Sampling invariant distributions of SPDEs

Charles-Edouard Bréhier

CNRS & Université Lyon 1, Institut Camille Jordan

Advances in Computational Statistical Physics, September 2018
Setting

Consider **Stochastic Partial Differential Equations**

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + f(u) + \xi, \quad \text{in } (0, \infty) \times \mathcal{D} \\
u(t, \cdot) &= 0, \quad \text{in } \partial \mathcal{D} \\
u(0, \cdot) &= u_0.
\end{aligned}
\]

\(\xi\) is (Gaussian) space-time white noise.

Domain: \(\mathcal{D} = (0, 1)^d\).

Example: \(f(\phi) = \phi - \phi^3\) (\(\Phi^4\) models).
Setting

Consider Stochastic Partial Differential Equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + f(u) + \xi, \quad \text{in } (0, \infty) \times \mathcal{D} \\
u(t, \cdot) &= 0, \quad \text{in } \partial \mathcal{D} \\
u(0, \cdot) &= u_0.
\end{aligned}
\]

\(\xi\) is (Gaussian) space-time white noise.

Domain: \(\mathcal{D} = (0, 1)^d\).

Example: \(f(\phi) = \phi - \phi^3\) (\(\Phi^4\) models).

In this talk:

- \(d = 1\): well-posedness in a “classical” sense
- \(f\) is globally Lipschitz continuous.
Abstract setting

Consider $u(t) = u(t, \cdot)$ as a process with values in $H = L^2(0, 1)$. It satisfies a Stochastic Evolution Equation

$$du(t) = Au(t)dt + F(u(t))dt + dW(t),$$

with solutions interpreted in the mild sense

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} F(u(s)) ds + \int_0^t e^{(t-s)A} dW(s).$$
Abstract setting

Consider $u(t) = u(t, \cdot)$ as a process with values in $H = L^2(0, 1)$. It satisfies a Stochastic Evolution Equation

$$du(t) = Au(t)dt + F(u(t))dt + dW(t), \quad (1)$$

with solutions interpreted in the mild sense

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s))ds + \int_0^t e^{(t-s)A}dW(s).$$

If $\text{Lip}(F)$ is sufficiently small, this Markov process admits a unique invariant distribution $\mu$, and $u(t) \xrightarrow{t \to \infty} \mu$ in distribution.

Estimation of $\int \varphi d\mu$?
Cost in terms of temporal and spatial discretization errors?
Assumptions

- **Linear operator:** \( Ae_n = -\lambda_n e_n \), for \( n \in \mathbb{N} \), with \( \lambda_n = (\pi n)^2 \), \( e_n(z) = \sqrt{2} \sin(n \pi z) \).
  It generates an analytic semi-group, \( e^{tA} = \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \langle e_n, \cdot \rangle e_n \).

- **Nonlinear operator:** \( F \) is Lipschitz continuous and bounded.
  For ergodicity: assume \( \text{Lip}(F) < \lambda_1 \).

- **Noise:** \( W(t) = \sum_{n \in \mathbb{N}} \beta_n(t) e_n \) is a cylindrical Wiener process.
Assumptions

- Linear operator: $Ae_n = -\lambda_n e_n$, for $n \in \mathbb{N}$, with $\lambda_n = (\pi n)^2$, $e_n(z) = \sqrt{2} \sin(n \pi z)$.
  It generates an analytic semi-group, $e^{tA} = \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \langle e_n, \cdot \rangle e_n$.
- Nonlinear operator: $F$ is Lipschitz continuous and bounded. For ergodicity: assume $\text{Lip}(F) < \lambda_1$.
- Noise: $W(t) = \sum_{n \in \mathbb{N}} \beta_n(t) e_n$ is a cylindrical Wiener process.

Regularity of trajectories: $\frac{1}{4} - \epsilon$ Hölder in time, $\frac{1}{2} - \epsilon$ Hölder in space. In terms of Sobolev type spaces: $\mathbb{E}|(-A)^{\alpha} u(t)|^2 < \infty$ if and only if $\alpha < \frac{1}{4}$. 
The invariant distribution

**Ergodicity:** synchronous coupling

\[
\frac{1}{2} \frac{d|u(t, u_0^{(2)}) - u(t, u_0^{(1)})|^2}{dt} \leq (-\lambda_1 + \text{Lip}(F))|u(t, u_0^{(2)}) - u(t, u_0^{(1)})|^2.
\]
The invariant distribution

Ergodicity: synchronous coupling

\[
\frac{1}{2} \frac{d|u(t, u_0^{(2)}) - u(t, u_0^{(1)})|^2}{dt} \leq (\lambda_1 + \text{Lip}(F))|u(t, u_0^{(2)}) - u(t, u_0^{(1)})|^2.
\]

The gradient case: \( F = -DV \). Then

\[
\mu(du) = \frac{1}{Z} \exp(-2V(u))\nu(du)
\]

where \( \nu \) is a Gaussian distribution on \( H \):

\[
\nu = \mathcal{N}(0, \frac{1}{2}(-A)^{-1}).
\]

\( \nu \) is the distribution of the Brownian Bridge on \( (0, 1) \).

Sampling conditioned diffusions, cf works by Hairer, Stuart & Voss.
Numerical discretization

In time: linear implicit Euler scheme.

\[ u_{n+1} = u_n + \Delta t Au_{n+1} + \Delta t F(u_n) + \Delta W_n \]
\[ = S_{\Delta t} u_n + \Delta t S_{\Delta t} F(u_n) + S_{\Delta t} \Delta W_n, \]

with \( S_{\Delta t} = (I - \Delta t A)^{-1} \).

In space: spectral Galerkin method (\( M \) modes).

\[ du^{(M)}(t) = Au^{(M)}(t)dt + \Delta t P_M F(u^{(M)}(t)) + P_M \Delta W_n. \]

For simulations: finite differences (mesh size \( \Delta x \)).

Fully-discrete: ergodic Markov chain \( (u_n^{(M)})_n \), the invariant distribution is denoted by \( \mu^{\Delta t, M} \).
Orders of convergence

Theorem (B. 2014, B.-Kopec 2016)

For all $\alpha < \frac{1}{2}$, $\varphi : H \rightarrow \mathbb{R}$ of class $C^2_b$,

$$| \int \varphi d\mu^{\Delta t, M} - \int \varphi d\mu | \leq C_\alpha(\|\varphi\|_2)(\Delta t^\alpha + \lambda_M^{-\alpha}).$$
Orders of convergence

Theorem (B. 2014, B.-Kopec 2016)

For all $\alpha < \frac{1}{2}$, $\varphi : H \to \mathbb{R}$ of class $C^2_b$,

$$| \int \varphi d\mu^{\Delta t, M} - \int \varphi d\mu | \leq C_\alpha(\| \varphi \|_2)(\Delta t^\alpha + \lambda^{-\alpha}_M).$$

- Strategy: analysis of the weak error (Kolmogorov or Poisson equation approach), cf Talay, Mattingly-Stuart-Tretyakov, etc...
- Extension to SPDEs: specific regularity properties
Orders of convergence

Theorem (B. 2014, B.-Kopec 2016)

For all $\alpha < \frac{1}{2}$, $\varphi : H \to \mathbb{R}$ of class $C^2_b$,

$$\left| \int \varphi d\mu^{\Delta t,M} - \int \varphi d\mu \right| \leq C_\alpha(\|\varphi\|_2)(\Delta t^\alpha + \lambda_M^{-\alpha}).$$

- Strategy: analysis of the weak error (Kolmogorov or Poisson equation approach), cf Talay, Mattingly-Stuart-Tretyakov, etc...
- Extension to SPDEs: specific regularity properties
- If $\varphi$ is only $C^1_b$ or $C^0_b$: weak order is degraded,

$$\left| \int \varphi d\mu^{\Delta t,M} - \int \varphi d\mu \right| \leq C_\alpha(\|\varphi\|_1)(\Delta t^{\frac{\alpha}{2}} + \lambda_M^{-\frac{\alpha}{2}}),$$

$$\limsup_{\Delta t \to 0, M \to \infty} \sup_{\|\varphi\|_0 \leq 1} \left| \int \varphi d\mu^{\Delta t,M} - \int \varphi d\mu \right| \geq 1.$$
Cost of a Monte-Carlo simulation

Natural estimator of $\int \varphi d\mu$ is $\frac{1}{K} \sum_{k=1}^{K} \varphi(u_{N}^{(M),k})$.
To have a mean-square error of size $\epsilon^2$, cost is of size

$$K \frac{T}{\Delta t} M \propto \epsilon^{-2} \frac{1}{\epsilon^{-\frac{1}{\alpha}}} \epsilon^{-\frac{1}{2\alpha}} \propto \epsilon^{-5-}.$$

Cost of a Monte-Carlo simulation

Natural estimator of $\int \varphi \, d\mu$ is $\frac{1}{K} \sum_{k=1}^{K} \varphi(u_{N}^{(M),k})$. To have a mean-square error of size $\epsilon^2$, cost is of size

$$K \frac{T \Delta t}{M} \propto \epsilon^{-2} \frac{|\log(\epsilon)|^{1/\alpha}}{\epsilon^{1/2\alpha}} \epsilon^{-5} \propto \epsilon^{-5-}.$$  

Objective: improve the order of convergence with respect to the time-step size.

Analysis with no spatial discretization.

Two ideas: postprocessing and preconditioning.
First technique: Post-processing

Postprocessed scheme [B.-Vilmart 2016]:

\[
    u_{n+1} = S_{\Delta t} \left( u_n + \Delta t F \left( u_n + \frac{1}{2} S_{\Delta t} \Delta W_n \right) + \Delta W_n \right).
\]

Post-processing: \( \overline{u}_N = u_N + \frac{1}{2} J_{\Delta t} \Delta W_N \),

with linear operators

\[
    S_{\Delta t} = (I - \Delta t A)^{-1}, \quad J_{\Delta t} = (I - \frac{\Delta t}{2} A)^{-1/2}.
\]

New approximation for \( \int \varphi d\mu_\infty \): \( \mathbb{E}[\varphi(\overline{u}_n)] \xrightarrow{n \to \infty} \int \varphi d\overline{\mu}_\infty^{\Delta t} \).
First technique: Post-processing

Postprocessed scheme [B.-Vilmart 2016]:

\[ u_{n+1} = S_{\Delta t} \left( u_n + \Delta t F \left( u_n + \frac{1}{2} S_{\Delta t} \Delta W_n \right) + \Delta W_n \right) \]

Post-processing: \( \overline{u}_N = u_N + \frac{1}{2} J_{\Delta t} \Delta W_N \),

with linear operators

\[ S_{\Delta t} = (I - \Delta t A)^{-1}, \quad J_{\Delta t} = (I - \frac{\Delta t}{2} A)^{-1/2}. \]

New approximation for \( \int \varphi d\mu_\infty \): \( \mathbb{E}[\varphi(\overline{u}_n)] \rightarrow n \rightarrow \infty \) \( \int \varphi d\overline{\mu}_{\Delta t} \).

If \( F = 0 \), then \( \overline{\mu}_{\Delta t} = \mu = \nu \).

Proof: \( \varphi(u) = \langle u, e_p \rangle^2 \),

\[ \int \varphi d\mu_\infty - \int \varphi d\mu_{\Delta t} = \frac{1}{2\lambda_p} \left( 1 - \frac{2}{2 + \lambda_p \Delta t} \right) = \mathbb{E}\left\langle \frac{1}{2} J_{\Delta t} \Delta W_N, e_p \right\rangle^2. \]
Numerical simulations. Conjecture: order $\frac{3}{2}$.

Case $f(u) = 0$

<table>
<thead>
<tr>
<th>relative error</th>
<th>Euler method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>slope $1/2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>new method</th>
<th>trap. meth.</th>
</tr>
</thead>
<tbody>
<tr>
<td>stepsize $h$</td>
<td></td>
</tr>
</tbody>
</table>

Case $f(u) = -u$

<table>
<thead>
<tr>
<th>relative error</th>
<th>Euler method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>slope $1/2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>trap. meth. slope $1$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>new method slope $3/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>stepsize $h$</td>
</tr>
</tbody>
</table>

Case $f(u) = -u - \sin(u)$

<table>
<thead>
<tr>
<th>relative error</th>
<th>Euler method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>slope $1/2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>trap. meth. slope $1/2$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>new method slope $3/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>stepsize $h$</td>
</tr>
</tbody>
</table>

Case $f(u) = -2u - u^3$

<table>
<thead>
<tr>
<th>relative error</th>
<th>Euler method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>slope $1/2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>trap. meth. slope $1$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>new method slope $3/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>stepsize $h$</td>
</tr>
</tbody>
</table>

Figure: Orders of convergence, $\varphi(u) = \exp(-\|u\|^2)$. 
Qualitative behavior

One realization of the standard and of the postprocessed schemes:

Figure: $f(u) = -u - \sin(u), \ T = 1, \ \Delta t = 0.01, \ \Delta x = 0.01.$
Qualitative behavior

One realization of the standard and of the postprocessed schemes:

Figure: \( f(u) = -u - \sin(u), \; T = 1, \; \Delta t = 0.01, \; \Delta x = 0.01. \)

Noise \((I - \frac{\Delta t}{2} A)^{-\frac{1}{2}} \Delta W_n\) is rougher than \((I - \Delta A)^{-1} \Delta W_n\).
Qualitative behavior

One realization of the standard and of the postprocessed schemes:

Figure: $f(u) = -u - \sin(u)$, $T = 1$, $\Delta t = 0.01$, $\Delta x = 0.01$.

Noise $(I - \frac{\Delta t}{2} A)^{-\frac{1}{2}} \Delta W_n$ is rougher than $(I - \Delta A)^{-1} \Delta W_n$.

The postprocessing gives the correct regularity.
Analysis of Postprocessing for SDEs

Stochastic Differential Equation: \( dX_t = f(X_t)dt + dB_t \).

Infinitesimal generator: \( \mathcal{L}\phi(x) = \langle f(x), D\phi(x) \rangle + \frac{1}{2} \Delta \phi(x) \).

Weak Taylor expansion:

\[
\mathbb{E}_x [\varphi(X(h))] = (e^{h\mathcal{L}} \varphi)(x) = \varphi(x) + h\mathcal{L}\varphi(x) + \frac{h^2}{2} \mathcal{L}^2 \varphi(x) + O(h^3)
\]

One-step integrator \( x \mapsto X_1 \):

\[
\mathbb{E}_x [\varphi(X_1)] = \varphi(x) + h\mathcal{L}\varphi(x) + h^2 A_1 \varphi(x) + O(h^3)
\]

Postprocessing: \( x \mapsto \overline{X}_1 \)

\[
\mathbb{E}_x [\varphi(\overline{X}_1)] = \varphi(x) + h\overline{A}_1 \varphi(x) + O(h^2)
\]
Error analysis

- **Integrator** \( X_k = \Phi^h(X_{k-1}) = (\Phi^h)^k(X_0) \); compute \( X_1, \ldots, X_n \).
- **Postprocessing**: \( \overline{X}_n = \overline{\Phi}^h(X_n) \).
- **Convergence to invariant distributions**: \( X_n \xrightarrow{n \to \infty} \mu^h \) and \( \overline{X}_n \xrightarrow{n \to \infty} \overline{\mu}^h \).
Error analysis

- Integrator \( X_k = \Phi^h(X_{k-1}) = (\Phi^h)^k(X_0) \); compute \( X_1, \ldots, X_n \).
- Postprocessing: \( \overline{X}_n = \overline{\Phi}^h(X_n) \).
- Convergence to invariant distributions: \( X_n \xrightarrow{n \to \infty} \mu^h \) and \( \overline{X}_n \xrightarrow{n \to \infty} \overline{\mu}^h \).

Results

- Weak order is equal to 1: \( \int \varphi \, d\mu^h_\infty - \int \varphi \, d\mu_\infty = O(h) \)
- Leading order term [Talay-Tubaro 1990]:
  \[
  \int \varphi \, d\mu^h_\infty - \int \varphi \, d\mu_\infty = C(\varphi)h + O(h^2)
  \]
- With a postprocessing [Vilmart 2015]:
  \[
  \int \varphi \, d\overline{\mu}^h_\infty - \int \varphi \, d\mu_\infty = \overline{C(\varphi)}h + O(h^2)
  \]
Error analysis

Results:

- Weak order 1: $\int \varphi d\mu_\infty - \int \varphi d\mu_\infty = O(h)$
- Leading order: $\int \varphi d\mu_\infty^h - \int \varphi d\mu_\infty = C(\varphi)h + O(h^2)$
- Postprocessing: $\int \varphi d\overline{\mu}_\infty^h - \int \varphi d\mu_\infty = \overline{C(\varphi)}h + O(h^2)$

with the expressions

$$C(\varphi) = -\int A_1 \Psi d\mu_\infty, \quad \overline{C(\varphi)} = -\int (A_1 + [\mathcal{L}, \overline{A}_1]) \Psi d\mu_\infty,$$

where $\Psi$ is solution of the Poisson equation $\mathcal{L}\Psi = \varphi - \int \varphi d\mu_\infty$:

$$\Psi(x) = -\int_0^\infty \mathbb{E}_x [\varphi(X(t)) - \int \varphi d\mu_\infty] dt$$
Order conditions

\[ C(\varphi) = - \int A_1 \psi d\mu_\infty, \quad \overline{C(\varphi)} = - \int (A_1 + [L, \overline{A}_1]) \psi d\mu_\infty, \]

Order conditions

- \( \int \varphi d\mu^h_\infty - \int \varphi d\mu_\infty = O(h^2) \) for all \( \varphi \) iff \( A^*_1 \mu_\infty = 0 \).
- \( \int \varphi d\overline{\mu}^h_\infty - \int \varphi d\mu_\infty = O(h^2) \) for all \( \varphi \) iff \( (A_1 + [L, \overline{A}_1])^* \mu_\infty = 0 \).

These conditions are weaker than requiring order 2, i.e. \( A_1 = \frac{L^2}{2} \).
Examples: gradient SDEs

Gradient SDE: \( dX_t = -\nabla V(X_t)dt + dB_t \): 

\[
X_{n+1} = X_n - h\nabla V(X_n + \frac{1}{2}\sqrt{h}\xi_n) + \sqrt{h}\xi_n , \quad X_n = X_n + \frac{1}{2}\sqrt{h}\xi_n.
\]

Introduced in [Leimkuhler-Matthews 13], in the non-Markovian form 

\[
\overline{X}_{n+1} = \overline{X}_n - h\nabla V(\overline{X}_n) + \frac{\sqrt{h}}{2}(\xi_n + \xi_{n+1}).
\]
Examples: gradient SDEs

Gradient SDE: $dX_t = -\nabla V(X_t)dt + dB_t$:

$$X_{n+1} = X_n - h\nabla V(X_n + \frac{1}{2}\sqrt{h}\xi_n) + \sqrt{h}\xi_n, \quad \overline{X}_n = X_n + \frac{1}{2}\sqrt{h}\xi_n.$$  

Introduced in [Leimkuhler-Matthews 13], in the non-Markovian form

$$\overline{X}_{n+1} = \overline{X}_n - h\nabla V(\overline{X}_n) + \frac{\sqrt{h}}{2}(\xi_n + \xi_{n+1}).$$  

Construction of an IMEX scheme for gradient SPDEs:

$$u_{n+1} = S_{\Delta t}\left(u_n - \Delta t\nabla V\left(u_n + \frac{1}{2}S_{\Delta t}\Delta W_n\right) + S_{\Delta t}\Delta W_n\right),$$

$$\overline{u}_N = u_N + \frac{1}{2}J_{\Delta t}\Delta W_N,$$

Proved results:

- Order 2 for a SDE $dX_t = AX_t dt - \nabla V(X_t) dt + dB_t$.
- Well-posed, ergodic for SPDEs.
Second technique: Preconditioning

Gradient SPDE:

\[ du(t) = Au(t)dt - DV(u(t))dt + dW(t). \]
Second technique: Preconditioning

**Gradient SPDE:**

\[ du(t) = Au(t)dt - DV(u(t))dt + dW(t). \]

**Preconditioned SPDE:**

\[ dv(t) = -v(t)dt + (-A)^{-1}DV(v(t))dt + (-A)^{-\frac{1}{2}}dW(t). \]

**Key observation**

The unique invariant distribution of \((v(t))_{t \geq 0}\) is equal to \(\mu_\infty\).

**Improvements:**

- \(A\) is replaced with the bounded operator \(-I\),
- noise is trace-class, \(\text{Tr}(A) < \infty\).
- Trajectories of \(v\) are almost \(\frac{1}{2}\)-Hölder continuous in time.

\(\hat{\text{C}}\)
Construction of numerical schemes

**Strategy**

SDE integrators (and postprocessing) may be applied to the preconditioned SPDE to sample the invariant distribution $\mu_\infty$. 
Construction of numerical schemes

**Strategy**

SDE integrators (and postprocessing) may be applied to the preconditioned SPDE to sample the invariant distribution $\mu_\infty$.

Example: **preconditioned Crank-Nicolson (pCN)**

\[
\begin{align*}
    v_{n+1} &= \frac{1 - \frac{\Delta t}{2}}{1 + \frac{\Delta t}{2}} v_n + \frac{\Delta t}{1 + \frac{\Delta t}{2}} (-A)^{-1} F(v_n) + \frac{1}{1 + \frac{\Delta t}{2}} (-A)^{-\frac{1}{2}} \Delta W_n.
\end{align*}
\]

*With $F = 0$, this is the only known proposal kernel for MCMC simulation in infinite dimension, cf Stuart & al.*
Three schemes of order 2 – ongoing work

Explicit Euler scheme and postprocessing

\[
\begin{aligned}
  v_{n+1} &= v_n - \Delta t (v_n + \frac{1}{2}(-A)^{-\frac{1}{2}} \Delta W_n) + \Delta t (-A)^{-\frac{1}{2}} F (v_n + \frac{1}{2}(-A)^{-\frac{1}{2}} \Delta W_n) \\
  &\quad + (-A)^{-\frac{1}{2}} \Delta W_n, \\
  \overline{v}_N &= v_N + \frac{1}{2} (-A)^{-\frac{1}{2}} \Delta W_N.
\end{aligned}
\]
Three schemes of order 2 – ongoing work

Linear-Implicit Euler scheme and postprocessing

\[
\begin{align*}
\nu_{n+1} &= \frac{1}{1+\Delta t} \nu_n + \frac{\Delta t}{1+\Delta t} (-A)^{-1} \nu (\nu_n + \frac{1}{2(1+\Delta t)} (-A)^{-\frac{1}{2}} \Delta W_n) \\
&\quad + \frac{1}{1+\Delta t} (-A)^{-\frac{1}{2}} \Delta W_n \\
\bar{\nu}_N &= \nu_N + \frac{1}{2(1+\Delta t)^{\frac{1}{2}}} (-A)^{-\frac{1}{2}} \Delta W_N.
\end{align*}
\]

- Weak error
  - Linear Implicit Euler scheme
    - Standard
    - Postprocessed scheme
    - Order 1
    - Order 2
Three schemes of order 2 – ongoing work

Runge-Kutta method with weak order 2

\[
\begin{aligned}
K_n &= (1 - \Delta t)v_n + \Delta t(-A)^{-1}F(v_n) + (-A)^{-\frac{1}{2}}\Delta W_n, \\
v_{n+1} &= v_n + \frac{\Delta t}{2}(-v_n - K_n + (-A)^{-1}F(v_n) + (-A)^{-1}F(K_n)) + (-A)^{-\frac{1}{2}}\Delta W_n.
\end{aligned}
\]
The three methods of order 2

Data of the experiment: $\Delta x = 0.02$, $F(u) = -u - 5 \cos(u) - 2 \sin(2u)$, $\varphi(u) = \int_0^1 e^{-5u(x)} \, dx$.

Multilevel Monte-Carlo is used to reduce the variance ($10^7$ realizations per level).
Conclusion

For gradient SPDEs, methods of order 2 for the invariant distribution can be built using preconditioning and postprocessing techniques.
Conclusion

For gradient SPDEs, methods of order 2 for the invariant distribution can be built using preconditioning and postprocessing techniques.

Improvement of the Monte-Carlo cost: with methods of order 2,

\[ \epsilon^{-2} \frac{T}{\Delta t} M = \epsilon^{-3 - \frac{1}{2}}. \]
Conclusion

For **gradient SPDEs**, methods of order 2 for the invariant distribution can be built using **preconditioning and postprocessing techniques**.

Improvement of the Monte-Carlo cost: with methods of order 2,

\[ \epsilon^{-2} \frac{T}{\Delta t} M = \epsilon^{-3-rac{1}{2}}. \]

Ongoing work and perspectives:

- theoretical analysis
- non-globally Lipschitz drift \( F \)?
- improve results for non-smooth test functions?

**Thanks for your attention.**