Central Limit Theorem for stationary Fleming-Viot particle systems in finite spaces

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Joint work with Tony Lelièvre (CERMICS/Inria) and Loucas Pillaud-Vivien (Inria)

Quasistationary distribution in finite spaces

- Let E be a **finite** space.
- Let $(\mathrm{x}_t)_{t\geq 0}$ be the **continuous time** Markov chain with infinitesimal generator

$$Lf(x) = \sum_{y\in\mathsf{E}} p(x,y)[f(y)-f(x)], \qquad p(x,y)\geq 0, \quad \sum_{y\in\mathsf{E}} p(x,y) = 1.$$

- Let $\mathsf{D}\subset\mathsf{E}$ be nonempty, and let $\tau_{\mathsf{D}} = \inf\{t\geq 0 : \mathrm{x}_t \notin \mathsf{D}\}$.

Quasistationary distribution (QSD)

A probability measure π on D is called **quasistationary** in D if

$$\forall t\geq 0, \qquad \mathbb{P}_\pi(\mathrm{x}_t\in\cdot|t<\tau_{\mathsf{D}}) = \pi(\cdot).$$

Perron–Frobenius Theorem

If the substochastic matrix[1] $P_{\mathsf{D}} = \{p(x,y), x,y\in\mathsf{D}\}$ is **irreducible**, then:

- **there is a unique QSD** π;
- the spectral radius $1-\lambda\in(0,1]$ of P_{D} is a single eigenvalue;
- $P_{\mathsf{D}}^*\pi = (1-\lambda)\pi$.

[1] P_{D} is seen as an operator on the functions $\mathsf{D}\to\mathbb{R}$, and P_{D}^* as an operator on the measures on D.

Yaglom limit

In the sequel we always assume that P_{D} is irreducible.

Yaglom limit

Darroch, Seneta – J. Appl. Probab. '67: for any initial distribution μ on D,

$$\lim_{t \to +\infty} \mathbb{P}_{\mu}(\mathrm{x}_t \in \cdot | t < \tau_{\mathrm{D}}) = \pi(\cdot).$$

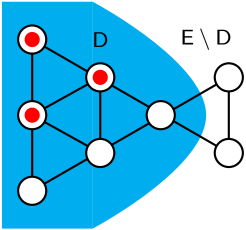
- The QSD is particularly relevant in the study of **metastability**, where convergence to the **Yaglom limit** occurs on a **shorter time scale** than **exit from** D.
- From **Kramers – Physica '40** to many works by people at CIRM this week!

Nontrivial computational issue: **how to sample from the QSD**?

- **Rejection** Monte-Carlo fails in almost surely finite time.
- $t \mapsto \mathbb{P}_{\mu}(\mathrm{x}_t \in \cdot | t < \tau_{\mathrm{D}})$ obeys a **nonlinear** evolution.
- Occupation measure-based algorithm proposed by **Aldous, Flannery, Palacios – Probab. Engrg. Inform. Sci. '88**, see **M. Benaïm**'s talk tomorrow.
- Particle system-based algorithm: **Fleming–Viot particle system** (**Burdzy, Hołyst, March – Comm. Math. Phys. '00**).

Sampling from the QSD: the Fleming–Viot particle system

- Take n copies (**particles**) of the process $(\mathrm{x}_t)_{t\geq 0}$ started iid according to μ on D.
- When one attempts to exit from D, pick its next position uniformly among the positions of the $n-1$ remaining particles.



We get a well-defined D^n-valued **exchangeable** continuous time Markov chain $(\mathrm{x}_t^1, \ldots, \mathrm{x}_t^n)_{t\geq 0}$, with empirical measure

$$\eta_t^n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{\mathrm{x}_t^i = x\}}, \qquad x \in \mathsf{D},$$

random process in the set $\mathcal{P}(\mathsf{D})$ of probability measures on D.

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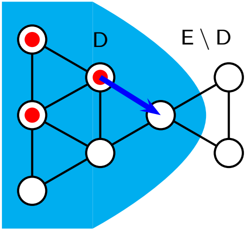

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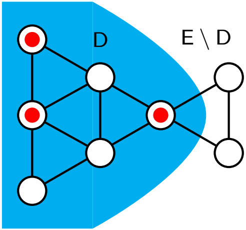

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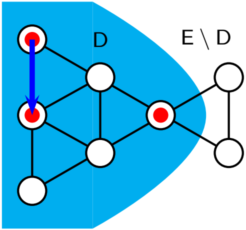

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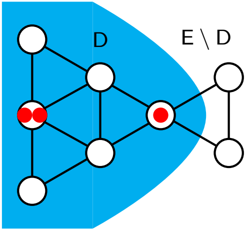

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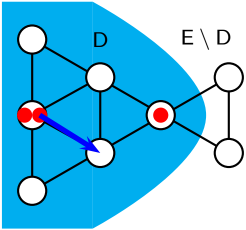

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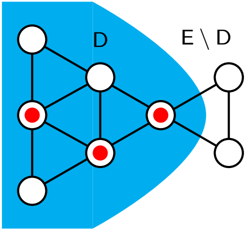

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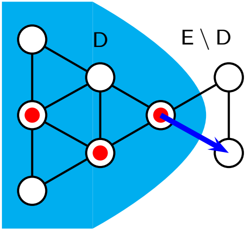

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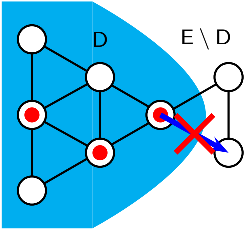

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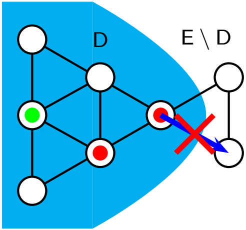

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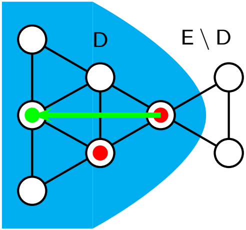

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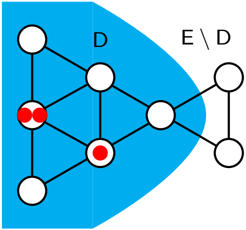

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LLN and CLT

What about the $n \to +\infty$ limit?

Laws of Large Numbers by **Asselah, Ferrari, Groisman – J. Appl. Probab. '11**.

- For all $t \geq 0$, $\eta_t^n \to \mathbb{P}_\mu(\mathrm{x}_t \in \cdot | t < \tau_\mathsf{D})$.
- For the stationary distribution: $\eta_\infty^n \to \pi$.

Central Limit Theorem by **Cérou, Delyon, Guyader, Rousset – arXiv '16, '17**, see **M. Rousset**'s talk on Friday.

- For all $t \geq 0$, $\sqrt{n}(\eta_t^n - \mathbb{P}_\mu(\mathrm{x}_t \in \cdot | t < \tau_\mathsf{D}))$ converges in distribution to $\mathcal{N}(0, K_t^\mu)$.
- If the system starts from the QSD, the covariance operator writes

$$\langle K_t^\pi f, f \rangle = \mathrm{Var}_\pi(f) + 2\lambda \int_{s=0}^{t} \mathrm{e}^{2\lambda s} \mathrm{Var}_\pi(Q_s f) \mathrm{d}s, \qquad \langle \pi, f \rangle = 0,$$

  where $Q_s f(x) = \mathbb{E}_x[f(\mathrm{x}_s) \mathbb{1}_{\{s < \tau_\mathsf{D}\}}]$.
- Extension to infinite time horizon not so straightforward.

Our result: stationary Central Limit Theorem.

- $\sqrt{n}(\eta_\infty^n - \pi)$ converges in distribution to $\mathcal{N}(0, K)$.
- The covariance operator writes

$$\langle K f, f \rangle = \mathrm{Var}_\pi(f) + 2\lambda \int_{s=0}^{\infty} \mathrm{e}^{2\lambda s} \mathrm{Var}_\pi(Q_s f) \mathrm{d}s, \qquad \langle \pi, f \rangle = 0.$$

Asymptotics of the Fleming–Viot particle system

FV particle system

η_t^n

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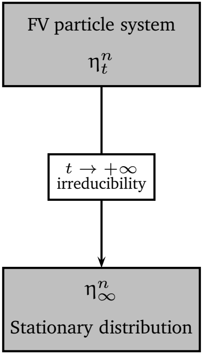

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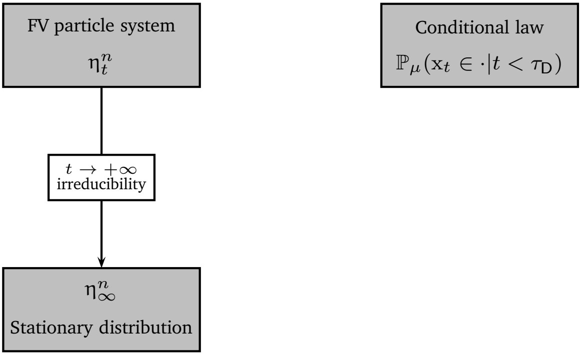

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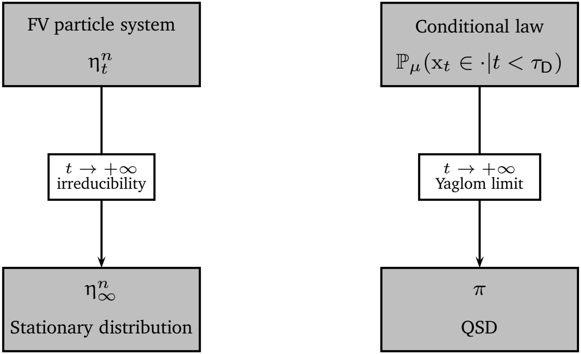

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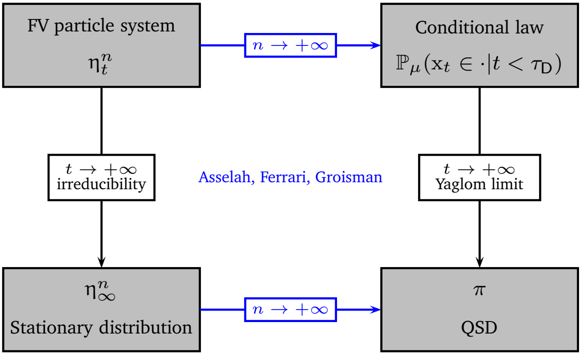

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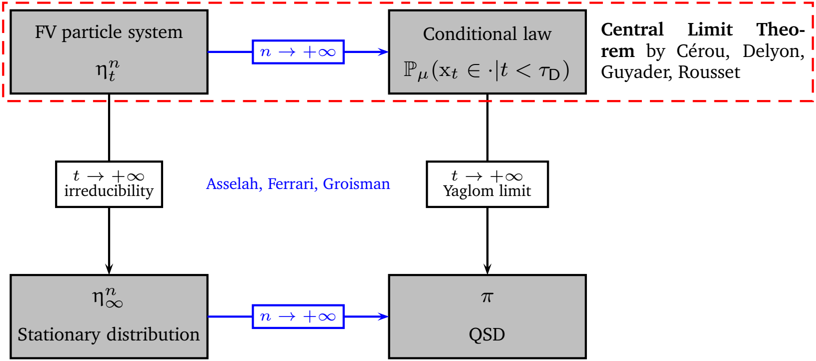

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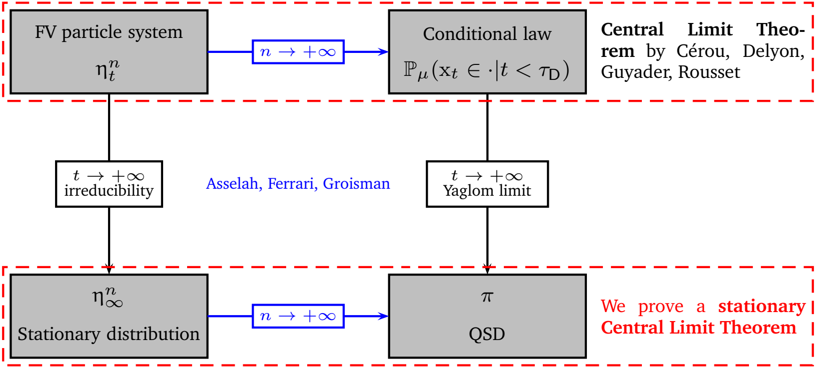

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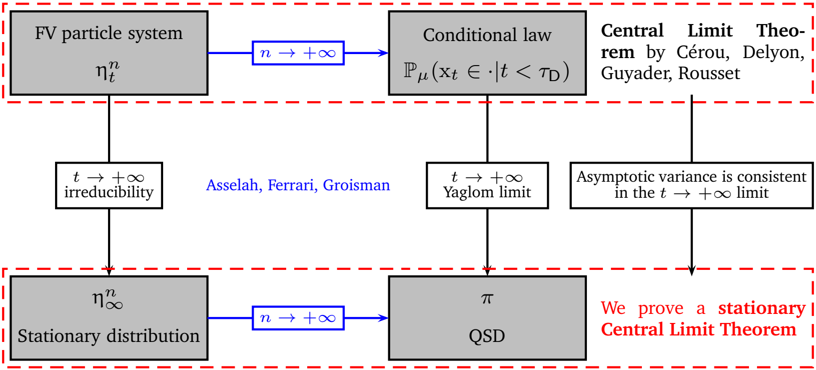

Sketch of the proof

- We aim to prove that $\sqrt{n}(\eta^n_\infty - \pi) \to \mathcal{N}(0, K)$,
- in the space $\mathcal{M}_0(\mathsf{D})$ of signed measures ξ on D such that $\sum_{x \in \mathsf{D}} \xi(x) = 0$.

1. We write the infinitesimal generator $\mathbf{M}^n$ of the process $\sqrt{n}(\eta^n_t - \pi)$ in $\mathcal{M}_0(\mathsf{D})$.
2. We compute the limit $\overline{\mathbf{M}}$ of $\mathbf{M}^n$.
   $\rightsquigarrow$ Any limit of $\sqrt{n}(\eta^n_\infty - \pi)$ is thus a stationary distribution for $\overline{\mathbf{M}}$.
3. We identify $\overline{\mathbf{M}}$ as the infinitesimal generator of a linear diffusion process $(\overline{\xi}_t)_{t \geq 0}$ on $\mathcal{M}_0(\mathsf{D})$, whose unique stationary distribution is a Gaussian measure $\mathcal{N}(0, \overline{K})$.
4. We solve a Lyapunov equation to compute K explicitly.
5. We prove that the sequence $\sqrt{n}(\eta^n_\infty - \pi)$ is tight.

Step 1 of the proof: infinitesimal generators

- The law of η_∞^n is the stationary distribution of the $\mathcal{P}(\mathsf{D})$-valued process $(\eta_t^n)_{t\geq 0}$ with infinitesimal generator

$$\mathbf{L}^n \phi(\eta) = \sum_{x,y\in\mathsf{D}} n\eta(x)\left(p(x,y) + q(x)\frac{n\eta(y)}{n-1}\right)\left[\phi\left(\eta + \frac{1}{n}\theta^{x,y}\right) - \phi(\eta)\right],$$

where $q(x) = \sum_{y\in\mathsf{E}\setminus\mathsf{D}} p(x,y)$ and $\theta^{x,y} = \mathbb{1}_y - \mathbb{1}_x$.

- The law of $\xi_\infty^n := \sqrt{n}(\eta_\infty^n - \pi)$ is the stationary distribution of the $\mathcal{M}_0(\mathsf{D})$-valued process $(\xi_t^n)_{t\geq 0} := (\sqrt{n}(\eta_t^n - \pi))_{t\geq 0}$ with infinitesimal generator

$$\mathbf{M}^n \psi(\xi) = \sum_{x,y\in\mathsf{D}} n\left(\pi(x) + \frac{\xi(x)}{\sqrt{n}}\right)\left(p(x,y) + q(x)\frac{n}{n-1}\left(\pi(y) + \frac{\xi(y)}{\sqrt{n}}\right)\right)$$
$$\times\left[\psi\left(\xi + \frac{1}{\sqrt{n}}\theta^{x,y}\right) - \psi(\xi)\right].$$

Step 2 of the proof: taking the limit

Take $\psi \in C_c^\infty(\mathcal{M}_0(\mathsf{D}))$. When $n \to +\infty$,

$$\mathbf{M}^n\psi(\xi) \to \overline{\mathbf{M}}\psi(\xi) = \langle ((P_{\mathsf{D}}^{\pi})^* - (1-\lambda)I)\xi, \nabla\psi\rangle + A_{\mathsf{D}}^{\pi} :: \nabla^2\psi(x), \quad \text{uniformly in } \xi,$$

with the following notation:

- P_{D}^{π} is the stochastic matrix with coefficients

$$p_{\mathsf{D}}^{\pi}(x,y) = p(x,y) + q(x)\pi(y), \qquad x,y \in \mathsf{D},$$

  which defines the **π-return process**.
- The π-return process describes the **pathwise limit** of the (stationary) Fleming–Viot particle system, it is irreducible and ergodic with respect to π.
- A_{D}^{π} is the symmetric operator defined by

$$\langle A_{\mathsf{D}}^{\pi} f, f\rangle = \frac{1}{2}\sum_{x,y\in\mathsf{D}} \pi(x)p_{\mathsf{D}}^{\pi}(x,y)[f(y)-f(x)]^2. \qquad \text{(energy / Dirichlet form)}$$

Consequence: assume **tightness**, so that $\xi_\infty^n \to \overline{\xi}_\infty$ (up to a subsequence). Then

$$0 = \mathbb{E}\left[\mathbf{M}^n\psi(\xi_\infty^n)\right] = \mathbb{E}[\underbrace{(\mathbf{M}^n - \overline{\mathbf{M}})\psi(\xi_\infty^n)]}_{\to 0 \quad \text{uniformly}} + \mathbb{E}\left[\overline{\mathbf{M}}\psi(\xi_\infty^n)\right] \to \mathbb{E}[\overline{\mathbf{M}}\psi(\overline{\xi}_\infty)],$$

therefore **the law of $\overline{\xi}_\infty$ is a stationary distribution for $\overline{\mathbf{M}}$**.

Step 3 of the proof: identification of the limit

The operator

$$\overline{\mathbf{M}}\psi(\xi) = \langle ((P^\pi_{\mathsf{D}})^* - (1-\lambda)I)\xi, \nabla\psi\rangle + A^\pi_{\mathsf{D}} :: \nabla^2\psi(x)$$

is the infinitesimal generator of the **linear diffusion process** $(\overline{\xi}_t)_{t\geq 0}$ **on** $\mathcal{M}_0(\mathsf{D})$

$$\mathrm{d}\overline{\xi}_t = B_0\overline{\xi}_t\mathrm{d}t + \Sigma\mathrm{d}\mathrm{w}_t,$$

with $(\mathrm{w}_t)_{t\geq 0}$ BM in $\mathbb{R}^k$, $\Sigma : \mathbb{R}^k \to \mathcal{M}_0(\mathsf{D})$ and $B_0 : \mathcal{M}_0(\mathsf{D}) \to \mathcal{M}_0(\mathsf{D})$ defined by

$$\frac{1}{2}\Sigma\Sigma^* := A^\pi_{\mathsf{D}} \qquad \text{and} \qquad B_0 = (P^\pi_{\mathsf{D}})^* - (1-\lambda)I.$$

- By irreducibility of the π-return process, A^π_{D} is **positive definite** on $\mathcal{M}_0(\mathsf{D})$.
- There exists $\gamma > 0$ such that any eigenvalue $\tau \in \mathbb{C}$ of B_0 satisfies $\mathrm{Re}\,\tau \leq -\gamma$.

Consequence

The **unique** stationary distribution of $\overline{\mathbf{M}}$ is the **centered Gaussian measure** on $\mathcal{M}_0(\mathsf{D})$ with covariance operator K defined as the unique solution to the **Lyapunov equation**

$$B_0K + KB_0^* + 2A^\pi_{\mathsf{D}} = 0.$$

- Uniqueness follows from uniform ellipticity.
- Linearity of evolution preserves Gaussian measures.
- Lyapunov equation is Ito's formula for the covariance of $(\overline{\xi}_t)_{t\geq 0}$.
- Existence of a solution for Lyapunov equation follows from spectral stability of the drift.

Step 4 of the proof: computation of the covariance

- The solution K to Lyapunov equation $B_0 K + K B_0^* + 2A_{\mathsf{D}}^\pi = 0$ is known to write
$$K = 2\int_{s=0}^{+\infty} \mathrm{e}^{sB_0} A_{\mathsf{D}}^\pi \mathrm{e}^{sB_0^*}\mathrm{d}s.$$
- Take $f : \mathsf{D} \to \mathbb{R}$ such that $\langle \pi, f\rangle = 0$, and compute
$$\langle Kf, f\rangle = 2\int_{s=0}^{+\infty} \langle A_{\mathsf{D}}^\pi \mathrm{e}^{sB_0^*} f, \mathrm{e}^{sB_0^*} f\rangle \mathrm{d}s.$$
- Since $B_0 = (P_{\mathsf{D}}^\pi)^* - (1-\lambda)I$, $\mathrm{e}^{sB_0^*} f = \mathrm{e}^{\lambda s}\mathrm{e}^{sL_{\mathsf{D}}^\pi} f = \mathrm{e}^{\lambda s} P_{s,\mathsf{D}}^\pi f$, where:
  - $L_{\mathsf{D}}^\pi = P_{\mathsf{D}}^\pi - I$ is the infinitesimal generator of the π-return process,
  - $P_{s,\mathsf{D}}^\pi = \mathrm{e}^{sL_{\mathsf{D}}^\pi}$ is the semigroup of the π-return process.
- As a consequence,
$$\langle Kf, f\rangle = 2\int_{s=0}^{+\infty} \mathrm{e}^{2\lambda s} \underbrace{\langle A_{\mathsf{D}}^\pi P_{s,\mathsf{D}}^\pi f, P_{s,\mathsf{D}}^\pi f\rangle}_{-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Var}_\pi(P_{s,\mathsf{D}}^\pi f)} \mathrm{d}s$$
$$= \mathrm{Var}_\pi(f) + 2\lambda \int_{s=0}^{+\infty} \mathrm{e}^{2\lambda s}\mathrm{Var}_\pi(P_{s,\mathsf{D}}^\pi f)\mathrm{d}s.$$
- $\langle \pi, f\rangle = 0$ then ensures that $P_{s,\mathsf{D}}^\pi f(x) = Q_s f(x) = \mathbb{E}_x[f(\mathrm{x}_s)\mathbb{1}_{\{s<\tau_{\mathsf{D}}\}}]$.

Step 5 of the proof: tightness of ξ^n_∞

- Recall that we denote by $\mathbf{L}^n$ the infinitesimal generator of the empirical distribution $(\eta^n_t)_{t\geq 0}$ of the FV particle system, so that $\mathbb{E}[\mathbf{L}^n\phi(\eta^n_\infty)] = 0$.
- Take $\phi(\eta) = \frac{1}{2}\langle \eta - \pi, R(\eta - \pi)\rangle$, where R is a symmetric operator.
- Little algebra yields the inequality

$$\mathbb{E}\left[\langle -B'[\eta^n_\infty]\xi^n_\infty, R\xi^n_\infty\rangle\right] \leq C(R), \qquad \xi^n_\infty = \sqrt{n}(\eta^n_\infty - \pi),$$

  where $B'[\eta] := B_0 + \langle \eta - \pi, q\rangle$, $q(x) = \sum_{y \in \mathsf{E}\setminus\mathsf{D}} p(x,y)$.
  - Quadratic part consistent with the limit $\mathbf{M}^n\psi \to \overline{\mathbf{M}}\psi$,
  - cubic nonlinearity originates from the fact that ϕ is not bounded.
- If there were no cubic nonlinearity, then:
  - take $R = N^{-1}$ where N solves the Lyapunov equation $B_0 N + N B_0^* + 2I = 0$,
  - this yields $\langle -B_0\xi, R\xi\rangle = -\frac{1}{2}\langle (N^{-1}B_0 + B_0^* N^{-1})\xi, \xi\rangle = \|N^{-1}\xi\|^2$,
  - from which you deduce the variance control $c_{N^{-1}}\mathbb{E}[\|\xi^n_\infty\|^2] \leq C(N^{-1})$.
- **LLN on η^n_∞** by Asselah, Ferrari, Groisman: **with large probability**,
  - $\langle \eta^n_\infty - \pi, q\rangle$ is small,
  - the Lyapunov equation with $B'[\eta^n_\infty] := B_0 +$ perturbation remains solvable.
- We get **tightness** but **no variance control**.

Some questions remain: variance control

In our **finite space** setting, can we obtain **uniform variance control** of the form

$$\mathbb{E}\left[\|\eta_\infty^n - \pi\|^2\right] \leq \frac{C}{n} \qquad ?$$

Such an estimate is known (at least) for:

- **diffusions** with **soft killing** (**Rousset – SIAM J. Math. '06**),
- **discrete space** Markov chains with **strong mixing condition** (**Cloez, Thai – Stoch. Proc. Appl. '16**).

A possible approach: uniform and quantitative control of correlations.

Some questions remain: extension to general state spaces

Can we **extend our CLT** to **more general Markov processes**?
(ideally, the same level of generality as **Cérou, Delyon, Guyader, Rousset – arXiv '16, '17**)

- An easy conjecture: any limit of $\xi^n_\infty \in \mathcal{M}_0(\mathsf{D})$ is a stationary distribution of the 'measure-valued' linear diffusion

$$\mathrm{d}\overline{\xi}_t = ((L^\pi_{\mathsf{D}})^* + \lambda)\overline{\xi}_t \mathrm{d}t + \mathrm{dm}_t,$$

  where:
  - L^π_{D} is the infinitesimal generator of the π-return process,
  - $(\mathrm{m}_t)_{t \geq 0}$ is a 'measure-valued' martingale with quadratic variation given by the Dirichlet form A^π_{D} of the π-return process.
- Do spectral properties of the π-return process still hold? What about tightness?
- How to make sense of all this?