A Central Limit Theorem for Fleming-Viot Particle Systems

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The killed Markov process

- Consider a 'killed' Markov process $X = (X_t)_{t \geqslant 0}$ in state space $F \sqcup \{\partial\}$. The process is stopped when hitting the cemetary ∂ with $\partial \cap F = \emptyset$.
- The killing time is denoted τ_∂ :

$$\tau_\partial := \inf\{t \geqslant 0, X_t = \partial\}.$$

- Goal: Simulate the conditional distribution

$$\eta_t := \mathcal{L}(X_t | \tau_\partial > t),$$

and the probability of the rare event

$$p_t := \mathbb{P}(\tau_\partial > t) \ll 1.$$

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- *and so on until final time* T.

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- Denote $\mathcal{N}_t :=$ the average number $(= \mathcal{O}_N(1))$ of branchings per particle at time t.
- Then

$$\mathbb{E}\left[(1-\frac{1}{N})^{N\mathcal{N}_t}\right] = p_t (= \mathbb{P}[X_t \neq \partial])$$

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- Recap of notation

$$\eta_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}, \quad p_t^N = \left(1-\frac{1}{N}\right)^{N\mathcal{N}_t}, \quad \gamma_t^N = p_t^N \eta_t^N.$$

$$\downarrow N \to \infty$$

$$\eta_t = \mathcal{L}(X_t | X_t \neq \partial) \qquad p_t = \mathbb{P}(X_t \neq \partial) \qquad \gamma_t := p_t \eta_t$$

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- Consider the sub-Markovian semi-group

$$Q^t(\varphi)(x) := \mathbb{E}[\varphi(X_t)\mathbf{1}_{\tau_\partial > t} | X_0 = x].$$

N.B.: $Q^t(\varphi)(x) = \mathbb{E}[\varphi(X_t)|X_0 = x]$ if $\varphi|_\partial \equiv 0$ by convention, which will always be the case here.

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- Remark that:

$$\gamma_t = \eta_0 Q^t := \int_{x \in F} Q^t(x, \, . \,) \eta_0(dx)$$

as well as

$$\eta_t = \frac{\eta_0 Q^t}{\eta_0 Q^t(1)}$$

Central Limit Theorems

Theorem (CLT)

Under **Ass. 1** *'non-synchronous jumps' and* **Ass. 2** *'non-explosion' below, for any* $\varphi \in C_b(F)$,

$$\sqrt{N}\left(p_T^N - p_T\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 := \underbrace{p_T^2 \ln(p_T)}_{\text{Universal part}} + 2\int_0^T \mathbf{Var}_{\eta_t}(Q^{T-t}(1_F))p_t^2 d\left(\ln \frac{1}{p_t}\right).$$

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Under **Ass. 1** *'non-synchronous jumps' and* **Ass. 2** *'non-explosion', one has for any* $\varphi \in C_b(F)$

$$\sqrt{N}\left(\eta_T^N(\varphi) - \eta_T(\varphi)\right) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma_T^2(\varphi)).$$

where in the case $\eta_T(\varphi) = 0$

$$\sigma_T^2(\varphi) := \underbrace{\mathbf{Var}_{\eta_T}(\varphi)}_{\textit{Universal part}} + \int_0^T \mathbf{Var}_{\eta_t}(Q^{T-t}(\varphi))\frac{p_t^2}{p_T^2} d\left(\ln \frac{1}{p_t}\right).$$

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$$\lim \mathbb{P}_x(X_T \notin \partial) \in \{0,1\}\,.$$

- Dominant term $2(1 - 1/p_T)$ is twice the naive Monte Carlo variance.

Regularity Assumption

Assumption 1 (Non-synchronous jumps of X)

For any initial condition and any $\varphi \in C_b(F)$:

(i) *the jump times of the càdlàg version of the martingale process $t \mapsto \mathbb{L}_t := Q^{T-t}(\varphi)(X_t)$ have an atomless distribution:*

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(ii) *The killing time τ_∂ has also an atomless distribution.*

The 'non-synchronous jumps' Assumption is 'morally' equivalent to: all "martingale jumps" and/or branchings in the Fleming-Viot system are never simultaneous.

Non-explosion Assumption

In addition we ask:

Assumption (Non-explosion)

The Fleming-Viot system is non-explosive in the sense that the number of branching at any finite time is almost surely finite $\mathbb{P}(\mathcal{N}_T < +\infty) = 1$.

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- *Let* $F \subset \mathbb{R}^d$ *be a bounded open domain with smooth boundary* $\partial F = \overline{F} \setminus F$. *Let* τ_∂ *be the hitting time of* $\overline{F}$.

$$X_t = \widetilde{X}_t \quad \textit{for} \;\; t < \tau_\partial, \quad \textit{else} = \partial$$

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- *Then Assumption 'non explosion' holds true ([Grigorescu and Kang, 2012]), as well as Assumption 'no synchronuous jumps'.*

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- We then have the chain rule

$$d(X_t Y_t) = Y_{t^-} dX_t + X_{t^-} dY_t + d[X, Y]_t$$

where $t \mapsto [X, X]_t$ is an increasing process, bilinear with respect to vector space structure on X called the quadratic variation. Broadly speaking

$$[X, X]_t = \lim_{|t_{i+1} - t_i| \mapsto 0}^{\mathbb{P}} \sum_i (X_{t_{i+1}} - X_{t_i})^2$$

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- If X is monotonous, $[X, X]_t$ is the sum of the squares of the jumps.

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- Example: let $t \mapsto M_t \in \{-1, 1\}$ be Poisson-like Markov + Martingale random walk process jumping up or down with proba $1/2$ at indep. expo. times. Then

$$[M, M]_t = \sum_{jumps} \mathbf{Var}(jump) = 1$$

$$< M, M >_t = dt$$

CLT for martingales with jumps

Theorem (Martinagle CLT (Ethier-Kurtz))

On a filtered probability space, let $t \mapsto m_t^N$ denote a sequence of càdlàg local martingales indexed by $N \geqslant 1$. Assume moreover that

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(iii) *For each N, there exists an increasing càdlàg quadratic variation process $t \mapsto i_t^N$ i.e. $t \mapsto (m_t^N - m_0^N)^2 - i_t^N$ is a local martingale.*

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(iv) *Vanishing jump: The process $t \mapsto i_t^N$ satisfies* $\lim_{N \to +\infty} \mathbb{E}\left[\sup_{t \in [0,T]} i_t^N - i_{t^-}^N\right] = 0.$

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For the increasing càdlàg process $t \mapsto i_t^N$ (with vanishing jumps) such that

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(v) *!! Main Assumption !!:* *There is a cont. and incr. det. function $t \mapsto i_t$ s. t., $\forall t \in [0, T]$, $i_t^N \xrightarrow[N \to +\infty]{\mathbb{P}} i_t$.*

Then $(m_t^N)_{t \in [0,T]}$ converges in law (under the Skorokhod topology) to $(M_t)_{t \in [0,T]}$, where $M_0 \sim \mu_0$ and $(M_t - M_0)_{t \in [0,T]}$ is a Gaussian martingale, independent of M_0, with independent increments and variance function i_t (time changed Brownian motion).

CLT for martingales with jumps

In short, we need to construct martingales of order $1/\sqrt{N}$ from the particle system and ensure the convergence of 'a' quadratic variation of those martingales of order $1/N$.

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- We will handle the distribution of $\gamma_T^N(Q)$ in the limit $N \to \infty$ by using a Central Limit Theorem for continuous time martingales.
- Not straightforward: the convergence of the quadratic variation $N[\gamma^N(Q), \gamma^N(Q)]_t$ is difficult (lots of IPPs !!).

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- $\mathbb{M}_t^n$ is the martingale contribution except for branching times of particle n.
- $\mathcal{M}_t^n$ is the martingale contribution at branchings only of particle n.
- No ambiguity, natural way to do this.

Orthogonality

The $2N$ martingales $\{\mathbb{M}_t^n, \mathcal{M}_t^m\}_{1\leqslant n,m\leqslant N}$ are mutually orthogonal. More specifically

(i) $[\mathbb{M}, \mathcal{M}]_t$ is a local martingale,

(ii)

$$[\mathcal{M}, \mathcal{M}]_t = \frac{1}{N}\sum_{n=1}^{N}[\mathcal{M}^n, \mathcal{M}^n]_t,$$

(iii) Moreover, if we note the 'intermediate' quadratic variation

$$(\mathbb{M}, \mathbb{M})_t = \frac{1}{N}\sum_{n=1}^{N}[\mathbb{M}^n, \mathbb{M}^n]_t,$$

then the process $[\mathbb{M}, \mathbb{M}]_t - (\mathbb{M}, \mathbb{M})_t$ is also a local martingale.

Ingredient (i): A 'key formula'

Lemma

The quadratic variation of martingales associated with the particles dynamics outside branchings can be related to

$$\gamma_t^N(Q^2) = \gamma_t^N([Q^{T-t}(\varphi)]^2)$$

through the key formula

$$p_{t^-}^N d(\mathbb{M},\mathbb{M})_t = d\gamma_t^N(Q^2) + \textit{Martingale}$$

Ingredient (ii): L^2 apriori estimates

Proposition (Villemonais 2014, CDGR 2017)

For any $\varphi \in \mathcal{D}$, we have

$$\mathbb{E}\left[\left(\gamma_T^N(\varphi) - \gamma_T(\varphi)\right)^2\right] \leqslant \frac{6\,\|\varphi\|_\infty^2}{N}.$$

Proof.

$$\gamma_T^N(\varphi) - \gamma_T(\varphi) = \frac{1}{\sqrt{N}} \int_0^T p_{t^-}^N \; d\mathbb{M}_t + \frac{1}{\sqrt{N}} \int_0^T p_{t^-}^N \; d\mathcal{M}_t + \gamma_0^N(Q^T\varphi) - \gamma_0(Q^T\varphi),$$

(i) Initial condition is OK by independence.

L^2 estimates

(ii) $\mathcal{M}$-terms. Using Ito's isometry and $d[\mathcal{M},\mathcal{M}]_t \leqslant 4\|\varphi\|_\infty^2 d\mathcal{N}_t$, we obtain

$$\mathbb{E}\left[\left(\int_0^T p_{t-}^N \, d\mathcal{M}_t\right)^2\right] = \mathbb{E}\left[\int_0^T \left(p_{t-}^N\right)^2 d[\mathcal{M},\mathcal{M}]_t\right]$$

$$\leqslant 4\|\varphi\|_\infty^2 \frac{1}{N}\sum_{j=1}^{\infty}\left(1-\tfrac{1}{N}\right)^{2(j-1)} \leqslant 4\|\varphi\|_\infty^2.$$

(iii) $\mathbb{M}$-terms. In the same way, applying Ito's isometry and the 'key formula' $p_{t-}^N d(\mathbb{M},\mathbb{M})_t = d\gamma_t^N(Q^2) +$ Martingale, we get

$$\mathbb{E}\left[\left(\int_0^T p_{t-}^N \, d\mathbb{M}_t\right)^2\right] = \mathbb{E}\left[\int_0^T \left(p_{t-}^N\right)^2 d[\mathbb{M},\mathbb{M}]_t\right]$$

$$\leqslant \mathbb{E}\left[\int_0^T p_{t-}^N \, d(\mathbb{M},\mathbb{M})_t\right] = \mathbb{E}\left[\gamma_T^N(Q^2)\right] \leqslant \|\varphi\|_\infty^2.$$

Ingredient (iii): Time uniform a priori estimate of p_t^N

Lemma

One has

$$\sup_{t\in[0,T]} \left| p_t^N - p_t \right| \xrightarrow[N\to\infty]{\mathbb{P}} 0.$$

Proof. Independent of the context. $t \mapsto p_t$ is continuous on $[0, T]$ by construction, it is cleat that $t \mapsto p_t^N$ is decreasing for all $N \geqslant 2$. The Lemma results of last Proposition and a from a probabilistic version of Second Dini (or Pólya) theorem: if a sequence of monotone functions converges pointwise on a compact interval and if the limit function is also continuous, then the convergence is uniform on that interval.

Increasing Process in general CLT

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- In order to use martingale CLT, we need a càdlàg increasing process i_t^N such that $(\gamma^N(Q)_t)^2 - i_t^N$ is a martingale (quadratic variation - like).

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$$i_t^N = \int_0^t \left(p_{u-}^N\right)^2 d(\mathbb{M},\mathbb{M})_u - \int_0^t \mathbf{Var}_{\eta_{u-}^N}(Q) p_{u-}^N \, dp_u^N + \frac{1}{N}\int_0^t \left(p_{u-}^N\right)^2 d\mathcal{R}_u.$$

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- NB: with the rest term is $O(1/N)$ with

$$\mathcal{R}_t = \sum_{n=1}^{N}\sum_{k=1}^{+\infty}\left(\left(1-\tfrac{1}{N}\right)^2 \mathbf{Var}_{\eta^{(n)}_{\tau_{n,k}^-}}(Q) - \mathbf{Var}_{\eta^N_{\tau_{n,k}^-}}(Q)\right) 1_{t\geqslant\tau_{n,k}}.$$

Integration by parts formulas

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- If $|z_{\tau_j^-}^N| \leqslant c(1-1/N)^j$,

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- If $|\Delta z_{\tau_j}^N| \leqslant c(1 - 1/N)^j / N$,

$$\int_0^t z_{s^-}^N \, d \ln p_s^N = z_t^N \ln p_t^N - \int_0^t \ln p_{s^-}^N \, dz_s^N + O(1/N).$$

Integration by parts for i_t^N

Using the intergation by parts formula abvoe and the key formula :

Lemma

The increasing process i_t^N can be integrated by parts and be rewritten as

$$i_t^N = p_t^N \gamma_t^N(Q^2) - \gamma_0^N(Q^2) + \left[\gamma_t^N(Q)\right]^2 \ln p_t^N - 2\int_0^t \gamma_{u^-}^N(Q^2)\,dp_u^N + O\Big(\frac{1}{\sqrt{N}}\Big).$$

Convergence of i_t^N and final proof of the CLT

By the non synchronous jump Assumption all the vanishing jump assumptions in the martingale CLT are verified.
The only remaining remaining part to prove is the following:

Proposition

For any $t \in [0, T]$, one has

$$i_t^N \xrightarrow[N\to\infty]{\mathbb{P}} i_t(\varphi). \qquad \textit{where}$$

$$i_t^N = p_t^N \gamma_t^N(Q^2) - \gamma_0^N(Q^2) + \left[\gamma_t^N(Q)\right]^2 \ln p_t^N - 2\int_0^t \gamma_{u^-}^N(Q^2)\, dp_u^N + O\left(\frac{1}{\sqrt{N}}\right.$$

$$i_t(\varphi) = p_t \gamma_t(Q^2) - \gamma_0(Q^2) + [\gamma_t(Q)]^2 \ln p_t - 2\int_0^t \gamma_u(Q^2)\, dp_u.$$

Proof of Convergence of i_t^N

After some calculations, all but one term can be treated using the L^2 a priori convergence estimate. The only remaing problem is the following:

$$i_t^N - i_t(\varphi) = \text{easy converging terms with } L^2\text{-estimate and IPP} + \\ -2\int_0^t (p_{u^-}^N - p_u)\, d\,\gamma_u^N(Q^2)$$

It is difficult to prove its convergence to 0 because the L^2 estimate is only pointwise. Hence handling the integrator is cumbersome !!.

Convergence of i_t^N

This difficult term is then treated as follows. Use the 'key formula'

$$\int_0^t (p_{u^-}^N - p_u)\, d\,\gamma_u^N(Q^2) = \int_0^t (p_{u^-}^N - p_u) p_{u^-}^N\, d(\mathbb{M},\mathbb{M})_u + O(\frac{1}{\sqrt{N}})$$

Since $(\mathbb{M},\mathbb{M})$ is an increasing process, it comes

$$\left| \int_0^t (p_{u^-}^N - p_u) p_{u^-}^N\, d(\mathbb{M},\mathbb{M})_u \right| \leqslant \sup_u |p_{u^-}^N - p_u| \times \left(\int_0^t p_{u^-}^N\, d(\mathbb{M},\mathbb{M})_u \right).$$

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- The 'key formula' back again implies:

$$\mathbb{E}\left[\int_0^t p_{u^-}^N\, d(\mathbb{M},\mathbb{M})_u \right] = \mathbb{E}\left[\gamma_t^N(Q^2) \right] \leqslant \|\varphi\|_\infty^2.$$

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- The a priori uniform estimate implies convergence of $\sup_u |p_{u^-}^N - p_u|$.

END OF PROOF

Adaptive Multilevel Splitting

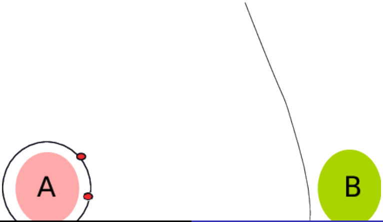
Lemma

The AMS algorithm mapped with the level-indexed process

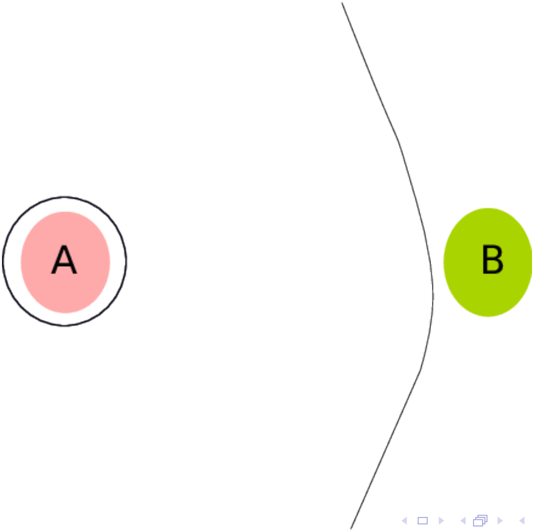
$$X_t := Y_{S_t}, \qquad S_t = \inf(s \geqslant 0 | \xi(Y_s) = t)$$

and stopped at iteration J_t (the first iteration when all particles reach $\{\xi > t\}$) is actually a Fleming-Viot particle system.

Proof Picture: level function ξ define a time change.



Adaptive Multilevel Splitting



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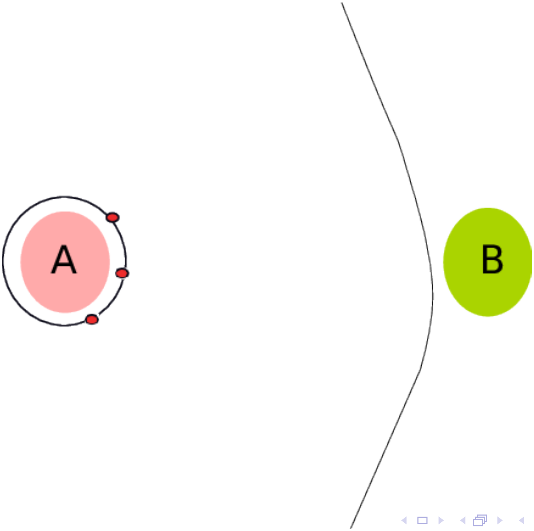

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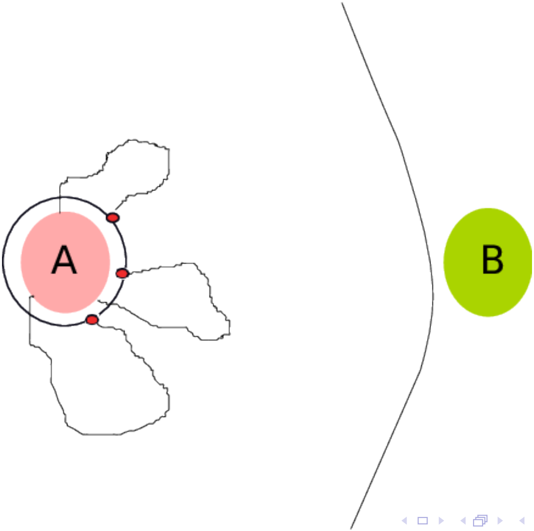

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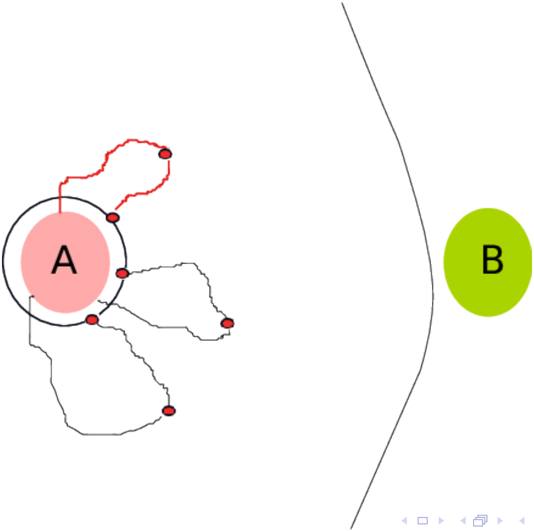

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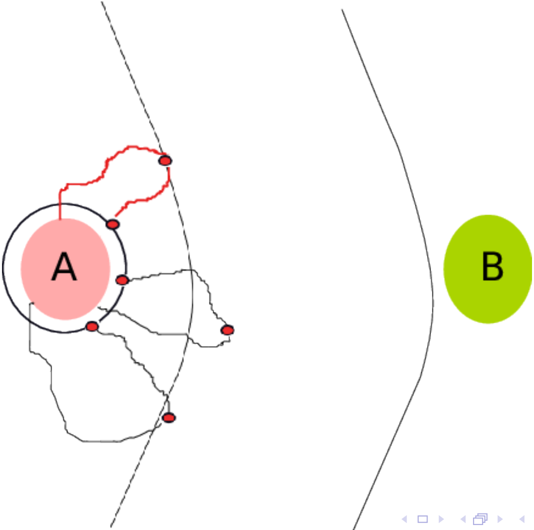

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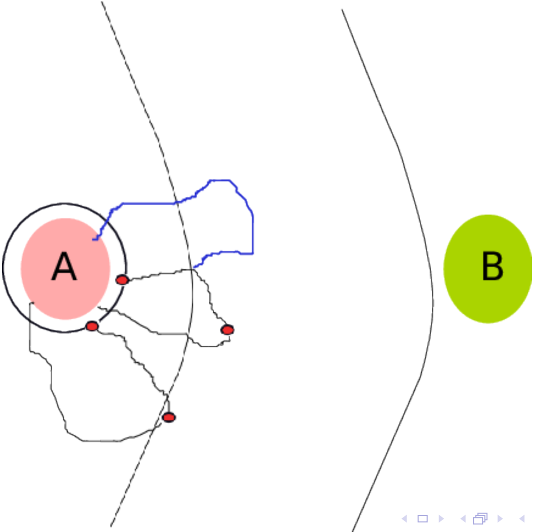

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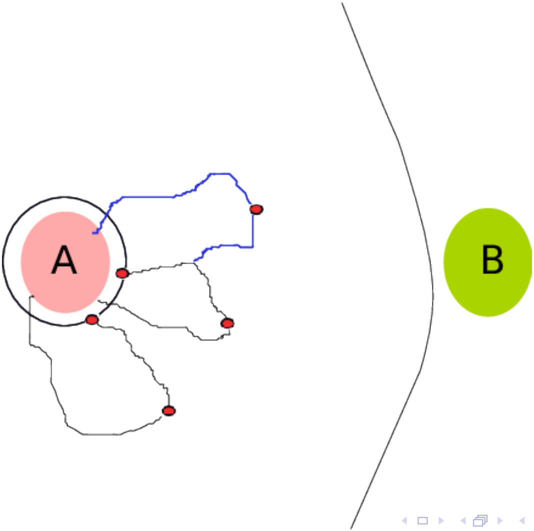

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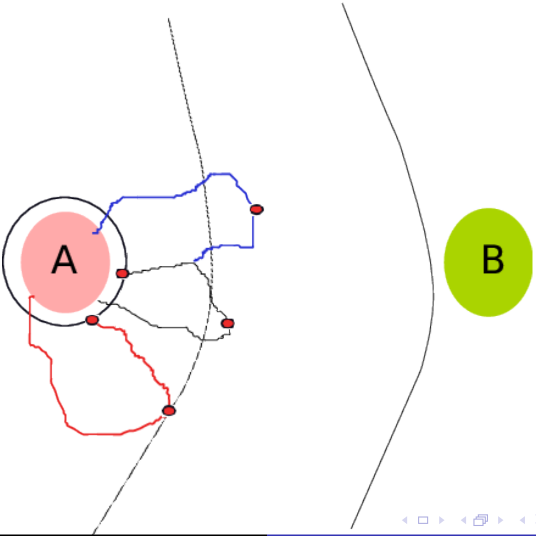

Arxiv

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