A Central Limit Theorem for Fleming-Viot Particle Systems

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The killed Markov process

- Consider a 'killed' Markov process $X = (X_t)_{t \geq 0}$ in state space $F \sqcup \{\partial\}$. The process is stopped when hitting the cemetary $\partial$ with $\partial \cap F = \emptyset$.
- The killing time is denoted $\tau_{\partial}$:

$$\tau_{\partial} := \inf\{t \geq 0, X_t = \partial\}.$$ 

- **Goal**: Simulate the conditional distribution

$$\eta_t := \mathcal{L}(X_t | \tau_{\partial} > t),$$

and the probability of the rare event

$$p_t := \mathbb{P}(\tau_{\partial} > t) \ll 1.$$
Fleming-Viot Particle System

**Definition (Fleming-Viot particle system)**

Fleming-Viot particle system \((X_t^1, \cdots, X_t^N)_{t \in [0,T]}\) is the Markov process with state space \(F^N\) defined by the rules
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A **Fleming-Viot particle system** $(X^1_t, \ldots, X^N_t)_{t \in [0,T]}$ is the Markov process with state space $F^N$ defined by the rules:

- **Initialization**: i.i.d. particles $X^1_0, \ldots, X^N_0 \sim \eta_0$,
- **Evolution and killing**: each particle evolves independently according to the law of the underlying Markov process until one of them hits $\partial$,
- **Splitting**: the killed particle is taken from $\partial$, and is given instantaneously the state of one of the $(N-1)$ other particles (randomly uniformly chosen), and so on until final time $T$. 

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CLT for AMS & Fleming-Viot
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- and so on until final time \(T\).
Unbiased Estimation of survival probability

At each branching, the total 'surviving mass' is multiplied by a factor \( \frac{(N - 1)}{N} = 1 - \frac{1}{N} \).

Denote \( N_t \) as the average number \( = O(N) \) of branchings per particle at time \( t \).

Then \( E[\left(1 - \frac{1}{N}\right)^N] = p_t \) (\( = P[X_t \neq \partial] \)).
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At each branching, the total 'surviving mass' is multiplied by factor \((N - 1)/N = 1 - 1/N\).

Denote \(\mathcal{N}_t := \) the average number (= \(O_N(1)\)) of branchings per particle at time \(t\).

Then

\[
\mathbb{E}\left[\left(1 - \frac{1}{N}\right)^{\mathcal{N}_t}\right] = p_t(= \mathbb{P}[X_t \neq \partial])
\]
Unbiased Estimators and Recap

\[ \eta_{Nt} := \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n t} \]

Non-normalized quantities unbiased estimation

\[ \gamma_{t}(\phi) := \mathbb{E}[\phi(X_t) \mathbb{I}_{\partial t}] = \mathbb{E}\left[\left(1 - \frac{1}{N}\right) N N_t \eta_{Nt}(\phi)\right]. \]

Recap of notation

\[ \eta_{Nt} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i t}, \quad \gamma_{Nt} = \eta_{Nt} \]

\[ \downarrow \quad N \to \infty \]

\[ \eta_t = L(X_t | X_t \neq \partial), \quad \gamma_t = P(X_t \neq \partial) \]

\[ \gamma_t = \gamma_{Nt} \]

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CLT for AMS & Fleming-Viot
Unbiased Estimators and Recap

- **Empirical measure** of particles $\eta^N_t := \frac{1}{N} \sum_{n=1}^N \delta_{X^n_t}$
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- **Empirical measure** of particles \( \eta_t^N := \frac{1}{N} \sum_{n=1}^{N} \delta_{X^n_t} \)
- **Non-normalized quantities** unbiased estimation

\[
\gamma_t(\varphi) := \mathbb{E}[\varphi(X_t)1_{\tau_\partial > 1}] = \mathbb{E}[(1 - 1/N)^{N\eta_t^N} \eta_t^N(\varphi)].
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Unbiased Estimators and Recap

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- **Recap of notation**

\[ \eta^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t}, \quad p^N_t = (1 - 1/N)^N \eta^N_t, \quad \gamma^N_t = p^N_t \eta^N_t. \]

\[ \downarrow \quad N \to \infty \]

\[ \eta_t = \mathcal{L}(X_t | X_t \neq \partial) \quad p_t = \mathbb{P}(X_t \neq \partial) \quad \gamma_t := p_t \eta_t \]
Consider the sub-Markovian semi-group \( Q_t(\phi)(x) := E\left[ \phi(X_t) \mid X_0 = x \right] \).

N.B.: \( Q_t(\phi)(x) = E\left[ \phi(X_t) \mid X_0 = x \right] \) if \( \phi \mid_{\partial} \equiv 0 \) by convention, which will always be the case here.

Remark that:

\[ \gamma_t = \eta_0 Q_t := \int_{x \in F} Q_t(x, \cdot) \eta_0(\cdot) \, dx \]

as well as

\[ \eta_t = \eta_0 Q_t \eta_0 Q_t(1) \]
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as well as

\[ \eta_t = \frac{\eta_0 Q^t}{\eta_0 Q^t(1)} \]
Central Limit Theorems

**Theorem (CLT)**

Under Ass. 1 'non-synchronous jumps' and Ass. 2 'non-explosion' below, for any \( \varphi \in C_b(F) \),

\[
\sqrt{N} \left( p_N^T - p_T \right) \xrightarrow{D} \mathcal{N}(0, \sigma^2),
\]

where

\[
\sigma^2 := p_T^2 \ln(p_T) + 2 \int_0^T \text{Var}_{\eta_t}(Q^{T-t}(1_F)) p_t^2 d\left( \ln \frac{1}{p_t} \right).
\]

Theorem CLT for AMS & Fleming-Viot
Theorem (CLT)

*Under Ass. 1 'non-synchronous jumps' and Ass. 2 'non-explosion', one has for any $\varphi \in C_b(F)$*

$$
\sqrt{N} \left( \eta^N_T(\varphi) - \eta_T(\varphi) \right) \overset{D}{\underset{N \to \infty}{\to}} N(0, \sigma^2_T(\varphi)).
$$

*where in the case $\eta_T(\varphi) = 0$*

$$
\sigma^2_T(\varphi) := \text{Var}_{\eta_T}(\varphi) + \int_0^T \text{Var}_{\eta_t}(Q^{T-t}(\varphi)) \frac{p_t^2}{p_T^2} d \left( \ln \frac{1}{p_t} \right).
$$
Remarks on Asymptotic Variances

Bounds on the relative asymptotic variance of survival probability estimator

\[
\log\left(\frac{1}{p^T}\right) \leq \frac{\sigma^2}{p^2} \leq 2\left(\frac{1}{p^T} - 1\right) + \log(p^T)
\]

Lower bound is sharp and formally obtained in the limit \(t_{\text{kill}} \gg t_{\text{mix}} \to 0\) (e.g. spectral radius of \(Q\ll\) large spectral gap of \(Q\)).

Upper bound is sharp and formally obtained in the limit \(t_{\text{kill}} \ll t_{\text{mix}} \to \infty\) with 'fate in initial condition' limit \(\lim P^{x}(X_T/\in \partial) \in \{0, 1\}\).

Dominant term \(2\left(1 - 1/p^T\right)\) is twice the naive Monte Carlo variance.
Remarks on Asymptotic Variances

- **Bounds on the relative asymptotic variance** of survival probability estimator

\[ \log \left( \frac{1}{p_T} \right) \leq \sigma^2 / p_T^2 \leq 2 \left( \frac{1}{p_T} - 1 \right) + \log(p_T) \]
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\lim P_x(X_T \notin \partial) \in \{0, 1\}.
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Remarks on Asymptotic Variances

- Bounds on the relative asymptotic variance of survival probability estimator

\[ \log(1/p_T) \leq \sigma^2/p_T^2 \leq 2(1/p_T - 1) + \log(p_T) \]

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\[ \lim P_x(X_T \notin \partial) \in \{0, 1\} . \]

- Dominant term \( 2(1 - 1/p_T) \) is twice the naive Monte Carlo variance.
Assumption 1 (Non-synchronous jumps of $X$)

For any initial condition and any $\varphi \in C_b(F)$:

(i) the jump times of the càdlàg version of the martingale process $t \mapsto L_t := Q^{T-t}(\varphi)(X_t)$ have an atomless distribution:

$$\mathbb{P}(L_{t^-} \neq L_t | X_0 = x) = 0 \quad \forall t \geq 0.$$
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(ii) The killing time $\tau_\partial$ has also an atomless distribution.

The 'non-synchronous jumps' Assumption is 'morally' equivalent to: all "martingale jumps" and/or branchings in the Fleming-Viot system are never simultaneous.
In addition we ask:

**Assumption (Non-explosion)**

The Fleming-Viot system is non-explosive in the sense that the number of branching at any finite time is almost surely finite

\[ \mathbb{P}(N_T < +\infty) = 1. \]
Example with Hard Obstacle (The originality of our result !)

Proposition

Let $F \subset \mathbb{R}^d$ be a bounded open domain with smooth boundary $\partial F = F \setminus \overline{F}$. Let $\tau_{\partial}$ be the hitting time of $F$.

If $X_t = \tilde{X}_t$ for $t < \tau_{\partial}$, then Assumption 'non explosion' holds true ([Grigorescu and Kang, 2012]), as well as Assumption 'no synchronuous jumps'.
Example with Hard Obstacle (The originality of our result !)

**Proposition**

\[ t \mapsto \tilde{X}_t \in \mathbb{R}^d \text{ be a diffusion with smooth and uniformly elliptic coefficients.} \]
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**Proposition**

- Let $t \mapsto \tilde{X}_t \in \mathbb{R}^d$ be a diffusion with *smooth and uniformly elliptic coefficients*.
- Let $F \subset \mathbb{R}^d$ be a *bounded open domain with smooth boundary* $\partial F = \overline{F} \setminus F$. Let $\tau_{\partial}$ be the hitting time of $\overline{F}$.

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Then Assumption 'non explosion' holds true ([Grigorescu and Kang, 2012]), as well as Assumption 'no synchronuous jumps'.
Proof: Stochastic calculus with jumps
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- Recall that one can integrate with respect to 'semi-martingales' $X = \text{monotoneous processes} + \text{martingales}$ as follows:

$$
\int Y_t^- dX_t \sim \int Y_t^- (X_{t+dt} - X_t)
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Recall that one can integrate with respect to 'semi-martingales' $X = \text{monotonous processes} + \text{martingales}$ as follows:

$$\int Y_t^- dX_t \simeq \int Y_t^- (X_{t+dt} - X_t)$$

We then have the chain rule

$$d(X_t Y_t) = Y_t^- dX_t + X_t^- dY_t + d[X, Y]_t$$

where $t \mapsto [X, X]_t$ is an increasing process, bilinear with respect to vector space structure on $X$ called the quadratic variation. Broadly speaking

$$[X, X]_t = \lim_{|t_{i+1} - t_i| \to 0} \sum_i (X_{t_{i+1}} - X_{t_i})^2$$
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  \[
  [X, X]_t = \lim_{|t_{i+1} - t_i| \to 0} \sum_i (X_{t_{i+1}} - X_{t_i})^2
  \]
  If $X$ is monotonous, $[X, X]_t$ is the sum of the squares of the jumps.
If \( t \mapsto M_t \) is a (local) martingale, then \( t \mapsto M_t^2 - [M, M]_t \) is again a local martingale. In the presence of jumps there are plenty of 'quadratic variation'-like increasing processes \( t \mapsto i(M_t) \) such that \( t \mapsto M_t^2 - i(M_t) \) is a local martingale. For instance there is a unique \( i(M_t) = \langle M, M \rangle_t \) which is predictable.

Example: let \( t \mapsto M_t \in \{-1, 1\} \) be Poisson-like Markov + Martingale random walk process jumping up or down with probability \( \frac{1}{2} \) at independent exponential times. Then 
\[
[M, M]_t = \sum jumps \text{Var}(jump) = 1 < M, M >_t = dt
\]
Stochastic calculus with jumps

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$$[M, M]_t = \sum_{\text{jumps}} \text{Var(jump)} = 1$$

$$\langle M, M \rangle_t = dt$$
Theorem (Martinagle CLT (Ethier-Kurtz))

On a filtered probability space, let \( t \mapsto m_t^N \) denote a sequence of càdlàg local martingales indexed by \( N \geq 1 \). Assume moreover that

(i) \( m_{N0} \xrightarrow{D} \infty \), where \( \mu_0 \) is a given probability on \( \mathbb{R} \).

(ii) Vanishing jumps: One has \( \lim_{N \to +\infty} E \left[ \sup_{t \in [0,T]} \left| m_t^N - m_0^N \right|^2 \right] = 0 \).

(iii) For each \( N \), there exists an increasing càdlàg quadratic variation process \( t \mapsto i_N^t \) i.e. \( t \mapsto (m_t^N - m_0^N)^2 - i_t^N \) is a local martingale.

(iv) Vanishing jump: The process \( t \mapsto i_t^N \) satisfies \( \lim_{N \to +\infty} E \left[ \sup_{t \in [0,T]} i_t^N \right] = 0 \).
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CLT for AMS & Fleming-Viot
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$$\lim_{N \to +\infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| m^N_t - m^N_{t-} \right|^2 \right] = 0.$$
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CLT for martingales with jumps

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\lim_{N \to +\infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |m_t^N - m_{t-}^N|^2 \right] = 0.
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(iv) **Vanishing jump:** The process \( t \mapsto i_t^N \) satisfies
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Theorem (Martinagle CLT (Ethier-Kurtz))

For the increasing càdlàg process \( t \mapsto i_t^N \) (with vanishing jumps) such that

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is a local martingale:

\((\nu) \text{ !! Main Assumption !!: There is a cont. and incr. det. function } t \mapsto i_t \text{ s. t., } \forall t \in [0, T], \, i_t^N \xrightarrow{\mathbb{P}} i_t.\)

Then \( (m_t^N)_{t \in [0, T]} \) converges in law (under the Skorokhod topology) to \( (M_t)_{t \in [0, T]} \), where \( M_0 \sim \mu_0 \) and \( (M_t - M_0)_{t \in [0, T]} \) is a Gaussian martingale, independent of \( M_0 \), with independent increments and variance function \( i_t \) (time changed Brownian motion).
In short, we need to construct martingales of order $1/\sqrt{N}$ from the particle system and ensure the convergence of 'a' quadratic variation of those martingales of order $1/N$. 
Overview of the proof

Key object: the càdlàg martingale $t \mapsto \gamma_N(t)$.

Initial condition treated separately (easy).

We will handle the distribution of $\gamma_N(T - t(\phi))$ in the limit $N \to \infty$ by using a Central Limit Theorem for continuous time martingales.

Not straightforward: the convergence of the quadratic variation $N[\gamma_N(Q), \gamma_N(Q)]$ is difficult (lots of IPPs !!).
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- **Key object: the càdlàg martingale**

\[ t \mapsto \gamma_t^N(Q) := \gamma_t^N\left(Q^{T-t}(\varphi)\right). \]
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  \[ t \mapsto \gamma^N_t(Q) := \gamma^N_t(Q^{T-t}(\varphi)). \]

- Initial condition treated separately (easy).
- We will handle the distribution of \( \gamma^N_T(Q) \) in the limit \( N \to \infty \) by using a Central Limit Theorem for continuous time martingales.
Overview of the proof

- Key object: the càdlàg martingale

\[ t \mapsto \gamma_t^N(Q) := \gamma_t^N \left( Q^{T-t}(\varphi) \right). \]

- Initial condition treated separately (easy).
- We will handle the distribution of \( \gamma_T^N(Q) \) in the limit \( N \to \infty \) by using a Central Limit Theorem for continuous time martingales.
- Not straightforward: the convergence of the quadratic variation \( N[\gamma^N(Q), \gamma^N(Q)]_t \) is difficult (lots of IPPs !!).
Martingale decomposition [Villemonais 2014]

\[ \gamma_N(t) = \gamma_N(0) + 1 \sqrt{\frac{N}{2}} \int_0^t p_N u - (dM_u + dM_u) \]

\[ M_t = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} M_n t \]

\[ M_n t \] is the martingale contribution except for branching times of particle \( n \).

\[ M_t \] is the martingale contribution at branchings only of particle \( n \).
Martingale decomposition [Villemonais 2014]

The key martingale decomposition is the following:

\[ \gamma_t^N(Q) = \gamma_0^N(Q) + \frac{1}{\sqrt{N}} \int_0^t p_u^N (dM_u + dM_u). \]
The key martingale decomposition is the following:

$$\gamma_t^N(Q) = \gamma_0^N(Q) + \frac{1}{\sqrt{N}} \int_0^t p_u^N (d\mathcal{M}_u + d\mathcal{M}_u).$$

With $\mathcal{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathcal{M}_t^n$ and $\mathcal{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathcal{M}_t^n$. 

$\mathcal{M}_t$ is the martingale contribution except for branching times of particle $n$. $\mathcal{M}_t$ is the martingale contribution at branchings only of particle $n$. No ambiguity, natural way to do this.
The key martingale decomposition is the following:

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The key martingale decomposition is the following:

\[ \gamma_t^N(Q) = \gamma_0^N(Q) + \frac{1}{\sqrt{N}} \int_0^t p_u^N (dM_u + dM_u). \]

With \( \mathbb{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathbb{M}^n_t \) and \( \mathcal{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathcal{M}^n_t \).

\( \mathbb{M}^n_t \) is the martingale contribution except for branching times of particle \( n \).

\( \mathcal{M}^n_t \) is the martingale contribution at branchings only of particle \( n \).
The key martingale decomposition is the following:

\[ \gamma^N_t(Q) = \gamma^N_0(Q) + \frac{1}{\sqrt{N}} \int_0^t \rho^N_{u^-}(d\mathcal{M}_u + d\mathcal{M}_u). \]

With \( \mathcal{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathcal{M}^n_t \) and \( \mathcal{M}_t := \frac{1}{\sqrt{N}} \sum_{n=1}^N \mathcal{M}^n_t. \)

\( \mathcal{M}^n_t \) is the martingale contribution except for branching times of particle \( n \).

\( \mathcal{M}^n_t \) is the martingale contribution at branchings only of particle \( n \).

No ambiguity, natural way to do this.
Orthogonality

The $2N$ martingales $\{M_t^n, M_t^m\}_{1 \leq n,m \leq N}$ are mutually orthogonal. More specifically

(i) $[M, M]_t$ is a local martingale,

(ii) 

$$[M, M]_t = \frac{1}{N} \sum_{n=1}^{N} [M^n, M^n]_t,$$

(iii) Moreover, if we note the 'intermediate' quadratic variation

$$\langle M, M \rangle_t = \frac{1}{N} \sum_{n=1}^{N} [M^n, M^n]_t,$$

then the process $[M, M]_t - \langle M, M \rangle_t$ is also a local martingale.
Lemma

The quadratic variation of martingales associated with the particles dynamics outside branchings can be related to

$$\gamma_t^N(Q^2) = \gamma_t^N([Q^{T-t}(\varphi)]^2)$$

through the key formula

$$p_t^N d(\mathbb{M}, \mathbb{M}_t) = d\gamma_t^N(Q^2) + \text{Martingale}$$
Ingredient (ii): $L^2$ apriori estimates

Proposition (Villemonais 2014, CDGR 2017)

For any $\varphi \in \mathcal{D}$, we have

$$
E\left[ \left( \gamma^N_T(\varphi) - \gamma_T(\varphi) \right)^2 \right] \leq \frac{6 \| \varphi \|^2_{\infty}}{N}.
$$

Proof.

$$
\gamma^N_T(\varphi) - \gamma_T(\varphi) = \frac{1}{\sqrt{N}} \int_0^T p^N_t \, dM_t + \frac{1}{\sqrt{N}} \int_0^T p^N_t \, dM_t \\
+ \gamma^N_0(Q^T \varphi) - \gamma_0(Q^T \varphi)
$$

(i) Initial condition is OK by independence.
(ii) $\mathcal{M}$-terms. Using Ito's isometry and $d[\mathcal{M}, \mathcal{M}]_t \leq 4\|\varphi\|_\infty^2 d\mathcal{N}_t$, we obtain

$$
E \left[ \left( \int_0^T p_t^N d\mathcal{M}_t \right)^2 \right] = E \left[ \int_0^T (p_t^N)^2 d[\mathcal{M}, \mathcal{M}]_t \right]
$$

\[
\leq 4\|\varphi\|_\infty^2 \frac{1}{N} \sum_{j=1}^{\infty} (1 - \frac{1}{N})^{2(j-1)} \leq 4\|\varphi\|_\infty^2.
\]

(iii) $\mathcal{M}$-terms. In the same way, applying Ito's isometry and the 'key formula' $p_t^N d(\mathcal{M}, \mathcal{M})_t = d\gamma_t^N(Q^2) + \text{Martingale}$, we get

$$
E \left[ \left( \int_0^T p_t^N d\mathcal{M}_t \right)^2 \right] = E \left[ \int_0^T (p_t^N)^2 d[\mathcal{M}, \mathcal{M}]_t \right]
$$

\[
\leq E \left[ \int_0^T p_t^N d(\mathcal{M}, \mathcal{M})_t \right] = E \left[ \gamma_t^N(Q^2) \right] \leq \|\varphi\|_\infty^2.
\]
Lemma

One has

$$\sup_{t \in [0, T]} \left| p^N_t - p_t \right| \overset{\mathbb{P}}{\longrightarrow} 0, \quad N \to \infty.$$ 

Proof. Independent of the context. $t \mapsto p_t$ is continuous on $[0, T]$ by construction, it is clear that $t \mapsto p^N_t$ is decreasing for all $N \geq 2$. The Lemma results of last Proposition and a from a probabilistic version of Second Dini (or Pólya) theorem: if a sequence of monotone functions converges pointwise on a compact interval and if the limit function is also continuous, then the convergence is uniform on that interval.
In order to use martingale CLT, we need a càdlàg increasing process \( i_N \) such that \((\gamma_N(Q_u))^2 - i_N(t)\) is a martingale (quadratic variation - like).

After tedious computations and trials and errors we chose a quadratic variation with only the branching jumps integrated.

\[
\begin{align*}
i_N(t) &= \int_0^t (p_N(u) - \cdot)^2 d(M(u), M(u) - Var \eta_N u) - \int_0^t Var \eta_N u - (Q_p N) - dp N u + \frac{1}{N} \int_0^t (p_N(u) - \cdot)^2 dR(u).
\end{align*}
\]

NB: with the rest term is \( O(1/N) \) with \( R(t) = N \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \left(1 - \frac{1}{N}\right)^2 Var \eta(n, \tau - n, k) - (Q) \right) \left( \frac{1}{t} \geq \tau(n, k) \right) \).
Increasing Process in general CLT

In order to use martingale CLT, we need a càdlàg increasing process $i^N_t$ such that $(\gamma^N(Q)_t)^2 - i^N_t$ is a martingale (quadratic variation - like). 

$$i^N_t = \int_0^t (p^N_u - \dot{\eta}^N_u)^2 d(M_u, M_u) - \int_0^t \text{Var} \eta^N_u - (Q_{p^N_u} - \text{Var} \eta^N_{\tau - n, k}(Q))\left(1 - \frac{1}{N}\right)^2 \text{Var} \eta(n, \tau - n, k)(Q) \cdot 1_{t \geq \tau - n, k}. $$

NB: with the rest term is $O(1/N)$ with 

$$R_t = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\left(1 - \frac{1}{N}\right)^2 \text{Var} \eta(n, \tau - n, k)(Q)\right) \cdot 1_{t \geq \tau - n, k}. $$
Increasing Process in general CLT

- In order to use martingale CLT, we need a càdlàg increasing process \( i_t^N \) such that \( (\gamma^N(Q)_t)^2 - i_t^N \) is a martingale (quadratic variation - like).
- After tedious computations and trials and errors we chose a quadratic variation with only the branching jumps integrated.

\[
\begin{align*}
  i_t^N &= \int_0^t \left( p_{u-}^N \right)^2 d(M_t, M_u) - \int_0^t \text{Var}_{\eta_{u-}^N}(Q) p_{u-}^N dp_u^N \\
  &+ \frac{1}{N} \int_0^t \left( p_{u-}^N \right)^2 d\mathcal{R}_u.
\end{align*}
\]
Increasing Process in general CLT

- In order to use martingale CLT, we need a càdlàg increasing process $i_t^N$ such that $(\gamma^N(Q)_t)^2 - i_t^N$ is a martingale (quadratic variation - like).

- After tedious computations and trials and errors we chose a quadratic variation with only the branching jumps integrated.

\[
i_t^N = \int_0^t (p_{u-}^N)^2 d(M, M)_u - \int_0^t \text{Var}_{\eta_{u-}^N}(Q) p_{u-}^N dp_{u-}^N \\
+ \frac{1}{N} \int_0^t (p_{u-}^N)^2 dR_u.
\]

- NB: with the rest term is $O(1/N)$ with

\[
R_t = \sum_{n=1}^N \sum_{k=1}^{+\infty} \left( (1 - \frac{1}{N})^2 \text{Var}_{\eta_{\tau_{n,k}}^N} (Q) - \text{Var}_{\eta_{\tau_{n,k}}^N} (Q) \right) 1_{t \geq \tau_{n,k}}.
\]
Integration by parts formulas

Let \( t \mapsto z^N_t \) be any càdlàg semi-martingale, \( c > 0 \) a deterministic constant, and assume for any branching time \( \tau_j, j \geq 1 \):

\[
\int_0^t p_N s - dz^N_s = p_N t z^N_t - z^N_0 - \int_0^t z^N_s dp_N s + O\left(\frac{1}{N}\right).
\]
Integration by parts formulas

Let $t \mapsto z_N^t$ be any càdlàg semi-martingale, $c > 0$ a deterministic constant, and assume for any branching time $\tau_j$, $j \geq 1$:

- If $|\Delta z_{\tau_j}^N| \leq c/N$, then
  \[
  \int_0^t p_s^N \, dz_s^N = p_t^N z_t^N - z_0^N - \int_0^t z_s^N \, dp_s^N + O(1/N).
  \]
Integration by parts formulas

Let $t \mapsto z^N_t$ be any càdlàg semi-martingale, $c > 0$ a deterministic constant, and assume for any branching time $\tau_j$, $j \geq 1$:

- If $|\Delta z^N_{\tau_j}| \leq c/N$, then
  \[
  \int_0^t p^N_{s-} \, dz^N_s = p^N_t z^N_t - z^N_0 - \int_0^t z^N_{s-} \, dp^N_s + O(1/N).
  \]

- If $|z^N_{\tau_j}| \leq c(1 - 1/N)^j$,
  \[
  \int_0^t z^N_{s-} \left( p^N_{s-} \right)^{-1} \, dp^N_s = \int_0^t z^N_{s-} \, d \ln p^N_s + O(1/N).
  \]
Integration by parts formulas

Let $t \mapsto z_t^N$ be any càdlàg semi-martingale, $c > 0$ a deterministic constant, and assume for any branching time $\tau_j$, $j \geq 1$:

- If $|\Delta z^N_{\tau_j}| \leq c/N$, then
  $$\int_0^t p_s^N d z_s^N = p_t^N z_t^N - z_0^N - \int_0^t z_s^N d p_s^N + O(1/N).$$

- If $|z^N_{\tau_j^-}| \leq c(1 - 1/N)^j$,
  $$\int_0^t z_s^N (p_s^N)^{-1} d p_s^N = \int_0^t z_s^N d \ln p_s^N + O(1/N).$$

- If $|\Delta z^N_{\tau_j}| \leq c(1 - 1/N)^j / N$,
  $$\int_0^t z_s^N d \ln p_s^N = z_t^N \ln p_t^N - \int_0^t \ln p_s^N d z_s^N + O(1/N).$$
Integration by parts for $i_t^N$

Using the integration by parts formula above and the key formula:

**Lemma**

The increasing process $i_t^N$ can be integrated by parts and be rewritten as

$$i_t^N = p_t^N \gamma_t^N (Q^2) - \gamma_0^N (Q^2) + \left[ \gamma_t^N (Q) \right]^2 \ln p_t^N$$

$$- 2 \int_0^t \gamma_u^N (Q^2) dp_u^N + O\left( \frac{1}{\sqrt{N}} \right).$$
Convergence of $i_t^N$ and final proof of the CLT

By the non synchronous jump Assumption all the vanishing jump assumptions in the martingale CLT are verified.

The only remaining part to prove is the following:

**Proposition**

For any $t \in [0, T]$, one has

$$i_t^N \overset{\mathbb{P}}{\underset{N \to \infty}{\longrightarrow}} i_t(\varphi).$$

where

$$i_t^N = p_t^N \gamma_t^N(Q^2) - \gamma_0^N(Q^2) + \left[\gamma_t^N(Q)\right]^2 \ln p_t^N - 2 \int_0^t \gamma_u^N(Q^2) dp_u^N + O\left(\frac{1}{\sqrt{N}}\right),$$

$$i_t(\varphi) = p_t \gamma_t(Q^2) - \gamma_0(Q^2) + \left[\gamma_t(Q)\right]^2 \ln p_t - 2 \int_0^t \gamma_u(Q^2) dp_u.$$
Proof of Convergence of $i_t^N$

After some calculations, all but one term can be treated using the $L^2$ a priori convergence estimate. The only remaining problem is the following:

$$i_t^N - i_t(\varphi) = \text{easy converging terms with } L^2\text{-estimate and IPP} +$$

$$-2 \int_0^t (p_{u^-}^N - p_u) d\gamma_u^N(Q^2)$$

It is difficult to prove its convergence to 0 because the $L^2$ estimate is only pointwise. Hence handling the integrator is cumbersome!!.
Convergence of $i_t^N$

This difficult term is then treated as follows. Use the 'key formula'

$$\int_0^t (p_{u_-}^N - p_u) d\gamma_u^N(Q^2) = \int_0^t (p_{u_-}^N - p_u)p_{u_-}^N d(\mathbb{M}, \mathbb{M})_u + O\left(\frac{1}{\sqrt{N}}\right)$$

Since $(\mathbb{M}, \mathbb{M})$ is an increasing process, it comes

$$\left| \int_0^t (p_{u_-}^N - p_u)p_{u_-}^N d(\mathbb{M}, \mathbb{M})_u \right| \leq \sup_u |p_{u_-}^N - p_u| \times \left( \int_0^t p_{u_-}^N d(\mathbb{M}, \mathbb{M})_u \right).$$
Convergence of $i^N_t$

This difficult term is then treated as follows. Use the 'key formula'

$$
\int_0^t (p^N_{u^-} - p_u) d\gamma^N_u(Q^2) = \int_0^t (p^N_{u^-} - p_u)p^N_{u^-} d(M, M)_u + O\left(\frac{1}{\sqrt{N}}\right)
$$

Since $(M, M)$ is an increasing process, it comes

$$
\left| \int_0^t (p^N_{u^-} - p_u)p^N_{u^-} d(M, M)_u \right| \leq \sup_u |p^N_{u^-} - p_u| \times \left( \int_0^t p^N_{u^-} d(M, M)_u \right).
$$

- The 'key formula' back again implies:

$$
\mathbb{E} \left[ \int_0^t p^N_{u^-} d(M, M)_u \right] = \mathbb{E} \left[ \gamma^N_t(Q^2) \right] \leq \|\varphi\|_\infty^2.
$$
Convergence of $i_t^N$

This difficult term is then treated as follows. Use the 'key formula'

$$\int_0^t (p_{u_-}^N - p_u) \, d\gamma_u^N(Q^2) = \int_0^t (p_{u_-}^N - p_u) p_{u_-}^N \, d(M, \tilde{M})_u + O\left(\frac{1}{\sqrt{N}}\right)$$

Since $(M, \tilde{M})$ is an increasing process, it comes

$$\left| \int_0^t (p_{u_-}^N - p_u) p_{u_-}^N \, d(M, \tilde{M})_u \right| \leq \sup_u |p_{u_-}^N - p_u| \times \left( \int_0^t p_{u_-}^N \, d(M, \tilde{M})_u \right).$$

- The 'key formula' back again implies:

$$\mathbb{E} \left[ \int_0^t p_{u_-}^N \, d(M, \tilde{M})_u \right] = \mathbb{E} \left[ \gamma_t^N(Q^2) \right] \leq \|\varphi\|_\infty^2.$$  

- The a priori uniform estimate implies convergence of

$$\sup_u |p_{u_-}^N - p_u|.$$  

END OF PROOF
Lemma

The AMS algorithm mapped with the level-indexed process

\[ X_t := Y_{S_t}, \quad S_t = \inf(s \geq 0 | \xi(Y_s) = t) \]

and stopped at iteration \( J_t \) (the first iteration when all particles reach \( \{\xi > t\}\) ) is actually a Fleming-Viot particle system.

Proof Picture: level function \( \xi \) define a time change.
Adaptive Multilevel Splitting
Adaptive Multilevel Splitting
Adaptive Multilevel Splitting
Adaptive Multilevel Splitting
Adaptive Multilevel Splitting
Adaptive Multilevel Splitting
Adaptive Multilevel Splitting

Bernard Delyon, Frédéric Cérou, Arnaud Guyader, Mathias Rousset. *Asymptotic normality of the AMS algorithm*. 2018