Beyond the Hörmander Condition

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CIRM, September 2018

Overview

- Motivation
  - Why studying UFG processes?
  - Examples of SDEs which are UFG but do not satisfy the Hörmander condition
- General Setting
- The Hörmander condition
  - PDE Theory and probability
  - Geometric - Control Theoretical point of view
- UFG condition
  - "Obtuse angle condition"
  - non-autonomous (hypoelliptic) processes "$\subseteq$" UFG diffusions
  - Some results on the long-time behaviours of UFG diffusions.

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- Non-ergodic theory???
- Multi-agent systems

$$\partial_t f_t + v\partial_x f_t + \partial_v \left[(G(M_f(t,x)) - v) f_t \right] - \sigma \partial_w f_t = 0$$

$f_t = f_t(x,v)$ particle density,

$$M(t,x) = \frac{\int_{\mathbb{T}} dy \int_{\mathbb{R}} dw\, f_t(y,w)\varphi(x-y)w}{\int_{\mathbb{T}} dy \int_{\mathbb{R}} dw\, f_t(y,w)\varphi(x-y)}$$

[Butta, Flandoli, Ottobre, Zegarlinski, arxiv, 2018]

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General Setting

- (Smooth) Vector field in $\mathbb{R}^N$, i.e. map $V : \mathbb{R}^N \to \mathbb{R}^N$, with

$$V(x) = (V^1(x), \ldots, V^j(x), \ldots, V^N(x)) \quad x \in \mathbb{R}^N$$

- Identified with first order differential operator

$$V(x) = \sum_j V^j(x)\partial_j$$

- For a collection of fields

$$V_0(x), V_1(x), \ldots, V_d(x)$$

- Example: in $\mathbb{R}^3$ consider

$$V := \partial_x - \frac{y}{2}\partial_z \quad \leftarrow \quad \text{differential operator}$$

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$$dX_t = V_0(X_t)dt + V_1(X_t) \circ dW_t, \quad X_0 = x$$

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- Take $\mathbb{E}f(X_t | X_0 = x) =: u(t, x)$. Then

$$\begin{aligned} \partial_t u(t,x) &= \mathcal{L}u(t,x), \\ u(0,x) &= f(x) \end{aligned}$$

- Operator $\mathcal{L}$ in *Hörmander's "Sum of squares" form*

$$\mathcal{L} = V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2$$

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I will use the word *distribution* in geometric sense i.e. as a map

$$M \ni x \xrightarrow{\mathcal{D}} \text{vector space} \subseteq T_x M$$

This can be produced by assigning a set of vector fields on M

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Hörmander and Parabolic Hörmander Condition

- Let $V_0, V_1, \dots, V_d$ be $d+1$ vector fields on $\mathbb{R}^N$
- Consider the following Lie Algebras

$$\Delta_0(x) = \mathrm{span}Lie\{V_0(x), V_1(x), \dots, V_d(x)\}$$
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- If $\Delta_0(x) = \mathbb{R}^N$ for every $x \in \mathbb{R}^N$ then $\mathcal{L}$ is hypoelliptic (on $\mathbb{R}^N$) – analytic viewpoint
- If $\Delta(x) = \mathbb{R}^N$ for every $x \in \mathbb{R}^N$ then $\partial_t - \mathcal{L}$ is hypoelliptic (on $\mathbb{R}_+ \times \mathbb{R}^N$) and the process X_t has a density – probabilistic perspective

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The HC in PDE Theory

- Under the PHC one can make sense of the PDE

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as $u(t,x)$ is smooth in both arguments (even when f is just continuous and bounded)

- In particular $u(t,x)$ is differentiable in the direction ∂_t, V_0, in all the directions $\{V_i\}_{i=1}^d$ and in the directions belonging to the Lie algebra

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The Control-Theoretical viewpoint

- In control theory the Hörmander condition is known as *Chow's Condition*:

  If $\boxed{span(Lie\{V_0(x), V_1(x), \ldots, V_d(x)\}) = \mathbb{R}^N \quad \text{for all } x \in \mathbb{R}^N}$

  then any two points of $\mathbb{R}^N$ are connected by integral curves of the fields $V_0(x), V_1(x), \ldots, V_d(x)$ (*reachability*)

Excursus

- The HC implies reachability, it *does not* imply that the corresponding stochastic dynamics goes everywhere (controllable)! E. g.:

$$dX_t = -sin(X_t)dt + \cos(X_t) \circ dW_t, \quad X_t \in \mathbb{R}$$

- Solution of apparent contradiction

$$dX_t = -sin(X_t)dt + \cos(X_t)u(t)dt$$

versus

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The Lie algebra

$$\Delta(x) = Lie\{V_1(x), \dots, V_d(x), [V_0, V_1], \dots, [V_0, V_d]\} \quad \text{is } \textit{finitely generated}$$

That is,

$$\left.\begin{array}{ll}
\text{Level 1} & \mathfrak{V}_0 = \{V_1, \dots, V_d\} \\
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\text{Level 3} & \mathfrak{V}_2 = \{[V_i, V], \quad 0 \leq i \leq d, V \in \mathfrak{V}_1\} \\
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\text{Level m} & \dots
\end{array}\right\} \mathcal{A}_m$$

- No assumption on the rank of the Lie algebra: the rank does not even need to be constant (and indeed, it is in general not constant)
- A bit more formally

$$V_{[\alpha]}(x) = \sum_{\beta \in \mathcal{A}_m} \varphi_\alpha^\beta(x) V_{[\beta]}(x) \, .$$

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- No assumption on the rank of the Lie algebra! the rank does not even need to be constant (and indeed, it is in general not constant)
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UFG condition

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PDE

- $\mathbb{E} f(X_t | X_0 = x) =: u(t,x)$

$$\partial_t u(t,x) = V_0 u(t,x) + \frac{1}{2} \sum_{i=1}^{d} V_i^2 u(t,x)$$

- Rewrite the above as

$$(\partial_t - V_0) u(t,x) = \frac{1}{2} \sum_{i=1}^{d} V_i^2 u(t,x)$$

  Suppose we can prove differentiability in the direction $\mathcal{V} := \partial_t - V_0$. Then we can still make sense of the above (Strook, Kusuoka, Crisan, Delarue)

- Extreme example of UFG condition: 1D transport equation

$$\begin{cases} \partial_t u(t,x) = \partial_x u(t,x) \\ u(0,x) = f(x) \end{cases} \implies u(t,x) = f(x+t)$$

  Solution is smooth in direction $\partial_t - \partial_x$ as $(\partial_t - \partial_x) u(t,x) = 0$.

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Geometric Brownian motion *does not* satisfy the HC!

$$dX_t = X_t dt + 2X_t dW_t, \quad \Rightarrow X_t = X_0\, e^{-t+2W_t} \quad X_t \in \mathbb{R}$$

with generator

$$\mathcal{L} = x\partial_x + 2x^2\partial_x^2$$

- Let $V_1 = x\partial_x$. Then

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Stochastic Geodesic Equation

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\begin{aligned}
dz_t &= V_0(z_t)dt + V_1(z_t) \circ dW_t, \quad z = (u, v) \in \mathbb{R}^6 \\
V_0 &= (v, -|v|^2 u), \quad V_1 = (0, u \times v) .
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$$z_t \in TS^2 := \{(u, v) : |u| = 1, v \perp u\}$$

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Some results on UFG diffusions

Known facts from theory of finitely generated distributions

- Under the UFG condition the space $\mathbb{R}^N$ is partitioned into manifolds $\mathcal{M}$.
- Each manifold $\mathcal{M}$ is the integral manifold of the distribution

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+ Strook and Varadhan control theorem

- If the process X_t starts on one manifold $\mathcal{M}$ it will stay on $\overline{\mathcal{M}}$ for every positive t:

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Exponential Decay of derivatives

- exponential decay of derivatives (+ some control theory) $\Rightarrow$ uniqueness of invariant measure (on the manifold)
- Obtuse Angle condition

$$[V_\alpha, V_0]f(x) \cdot V_\alpha f(x) \leq -\lambda |V_\alpha f(x)|^2$$

- Obtuse angle condition implies

$$|V_\alpha P_t f(x)| \leq ce^{-\lambda t}$$

- Suppose you know $(P_t f)(x) \to \nu$

$$(P_t f)(y) = (P_t f)(y) - (P_t f)(x) + (P_t f)(x) - \nu$$
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- Under the UFG condition there is a change of coordinates such that *locally* one can always express the SDE

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ dW_t^i, \quad X_0 = x$$

in the form “ODE + SDE”: $\tilde{X}_t = (Z_t, \zeta_t)$

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Non-autonomous hypoelliptic systems

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- Either $\zeta_t \to \pm\infty$ or $\zeta_t \to \bar{\zeta}$
- Let $\bar{\zeta}$ be a stationary point of U and suppose $\zeta_t \to \bar{\zeta}$
- Reasonable guess: $X_t = (Z_t, \zeta_t) \to (\bar{Z}_t, \bar{\zeta})$ where

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- Notice that in this case $V_0^{(\perp)}(x) = (0, \ldots, 0, U(z))$ hence

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Example 1

UFG- Heisenberg diffusion

$$
\begin{aligned}
dY_t &= -Y_t dt + \sqrt{2} dW_t^2 \\
dZ_t &= -2Z_t dt - \sqrt{2} Y_t \circ dW_t^1 + \sqrt{2} \zeta_t \circ dW_t^2 \\
d\zeta_t &= -\zeta_t dt
\end{aligned}
$$

Example 2 (less cheating)

- Random circles

$$dX_t = -Y_t dt + \sqrt{2} X_t \circ dB_t$$
$$dY_t = X_t dt + \sqrt{2} Y_t \circ dB_t.$$

$$V_0 = (-y, x), \quad V_1 = (x, y)$$

- After the change of coordinates one obtains

$$d\zeta_t = dt$$
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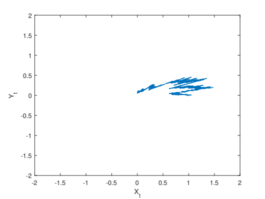
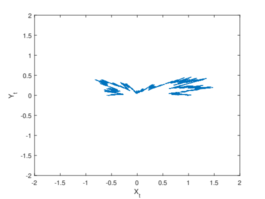
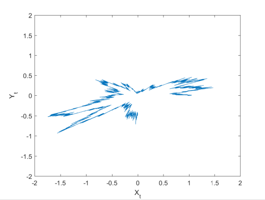
\end{aligned}
$$



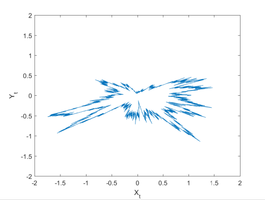
(a) $0 \leq t \leq \pi/2$.



(b) $0 \leq t \leq \pi$.



(c) $0 \leq t \leq \frac{3\pi}{2}$



(d) $0 \leq t \leq 2\pi$

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