Central Limit Theorem for stationary Fleming-Viot particle systems in finite spaces

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Joint work with Tony Lelièvre (CERMICS/Inria) and Loucas Pillaud-Vivien (Inria)
Quasistationary distribution in finite spaces

- Let $E$ be a finite space.
- Let $(x_t)_{t \geq 0}$ be the continuous time Markov chain with infinitesimal generator
  \[
  Lf(x) = \sum_{y \in E} p(x, y)[f(y) - f(x)], \quad p(x, y) \geq 0, \quad \sum_{y \in E} p(x, y) = 1.
  \]

- Let $D \subset E$ be nonempty, and let $\tau_D = \inf\{t \geq 0 : x_t \notin D\}$.

Quasistationary distribution (QSD)

A probability measure $\pi$ on $D$ is called quasistationary in $D$ if

\[
\forall t \geq 0, \quad \mathbb{P}_\pi(x_t \in \cdot | t < \tau_D) = \pi(\cdot).
\]

Perron–Frobenius Theorem

If the substochastic matrix\(^1\) $P_D = \{p(x, y), x, y \in D\}$ is irreducible, then:

- there is a unique QSD $\pi$;
- the spectral radius $1 - \lambda \in (0, 1]$ of $P_D$ is a single eigenvalue;
- $P_D^*\pi = (1 - \lambda)\pi$.

\(^1\) $P_D$ is seen as an operator on the functions $D \to \mathbb{R}$, and $P_D^*$ as an operator on the measures on $D$. 
Yaglom limit

In the sequel we always assume that $P_D$ is irreducible.

Yaglom limit

**Darroch, Seneta – J. Appl. Probab. ’67**: for any initial distribution $\mu$ on $D$,

$$\lim_{t \to +\infty} \mathbb{P}_\mu(x_t \in \cdot | t < \tau_D) = \pi(\cdot).$$

- The QSD is particularly relevant in the study of **metastability**, where convergence to the **Yaglom limit** occurs on a **shorter time scale** than exit from $D$.
- From **Kramers – Physica ’40** to many works by people at CIRM this week!

Nontrivial computational issue: **how to sample from the QSD**?

- **Rejection** Monte-Carlo fails in almost surely finite time.
- $t \mapsto \mathbb{P}_\mu(x_t \in \cdot | t < \tau_D)$ obeys a **nonlinear** evolution.
Take \( n \) copies (particles) of the process \((x_t)_{t \geq 0}\) started iid according to \( \mu \) on \( D \).

When one attempts to exit from \( D \), pick its next position uniformly among the positions of the \( n-1 \) remaining particles.

We get a well-defined \( D^n \)-valued exchangeable continuous time Markov chain \((x^1_t, \ldots, x^n_t)_{t \geq 0}\), with empirical measure

\[
\eta^n_t(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{x^i_t = x\}}, \quad x \in D,
\]

random process in the set \( \mathcal{P}(D) \) of probability measures on \( D \).

The chain is irreducible: \( \eta^n_t \to \eta^n_\infty \) in distribution (exponential/uniform rates in Cloez, Thai – Stoch. Proc. Appl. ’16).

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Sampling from the QSD: the Fleming–Virot particle system

- Take $n$ copies (particles) of the process $(x_t)_{t \geq 0}$ started iid according to $\mu$ on $D$.
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We get a well-defined $D^n$-valued **exchangeable** continuous time Markov chain $(x^n_1, \ldots, x^n_n)_{t \geq 0}$, with empirical measure

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LLN and CLT

What about the $n \to +\infty$ limit?

- For all $t \geq 0$, $\eta^n_t \to \mathbb{P}_\mu(x_t \in \cdot | t < \tau_D)$.
- For the stationary distribution: $\eta^n_\infty \to \pi$.

**Central Limit Theorem** by Cérou, Delyon, Guyader, Rousset – arXiv ’16, ’17, see M. Rousset’s talk on Friday.
- For all $t \geq 0$, $\sqrt{n}(\eta^n_t - \mathbb{P}_\mu(x_t \in \cdot | t < \tau_D))$ converges in distribution to $\mathcal{N}(0, K^\mu_t)$.
- If the system starts from the QSD, the covariance operator writes
  $$\langle K^\pi_t f, f \rangle = \text{Var}_\pi(f) + 2\lambda \int_{s=0}^{t} e^{2\lambda s} \text{Var}_\pi(Q_s f) ds, \quad \langle \pi, f \rangle = 0,$$
  where $Q_s f(x) = \mathbb{E}_x[f(x_{s})1_{\{s < \tau_D\}}]$.
- Extension to infinite time horizon not so straightforward.

Our result: **stationary Central Limit Theorem**.
- $\sqrt{n}(\eta^n_\infty - \pi)$ converges in distribution to $\mathcal{N}(0, K)$.
- The covariance operator writes
  $$\langle K f, f \rangle = \text{Var}_\pi(f) + 2\lambda \int_{s=0}^{\infty} e^{2\lambda s} \text{Var}_\pi(Q_s f) ds, \quad \langle \pi, f \rangle = 0.$$

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Asymptotics of the Fleming–Viot particle system

FV particle system

$\eta^n_t$
Asymptotics of the Fleming–Viot particle system

FV particle system
\[ \eta^n_t \]

\( t \to +\infty \)
irreducibility

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Conditional law
\[ \mathbb{P}_\mu(x_t \in \cdot | t < \tau_D) \]
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Yaglom limit

\[ \pi \]
QSD

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Asselah, Ferrari, Groisman

Stationary distribution
\[ \eta^n_{\infty} \]

Yaglom limit
\[ t \to +\infty \]

QSD
\[ \pi \]
Asymptotics of the Fleming–Viot particle system

- **FV particle system** \( \eta_t^n \)
- **Conditional law** \( \mathbb{P}_\mu(x_t \in \cdot | t < \tau_D) \)
- **Central Limit Theorem** by Cérou, Delyon, Guyader, Rousset

- \( n \to +\infty \)
- \( t \to +\infty \) irreducibility
- \( n \to +\infty \) Yaglom limit
- \( n \to +\infty \) stationary distribution

- \( \eta_{\infty}^n \)
- \( \pi \)
- QSD

Asselah, Ferrari, Groisman
Asymptotics of the Fleming–Viot particle system

- **FV particle system** $\eta^n_t$
  - $t \to +\infty$
  - Irreducibility
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- **Conditional law** $\mathbb{P}_\mu(x_t \in \cdot | t < \tau_D)$
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  - Yaglom limit

- **Stationary distribution** $\eta^n_{\infty}$
  - $n \to +\infty$

- **Central Limit Theorem** by Cérou, Delyon, Guyader, Rousset

- **Stationary distribution** $\pi$
  - $n \to +\infty$
  - QSD

- We prove a stationary Central Limit Theorem

**CLT for stationary Fleming-Viot**

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$t \to +\infty$

irreducibility

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Stationary distribution

$n \to +\infty$

$\eta^n_\infty$

$\pi$

QSD

We prove a stationary Central Limit Theorem

$t \to +\infty$

Yaglom limit

Asymptotic variance is consistent in the $t \to +\infty$ limit

Julien Reygner  CLT for stationary Fleming-Viot
Sketch of the proof

1. We aim to prove that $\sqrt{n}(\eta^n_\infty - \pi) \to \mathcal{N}(0, K)$,
2. in the space $\mathcal{M}_0(D)$ of signed measures $\xi$ on $D$ such that $\sum_{x \in D} \xi(x) = 0$.
3. We write the infinitesimal generator $M^n$ of the process $\sqrt{n}(\eta^n_t - \pi)$ in $\mathcal{M}_0(D)$.
4. We compute the limit $\overline{M}$ of $M^n$.
5. Any limit of $\sqrt{n}(\eta^n_\infty - \pi)$ is thus a stationary distribution for $\overline{M}$.
6. We identify $\overline{M}$ as the infinitesimal generator of a linear diffusion process $(\overline{\xi}_t)_{t \geq 0}$ on $\mathcal{M}_0(D)$, whose unique stationary distribution is a Gaussian measure $\mathcal{N}(0, K)$.
7. We solve a Lyapunov equation to compute $K$ explicitly.
8. We prove that the sequence $\sqrt{n}(\eta^n_\infty - \pi)$ is tight.
Step 1 of the proof: infinitesimal generators

The law of $\eta^n_\infty$ is the stationary distribution of the $\mathcal{P}(D)$-valued process $(\eta^n_t)_{t \geq 0}$ with infinitesimal generator

$$L^n \phi(\eta) = \sum_{x,y \in D} n\eta(x) \left( p(x,y) + q(x) \frac{n\eta(y)}{n-1} \right) \left[ \phi \left( \eta + \frac{1}{n} \theta^x,y \right) - \phi(\eta) \right],$$

where $q(x) = \sum_{y \in E \setminus D} p(x,y)$ and $\theta^x,y = 1_y - 1_x$.

The law of $\xi^n_\infty := \sqrt{n}(\eta^n_\infty - \pi)$ is the stationary distribution of the $\mathcal{M}_0(D)$-valued process $(\xi^n_t)_{t \geq 0} := (\sqrt{n}(\eta^n_t - \pi))_{t \geq 0}$ with infinitesimal generator

$$M^n \psi(\xi) = \sum_{x,y \in D} n \left( \pi(x) + \frac{\xi(x)}{\sqrt{n}} \right) \left( p(x,y) + q(x) \frac{n}{n-1} \left( \pi(y) + \frac{\xi(y)}{\sqrt{n}} \right) \right) \times \left[ \psi \left( \xi + \frac{1}{\sqrt{n}} \theta^x,y \right) - \psi(\xi) \right].$$
Step 2 of the proof: taking the limit

Take \( \psi \in C^\infty_c(M_0(D)) \). When \( n \to +\infty \),

\[
M^n \psi(\xi) \to \overline{M}\psi(\xi) = \langle ((P_\pi^D)^* - (1 - \lambda)I)\xi, \nabla \psi \rangle + A_\pi^D \cdot \nabla^2 \psi(x), \quad \text{uniformly in } \xi,
\]

with the following notation:

- \( P_\pi^D \) is the stochastic matrix with coefficients
  \[
p_\pi^D(x, y) = p(x, y) + q(x)\pi(y), \quad x, y \in D,
\]
  which defines the \( \pi \)-return process.

- The \( \pi \)-return process describes the pathwise limit of the (stationary) Fleming–Viot particle system, it is irreducible and ergodic with respect to \( \pi \).

- \( A_\pi^D \) is the symmetric operator defined by
  \[
  \langle A_\pi^D f, f \rangle = \frac{1}{2} \sum_{x, y \in D} \pi(x) p_\pi^D(x, y) [f(y) - f(x)]^2.
  \]
  (energy / Dirichlet form)

Consequence: assume tightness, so that \( \xi_n \to \overline{\xi}_{\infty} \) (up to a subsequence). Then

\[
0 = \mathbb{E} [M^n \psi(\xi_{n\infty})] = \mathbb{E}[(M^n - \overline{M})\psi(\xi_{n\infty})] + \mathbb{E} [\overline{M}\psi(\xi_{n\infty})] \to \mathbb{E}[\overline{M}\psi(\overline{\xi}_{\infty})],
\]

\( \to 0 \) uniformly

therefore the law of \( \overline{\xi}_{\infty} \) is a stationary distribution for \( \overline{M} \).
Step 3 of the proof: identification of the limit

The operator
\[
\overline{M} \psi(\xi) = \langle ((P^\pi_D)^* - (1 - \lambda)I)\xi, \nabla \psi \rangle + A^\pi_D : \nabla^2 \psi(x)
\]
is the infinitesimal generator of the **linear diffusion process** \((\overline{\xi}_t)_{t \geq 0}\) on \(\mathcal{M}_0(D)\)
\[
d\overline{\xi}_t = B_0 \overline{\xi}_t dt + \Sigma dw_t,
\]
with \((w_t)_{t \geq 0}\) BM in \(\mathbb{R}^k\), \(\Sigma : \mathbb{R}^k \to \mathcal{M}_0(D)\) and \(B_0 : \mathcal{M}_0(D) \to \mathcal{M}_0(D)\) defined by
\[
\frac{1}{2} \Sigma \Sigma^* := A^\pi_D \quad \text{and} \quad B_0 = (P^\pi_D)^* - (1 - \lambda)I.
\]

- By irreducibility of the \(\pi\)-return process, \(A^\pi_D\) is **positive definite** on \(\mathcal{M}_0(D)\).
- There exists \(\gamma > 0\) such that any eigenvalue \(\tau \in \mathbb{C}\) of \(B_0\) satisfies \(\Re \tau \leq -\gamma\).

**Consequence**

The **unique** stationary distribution of \(\overline{M}\) is the **centered Gaussian measure** on \(\mathcal{M}_0(D)\) with covariance operator \(K\) defined as the unique solution to the **Lyapunov equation**
\[
B_0 K + K B_0^* + 2A^\pi_D = 0.
\]

- Uniqueness follows from uniform ellipticity.
- Linearity of evolution preserves Gaussian measures.
- Lyapunov equation is Ito’s formula for the covariance of \((\overline{\xi}_t)_{t \geq 0}\).
- Existence of a solution for Lyapunov equation follows from spectral stability of the drift.
Step 4 of the proof: computation of the covariance

- The solution $K$ to Lyapunov equation $B_0 K + KB_0^* + 2A_D^\pi = 0$ is known to write

$$K = 2 \int_{s=0}^{+\infty} e^{sB_0} A_D^\pi e^{sB_0^*} ds.$$ 

- Take $f : D \to \mathbb{R}$ such that $\langle \pi, f \rangle = 0$, and compute

$$\langle Kf, f \rangle = 2 \int_{s=0}^{+\infty} \langle A_D^\pi e^{sB_0^*} f, e^{sB_0^*} f \rangle ds.$$ 

- Since $B_0 = (P_D^\pi)^* - (1 - \lambda)I$, $e^{sB_0^*} f = e^{sL_D^\pi} f = e^{sL_D} f$, where:
  - $L_D^\pi = P_D^\pi - I$ is the infinitesimal generator of the $\pi$-return process,
  - $P_{s,D}^\pi = e^{sL_D^\pi}$ is the semigroup of the $\pi$-return process.

As a consequence,

$$\langle Kf, f \rangle = 2 \int_{s=0}^{+\infty} e^{2\lambda s} \left( A_D^\pi P_{s,D}^\pi f, P_{s,D}^\pi f \right) ds - \frac{1}{2} \frac{d}{ds} \text{Var}_\pi(P_{s,D}^\pi f)$$

$$= \text{Var}_\pi(f) + 2\lambda \int_{s=0}^{+\infty} e^{2\lambda s} \text{Var}_\pi(P_{s,D}^\pi f) ds.$$ 

- $\langle \pi, f \rangle = 0$ then ensures that $P_{s,D}^\pi f(x) = Q_s f(x) = \mathbb{E}_x [f(x_s) 1_{\{s < \tau_D\}}].$
Step 5 of the proof: tightness of $\xi^n_{\infty}$

- Recall that we denote by $L^n$ the infinitesimal generator of the empirical distribution $(\eta^n_t)_{t \geq 0}$ of the FV particle system, so that $E[L^n \phi(\eta^n_{\infty})] = 0$.
- Take $\phi(\eta) = \frac{1}{2} \langle \eta - \pi, R(\eta - \pi) \rangle$, where $R$ is a symmetric operator.
- Little algebra yields the inequality

$$E \left[ \langle -B'[\eta^n_{\infty}]\xi^n_{\infty}, R\xi^n_{\infty} \rangle \right] \leq C(R), \quad \xi^n_{\infty} = \sqrt{n}(\eta^n_{\infty} - \pi),$$

where $B'[\eta] := B_0 + \langle \eta - \pi, q \rangle$, $q(x) = \sum_{y \in E \setminus D} p(x, y)$.

  - Quadratic part consistent with the limit $M^n \psi \to \bar{M} \psi$,
  - cubic nonlinearity originates from the fact that $\phi$ is not bounded.

- If there were no cubic nonlinearity, then:
  - take $R = N^{-1}$ where $N$ solves the Lyapunov equation $B_0 N + N B^*_0 + 2 I = 0$,
  - this yields $\langle -B_0 \xi, R\xi \rangle = -\frac{1}{2} \langle (N^{-1} B_0 + B^*_0 N^{-1}) \xi, \xi \rangle = \|N^{-1} \xi\|^2$,
  - from which you deduce the variance control $cN^{-1} E[\|\xi^n_{\infty}\|^2] \leq C(N^{-1})$.

- LLN on $\eta^n_{\infty}$ by Asselah, Ferrari, Groisman: with large probability,
  - $\langle \eta^n_{\infty} - \pi, q \rangle$ is small,
  - the Lyapunov equation with $B'[\eta^n_{\infty}] := B_0 + \text{perturbation}$ remains solvable.

- We get tightness but no variance control.
Some questions remain: variance control

In our finite space setting, can we obtain uniform variance control of the form

\[ \mathbb{E} \left[ \| \eta^n - \pi \|^2 \right] \leq \frac{C}{n} \]

Such an estimate is known (at least) for:

- **diffusions** with **soft killing** ([Rousset – SIAM J. Math. ’06]),
- **discrete space** Markov chains with **strong mixing condition** ([Cloez, Thai – Stoch. Proc. Appl. ’16]).

A possible approach: uniform and quantitative control of correlations.

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CLT for stationary Fleming-Viot
Some questions remain: extension to general state spaces

Can we extend our CLT to more general Markov processes? (ideally, the same level of generality as Cérou, Delyon, Guyader, Rousset – arXiv ’16, ’17)

- An easy conjecture: any limit of $\xi^n_\infty \in M_0(D)$ is a stationary distribution of the ‘measure-valued’ linear diffusion

$$d\xi_t = ((L_D^\pi)^* + \lambda)\xi_t dt + dm_t,$$

where:

- $L_D^\pi$ is the infinitesimal generator of the $\pi$-return process,
- $(m_t)_{t \geq 0}$ is a ‘measure-valued’ martingale with quadratic variation given by the Dirichlet form $A_D^\pi$ of the $\pi$-return process.

- Do spectral properties of the $\pi$-return process still hold? What about tightness?

- How to make sense of all this?