arithmetic class of examples. —

These (notably, properties (α) and (β)) are natural hypotheses, as revealed by the following class of examples.
If $\Gamma \subset \mathrm{Psl}(2,\mathbb{R})$ is a Fuchsian group of the first kind, which is geometrically finite and contains no elliptic element, then

$$\Sigma = \Gamma \backslash \mathbb{H}$$

can be compactified by adjunction of finitely many points.
Conversely, if $\Sigma = \bar{\Sigma} \setminus \{a_1, \ldots, a_N\}$ is such that (equivalently):

- $\tilde{\Sigma} = \mathbb{H}$,
- $2g_{\bar{\Sigma}} - 2 + N > 0$,
- Σ admits a Kähler-Einstein metric with negative scalar curvature, or
- $K_{\bar{\Sigma}}[D]$ $(D = \{a_1, \ldots, a_N\})$ is ample,

then: $\Gamma = \pi_1(\Sigma)$ is Fuchsian, first kind, geometrically finite, with no elliptic element.

II-a) Setting

An arithmetic class of examples. —

Easy case: the **principal congruence subgroup** *of level* 2

$$\Gamma = \bar{\Gamma}(2) = \ker\{\mathrm{Psl}(2,\mathbb{Z}) \to \mathrm{Sl}(2,\mathbb{Z}/2\mathbb{Z})\};$$

then as Riemann surfaces, $\bar{\Gamma}(2)\backslash\mathbb{H} = \mathbb{P}^1 \setminus \{0,1,\infty\}$.

In this context, $K_{\bar{\Sigma}}[D]$ is ample, and (the formal square root) of $(K_{\bar{\Sigma}}[D]_{|\Sigma}, \pi_*\omega_{\mathbb{H}} \otimes h_D)$ verifies (α) and (β) —
here, $\omega_{\mathbb{H}}$ descends to Σ, and h_D is defined on Σ by: $|\sigma_D|^2_{h_D} \equiv 1$ for some $\sigma_D \in \mathscr{O}([D])$ such that $D = \{\sigma_D = 0\}$.

II-Punctured Riemann surfaces

b) Application of Theorem 0

Assume $(\Sigma, \omega_\Sigma, L, h)$ verify (α) and (β); then, for $p \geq 2$,

$$H^0_{(2)}\big(\Sigma, L^p_{|\Sigma}\big) \hookrightarrow H^0\big(\bar{\Sigma}, L^p\big),$$

and more precisely, by Skoda's theorem:

$$H^0_{(2)}\big(\Sigma, L^p_{|\Sigma}\big) \simeq \big\{\sigma \in H^0\big(\bar{\Sigma}, L^p\big) \,\big|\, \sigma(a_j) = 0,\, j = 0, \ldots, N\big\};$$

in particular, $H^0_{(2)}(\Sigma, L^p_{|\Sigma})$ is of finite dimension, denoted by d_p.

Thus:

1- as $B_p^\Sigma(x) = \sum_{j=1}^{d_p} |\sigma_j^{(p)}(x)|^2_{h^p}$, for any fixed p,

$$B_p^\Sigma(x) \to 0 \qquad \text{as} \quad x \to D;$$

2- whereas for all $m \geq 1$ and all compact subsets K of Σ,

$$\Big\| \frac{2\pi}{p} B_p^\Sigma(x) - 1 \Big\|_{C^m(K)} \to 0 \qquad \text{as} \quad p \to \infty$$

by Theorem 0.

What happens in the transition region? How to describe it?

II-Punctured Riemann surfaces

c) Results

First, a *localization* result (comparison with the model $\mathbb{D}^*$):

Theorem 1

For any $m \geq 0$, $\ell \geq 0$ *and* $\delta > 0$, *there exists* $Q = Q(m, \delta)$ *such that for all* $p \gg 1$,

$$\forall z \in V_1^* \cup \ldots \cup V_N^*, \quad \left|\log(|z|^2)\right|^\delta \left|B_p^\Sigma(z) - B_p^{\mathbb{D}^*}(z)\right|_{C^m(\omega_\Sigma)} \leq Qp^{-\ell},$$

where $B_p^{\mathbb{D}^*}$ *is computed from the data* $\left(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \left|\log(|z|^2)\right| |\cdot|\right)$.

II-c) Results

Then, from Theorems 0, 1, and an explicit computation on the model $\mathbb{D}^*$, one can, among others, estimate precisely the *distorsion factor*:

Corollary 2

For $p \gg 1$,

$$\sup_{x\in\Sigma,\, \sigma\in H^0_{(2),p}\setminus\{0\}} \frac{|\sigma(x)|^2_{h^p}}{\|\sigma\|^2_{\boldsymbol{L}^2}} = \sup_{x\in\Sigma} B_p(x) = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p).$$

In the arithmetic situation evoked above, for non-cocompact Γ*, this translates as:*

$$\sup_{z\in\mathbb{H},\, f\in\mathcal{S}^{\Gamma}_{2p}\setminus\{0\}} \frac{(2\mathrm{Im} z)^{2p}|f(z)|^2}{\|f\|^2_{\mathrm{Pet}}} = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}(p),$$

where $\mathcal{S}^{\Gamma}_{2p}$ *is the space of* **cusp modular forms (Spitzenformen) of weight** $2p$.

Remarks: ▷ If Γ were cocompact, the $\sup$ above would be $\frac{p}{\pi} + \mathcal{O}(1)$.

- ▷ In the line of results by Abbes-Ullmo, Michel-Ullmo, Friedman-Jorgenson-Kramer.
- ▷ Version with Γ admitting elliptic elements.

II-c) Results

With a 2-variable version of Theorem 1, one can moreover get the following sharp asymptotics for B_p^{Σ}:

Corollary 3

With the same notations as above, for and $\ell \geq 0$, there exists $C = C(\ell)$ such that for all $p \gg 1$,

$$\sup_{z \in V_1^* \cup \ldots \cup V_N^*} \left| \frac{B_p^{\Sigma}}{B_p^{\mathbb{D}^*}} - 1 \right| \leq C p^{-\ell}.$$

(sharp indeed: we'll soon see that $B_p^{\mathbb{D}^*}$ can take extremely small values).

III-Proofs

a) Corollary 2

By Theorems 0 and 1, enough to establish the same result for $B_p^{\mathbb{D}^*}$ (close to $0 \in \mathbb{D}$). Observe that $\{z^\ell\}_{\ell \geq 1}$ is a complete orthogonal family of $H^0_{(2)}\big(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \big|\log(|z|^2)\big|^p |\cdot|\big)$; direct computations then lead to:

$$B_p^{\mathbb{D}^*}(z) = \frac{\big|\log(|z|^2)\big|^p}{2\pi(p-1)!} \sum_{\ell=1}^{\infty} \ell^{p-1}|z|^{2\ell}.$$

This is explicit enough to:

i) confirm the convergence given by Theorem 0, even near $\partial\mathbb{D}$, and with exponential rate; e.g. on annuli $\{a \leq |z| < 1\}$ $(a \in (0,1))$,

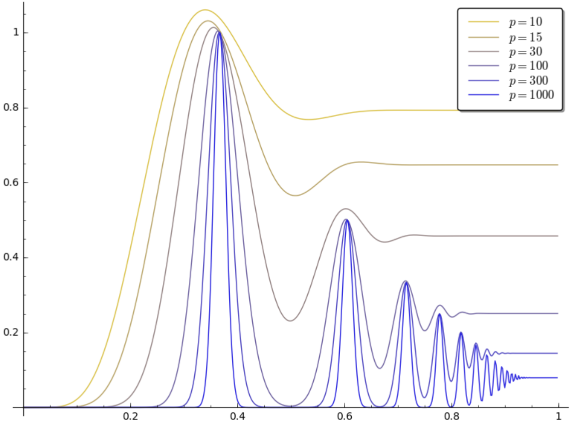
$$\Big\| B_p^{\Sigma}(x) - \frac{p-1}{2\pi} \Big\|_{C^m(\{a \leq |z| < 1\})} = \mathcal{O}(e^{-cp}) \qquad \text{for some } c = c(a) > 0;$$

ii) analyze $B_p^{\mathbb{D}^*}$ up to 0: setting $x = |z|^{2/p}$ and $f_p(x) = B_{p+1}^{\mathbb{D}^*}(z)$, one gets:

$$\Big(\frac{2\pi}{p}\Big)^{3/2} f_p = \sum_{\ell=1}^{\infty} [\text{Gaussian functions centered at } e^{-1/\ell}, \text{ of height } \frac{1}{\ell}].$$

III-a) Corollary 2

ii)



The scaled functions $\left(\frac{2\pi}{p}\right)^{3/2} f_p$ on $(0,1)$

From this, we infer $\sup_{[0,1]} f_p = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p)$, and this sup is reached near $x = e^{-1}$ (which corresponds to $|z| = e^{-p/2}$).

III-a) Corollary 2

For the translation to modular forms, recall:

- the definition of the *space of modular forms of weight* $2p$:

$$\mathcal{M}_{2p}^{\Gamma} = \left\{ f \in \mathscr{O}(\mathbb{H}) \,\middle|\, \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), f(\gamma \cdot z) = (cz+d)^{2p} f(z) \right\};$$

- *Mumford's isomorphism*:

$$\begin{array}{rccc} \Phi: & \mathcal{M}_{2p}^{\Gamma} & \xrightarrow{\sim} & H^0\big(\bar{\Sigma}, L^{2p}\big) \\ & f & \longmapsto & f(dz)^{\otimes p} \end{array},$$

  which restricts to an isometry

$$\mathcal{S}_{2p}^{\Gamma} \xrightarrow{\sim} H^0_{(2)}\big(\Sigma, L^{2p}\big)$$

  where $\mathcal{S}_{2p}^{\Gamma} = \{ f \in \mathcal{M}_{2p}^{\Gamma} \,|\, (\Phi f)(a_j) = 0,\, j = 1, \ldots, N \}$ is endowed with *Petersson's inner product:*

$$\langle f, g \rangle_{\mathrm{Pet}} = \int_{\text{fdmtl dmn}} f(z)\overline{g(z)}(2y)^{2p} \, d\mathrm{vol}_{\mathbb{H}}(z).$$

□

III-Proofs
b) Theorem 1

- Relies on Ma-Marinescu's technology, inspired by Bismut-Lebeau, and centered at the singularity!
- Based on:
  - i) finite propagation speed for the wave equations for Kodaira Laplacians;
  - ii) spectral gap for the Kodaira Laplacians.
- First get the estimate

$$\forall z \in V_1^* \cup \ldots \cup V_N^*, \quad \left|\log(|z|^2)\right|^{\delta} \left|B_p^{\Sigma}(z) - B_p^{\mathbb{D}^*}(z)\right|_{C^m(\omega_\Sigma)} \leq Qp^{-\ell},$$

  but with $\delta < -\frac{1}{2}$!
- Then improve to $\delta > 0$ with help of the holomorphicity of the sections. □

III-Proofs

c) Corollary 3

Assume $N = 1$ for simplicity, and then:

- truncate the orthonormal family $\{c_\ell^{(p)} z^\ell\}_{1\leq \ell\leq \delta_p}$ ($\delta_p \leq d_p$) far from 0, and use the trivialization near a_1 to see it as an *orthogonal* family on Σ;
- correct it into an *orthonormal family*, and complete it into an orthonormal *basis* $(\sigma_\ell^{(p)})_{1\leq \ell\leq d_p}$ of $H^0_{(2)}(\Sigma, L^p)$;
- carefully compare

$$B_p^\Sigma(z) = \big|\log(|z^2|)\big|^p \sum_{\ell=1}^{d_p} |\sigma_\ell^{(p)}|^2_{h_0^p} \quad \text{and} \quad B_p^{\mathbb{D}^*}(z) = \big|\log(|z^2|)\big|^p \sum_{\ell=1}^{\infty} (c_\ell^{(p)})^2 |z|^{2\ell}$$

  on a punctured disc of shape $\{0 < |z| \leq cp^{-A}\}$ (the estimate on the annulus $\{cp^{-A} < |z| < r\}$ follows at once from Theorem 1 and refined analysis on $B_p^{\mathbb{D}^*}$);
- for the comparison, adjust δ_p (linear in p) and A, and use a localization result on $\mathcal{B}_p^\Sigma$ (analogous to Theorem 1 for B_p^Σ) to estimate the error terms $\sigma_\ell^{(p)} - c_\ell^{(p)} z^\ell$, $\ell \in \{1, \dots, \delta_p\}$. □