Rare events in complex dynamical systems: the examples of the climate and the solar system dynamics

F. BOUCHET – ENS de Lyon and CNRS
With J. WOUTERS and F. RAGONE (heat waves), and E. WOILLEZ (astronomy)

INRIA – Rennes – June 2018.

Outline

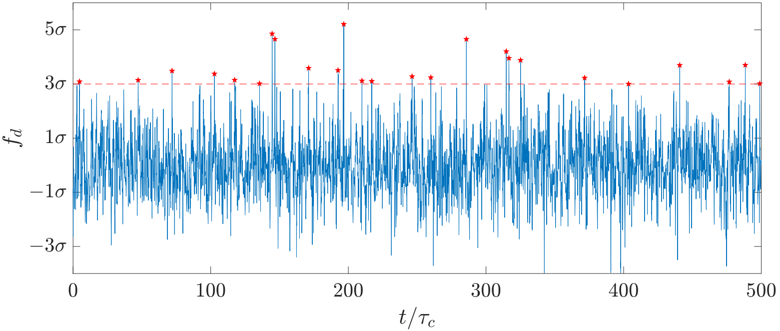
Outline

1. Rare events in complex dynamical systems and instanton phenomenology
   - Rare events with a huge impact: examples in turbulent flows
   - Rare and extreme events in astronomy
   - Rare events with a huge impact: extreme heat waves
2. Instanton for the destabilization of the solar system by Mercury
   - Is the solar system stable?
   - The dynamical mechanism for Mercury destabilization
   - First exit time and instanton for Mercury–Jupiter resonance
3. Probability and dynamics of extreme heat waves
   - The jet stream, blocking events, and heat waves
   - Large deviations for time averaged observables
   - Sampling extreme heat waves using a large deviation algorithm

Lattice Boltzmann Simulations of 2D Flows

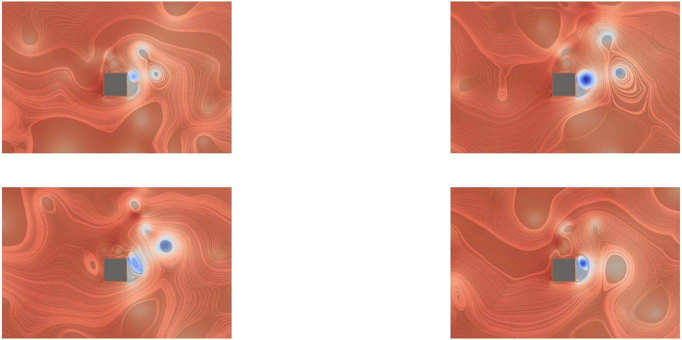
Velocity ($Re_G = 1200$, $Re_O \simeq 500$)

Extreme Drags for 2D Grid Turbulence Flows



Velocity ($Re_G = 1\,200$, $Re_O \simeq 500$) (with T. Lestang)

Predictability of Extreme Patterns for Turbulent Flows



Streamlines and pressure for the four most extreme drag events

- Extreme event patterns are often predictable. Instanton?

Freidlin–Wentzell Theory

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t)$$

- Path integral representation of transition probabilities:

$$P(x_t, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_T} \mathrm{e}^{-\frac{\mathscr{A}_T[x]}{\varepsilon}} \mathscr{D}[x]$$

with $\mathscr{A}_T[x] = \int_0^T \mathscr{L}[x,\dot{x}]\, dt$ and $\mathscr{L}[x,\dot{x}] = \frac{1}{4}[\dot{x} - \mathbf{b}(\mathbf{x})]^2$.

- We may consider the $\varepsilon \to 0$ limit, using a saddle point approximation (WKB), Then we obtain the large deviation result

$$P(x_T, T; x_0, 0) \underset{\varepsilon \to 0}{\asymp} e^{-\frac{\min_{\{x(t)\}}\left\{\mathscr{A}_T[x] \middle| x(0)=x_0 \text{ and } x(T)=x_T\right\}}{\varepsilon}}.$$

Most Transition Paths Follow the Instanton

- In the weak noise limit, most transition paths follow the most probable path (instanton)

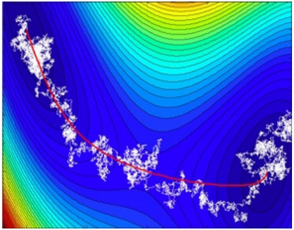
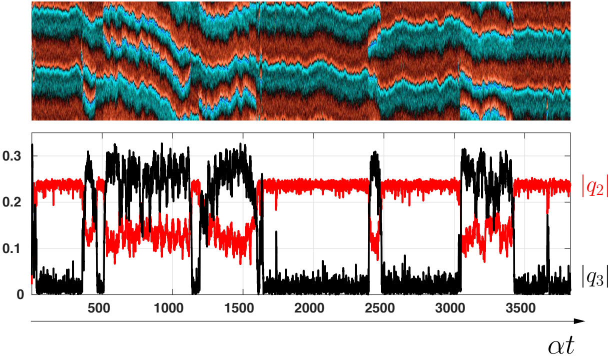


Figure by Eric Van den Eijnden

- For gradient dynamics, instantons are time reversed relaxation paths from a saddle to an attractor. Arrhenius law then follows

$$P\left(x_1, T; x_{-1}, 0\right) \underset{k_B T_e \to 0}{\asymp} \mathrm{e}^{-\frac{\Delta V}{k_B T_e}}.$$

Rare Transitions Between Quasigeostrophic Jets

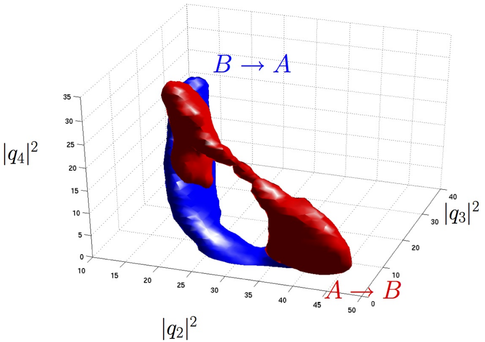


Rare transitions for quasigeostrophic jets (with E. Simonnet)

- How to predict those rare transitions transition rates? What is their probability? Which theoretical approach?

Atmosphere Jet "Instantons" Computed using the AMS

AMS: an algorithm to compute rare events, for instance rare reactive trajectories



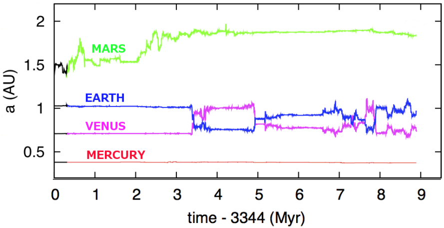
Transition trajectories between 2 and 3 jet states

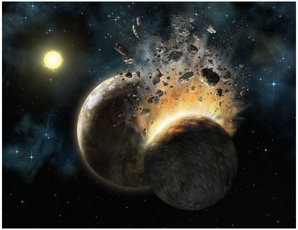
- The dynamics of turbulent transitions is predictable.
- Asymmetry between forward and backward transitions.

Outline

Collisional Trajectories in the Solar System

In collaboration with J. Laskar.





Distance from the sun vs time (J. Laskar)
(7.10^6 hours of CPU, p=1/100 000)

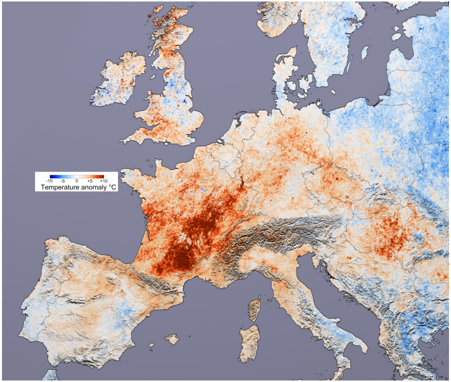
Collision probability?

- What are the probabilities of past and future qualitative changes of the solar system?

Outline

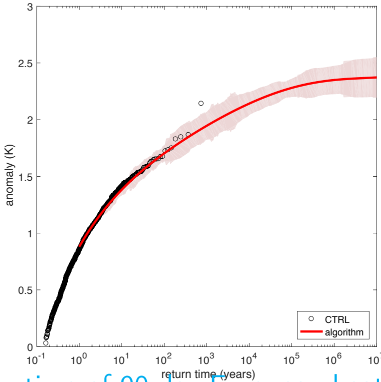
Extreme Heat Waves

Example: the 2003 heat wave over western Europe



July 20 2003-August 20 2003 land surface temperature minus the average for the same period for years 2001, 2002 and 2004 (TERRA MODIS).

The Return Time of Extreme Heat Waves

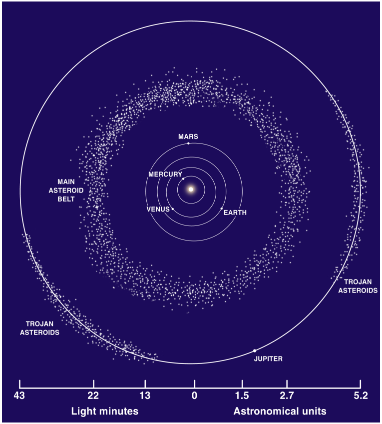


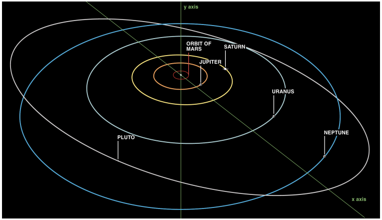
Return time of 90 day European heat waves

- At a fixed numerical cost, with the large deviation algorithm, we can study events which are several orders of magnitude rarer than the ones we could study with the control run.

Outline

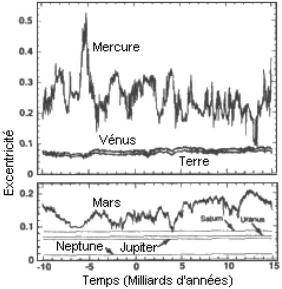
The Immutable Planetary Orbits





- The Laplace-Lagrange equation for the planet secular motion is integrable.

Is the Solar System Actually Stable?

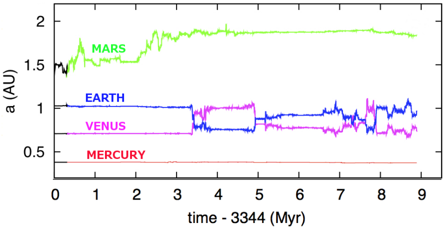


Mercury eccentricity (J. Laskar, 1989)

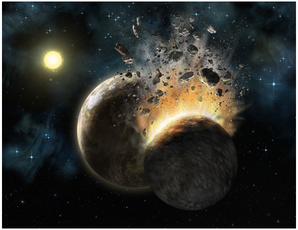
- Dynamical systems with more than two degrees of freedom are generically chaotic.
- The solar system is chaotic. The Lyapunov time is about 10 My.

Collisional Trajectories in the Solar System

In collaboration with J. Laskar.



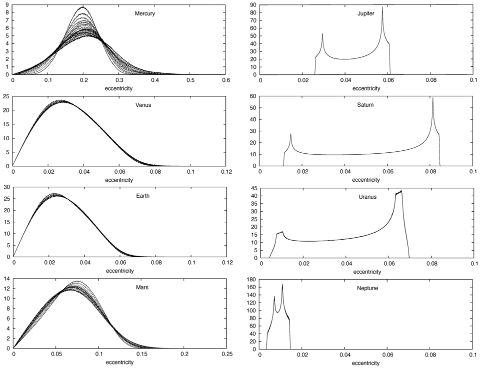
Distance from the sun vs time (J. Laskar)
(7.10^6 hours of CPU, p=1/100 000)



Collision probability?

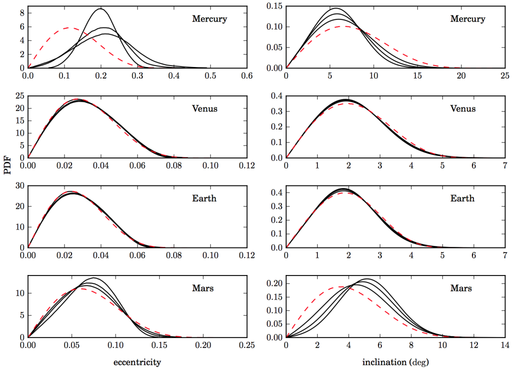
- What are the probabilities of past and future qualitative changes of the solar system?

Evolution of Orbital Element Distribution Functions



Eccentricity and inclination PDFs (J. Laskar, 2008)

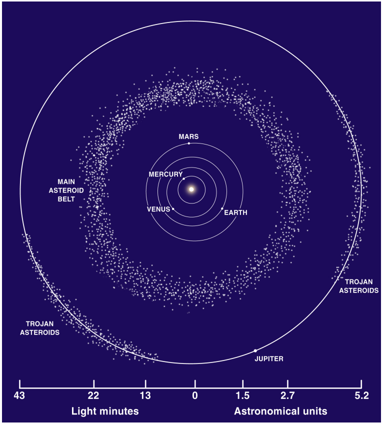
Evolution of Orbital Element: Mercury Specificity



Eccentricity and inclination PDFs (F. Mogavero, 2017)

Could we Apply Directly Freidlin–Wentzell Formalism?

What is the amplitude of the noise due to the asteroid chaotic motion?



Diffusion of Mars Longitude due to Asteroids?

$$\Delta\lambda = \left(\frac{t}{\tau_{diff}}\right)^{3/2}$$

$$\tau_{diff} = \left(\left(\frac{M_S}{m_a}\right)^2 \frac{a_{mars}^3 \tau_a}{GM_S}\right)^{1/3}$$

- An explicit formula for the order of magnitude for the stochastic effect of asteroid on Mars longitude.

E. Woillez and F. Bouchet, Astronomy and Astrophysics, 2018

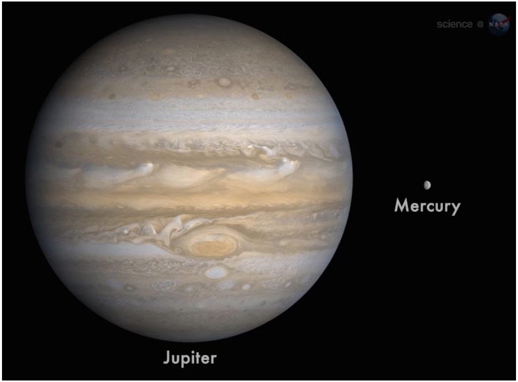
Internal Chaos Strongly Dominates Asteroid Noise

	mass $m_a/M_S \times 10^{-10}$	Lyapunov times (yr)	τ_{diff} (Ma)
Ceres	4.7	28900	22
Vesta	1.3	14282	24
Pallas	1.05	6283	33
total	7.05	-	17

E. Woillez and F. Bouchet, Astronomy and Astrophysics, 2018

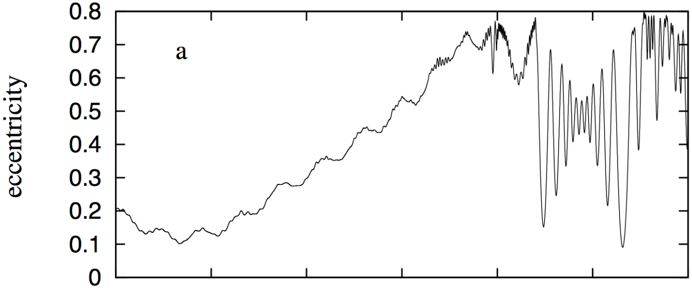
Outline

Mercury Ejection Dynamics



- $g_1 - g_5$ (Mercury–Jupiter) resonance plays a crucial role for Mercury destabilization (Laskar, Gastineau, Batygin and Laughlin (2008 and 2009)).

Jupiter–Mercury Resonance



Fast increase of Mercury eccentricity because of a resonance with Jupiter. (time scale 1My) (Boué, Laskar and Farago, 2012)

- Mercury--Jupiter resonance explains Mercury destabilization for $e > 0.7$.
- How does this all start?

The Batygin–Morbidelli–Holman Model for Mercury

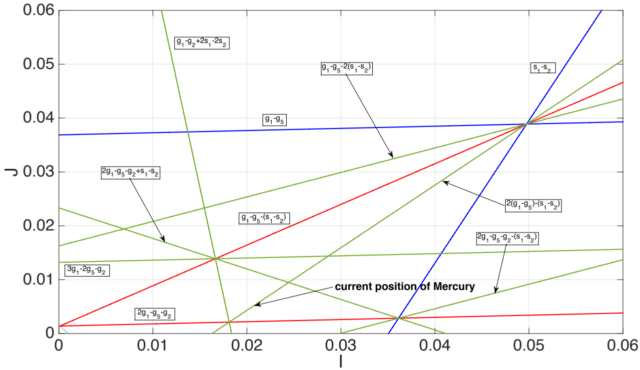
$s_1 - s_2$ and $g_1 - g_2$ resonances play a major role.

$$\begin{aligned} H := \quad & H_0(I,J) && \text{Ordre 4 secular expansion wrt } (e,i) \\ & +E_{g5}\sqrt{I}\cos(\varphi + g_5 t + \beta_5) && \text{main resonnances} \\ & +E_{s2}\sqrt{J}\cos(\psi + s_2 t + \theta_2) && \\ & +E_{g2}\sqrt{I}\cos(\varphi + g_2 t + \beta_2) && \end{aligned}$$

$$I \approx \frac{e^2}{2},\ J \approx \frac{i^2}{2},\ \varphi = -\varpi,\ \psi = -\Omega.$$

- A time dependent Hamiltonian system with 2 degrees of freedom (Batygin, Morbidelli and Holman, 2015).

Resonance Maps in the Action Space



Resonance map for the BMH model

Resonant and Non-Resonant Perturbations

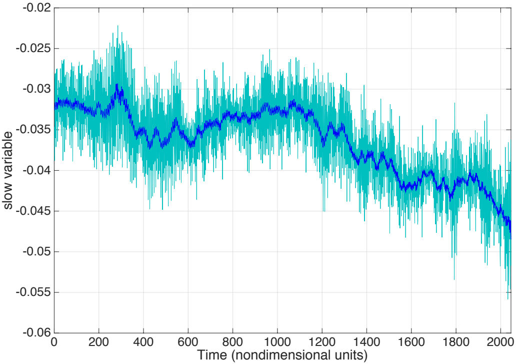
- **Canonical transform:** $\tilde{\varphi} = \varphi + g_5 t + \beta_5$, $\tilde{\psi} = \psi + s_2 t + \theta_2$,

$$H := \widetilde{H} + H_{pert} \text{ with}$$

$$\boxed{\widetilde{H} := H_0(I, J) + g_5 I + s_2 J + E_{g5}\sqrt{I}\cos(\tilde{\varphi}) + E_{s2}\sqrt{J}\cos(\tilde{\psi})}$$

- Key idea: higher order resonances are small perturbations. They have a small long term effect than can be dealt with using stochastic averaging.

Slow Diffusion of the Effective Hamiltonian



Hamiltonian (green) and effective Hamiltonian (blue) dynamics

Stochastic Averaging for the Slow Hamiltonian

$$\boxed{\widetilde{H} := H_0(I,J) + g_5 I + s_2 J + E_{g5}\sqrt{I}\cos(\tilde{\varphi}) + E_{s2}\sqrt{J}\cos(\tilde{\psi})}$$

is a good slow variable.

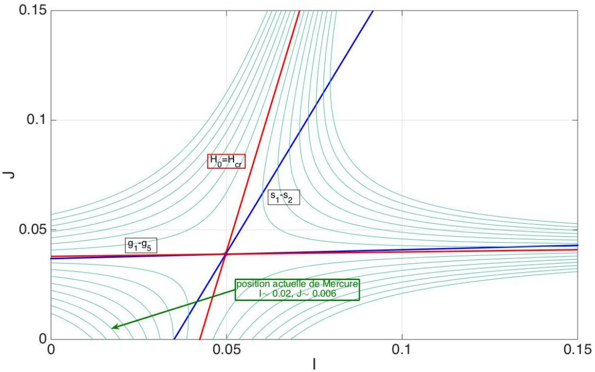
Stochastic averaging: we get diffusion equation

$$\frac{\mathrm{d}\left\langle \widetilde{H} \right\rangle}{\mathrm{d}t} = D\xi(t)$$

Using Lie transform, we can estimate the order of magnitude of D:

$$\boxed{\frac{D_{th}}{D_{empirical}} = 0.65}$$

Diffusion on the Level Lines of the Slow Hamiltonian



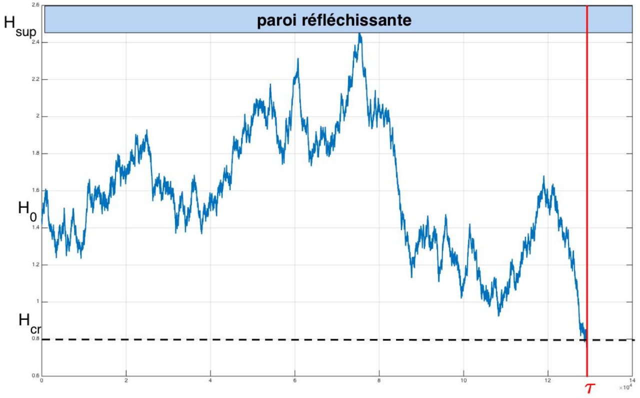
The effective Hamiltonian level lines in the action space

- We have mapped the dynamics on a first exit time problem for an effective diffusion.

Outline

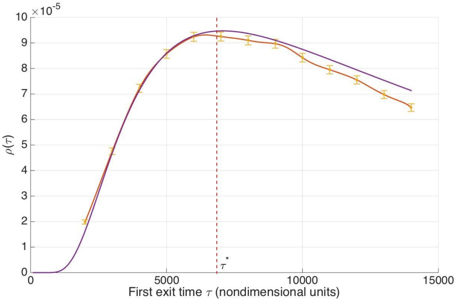
1. Rare events in complex dynamical systems and instanton phenomenology
   - Rare events with a huge impact: examples in turbulent flows
   - Rare and extreme events in astronomy
   - Rare events with a huge impact: extreme heat waves
2. Instanton for the destabilization of the solar system by Mercury
   - Is the solar system stable?
   - The dynamical mechanism for Mercury destabilization
   - First exit time and instanton for Mercury–Jupiter resonance
3. Probability and dynamics of extreme heat waves
   - The jet stream, blocking events, and heat waves
   - Large deviations for time averaged observables
   - Sampling extreme heat waves using a large deviation algorithm

The First Exit Time for a Diffusion



Boundary conditions for the effective diffusion

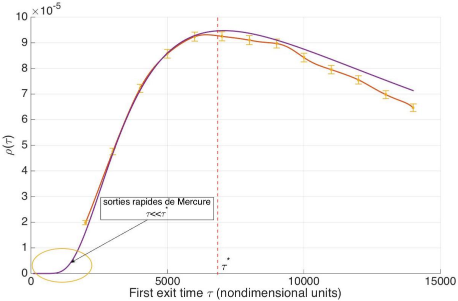
First Exit Times for Mercury–Jupiter Resonances



First exit time probability for the Mercury–Jupiter resonance - $\tau* \approx 12.4\,\mathrm{Gyr}$

- There is a pretty good agreement between the the HMG model and the diffusive model for the effective Hamiltonian.

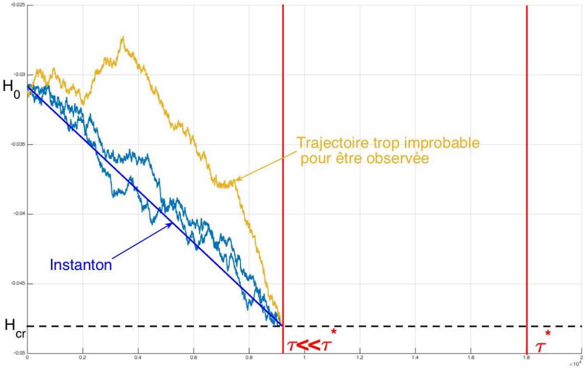
Should the Exit Dynamics be Predictable (Instantons)?



First exit time probability for the Mercury–Jupiter resonance

- We expect short time instantons for the dynamics towards Jupiter–Mercury resonance.

Short Time Instanton for the Brownian Motion

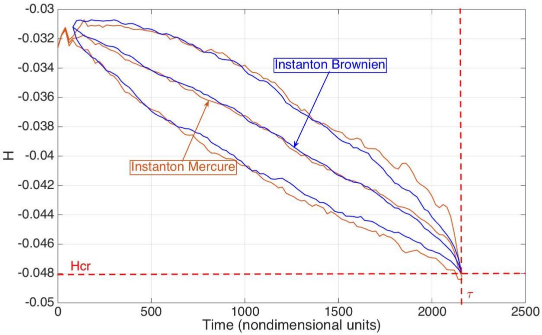


Samples of trajectories with exit time τ

- For a free diffusion, the instanton is a straight line.

Instanton for Mercury–Jupiter Resonance

For the Batygin–Morbidelli–Holman model



Instanton for sample paths of Mercury–Jupiter resonance

- There is an excellent agreement between the BMH model and the effective Hamiltonian diffusion.

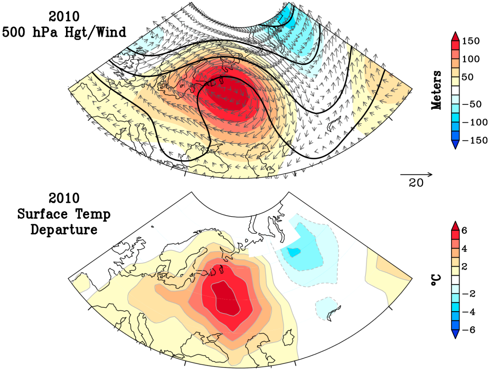
Summary and Perspectives

- The dynamical processes leading to Mercury–Jupiter resonance can be understood qualitatively.
- Within the Batygin–Morbidelli–Holman model, Mercury–Jupiter resonance typically occurs within 12 G.yr. and is rare for much shorter time scales.
- The several hundred thousand year destabilization of the inner solar system is rare and follows an instanton dynamics.
- We aim at quantifying this instanton behavior for realistic models, using rare event algorithms (with J. Laskar).

http://perso.ens-lyon.fr/freddy.bouchet/

Outline

Extreme Heat Waves and Anticyclonic Anomalies

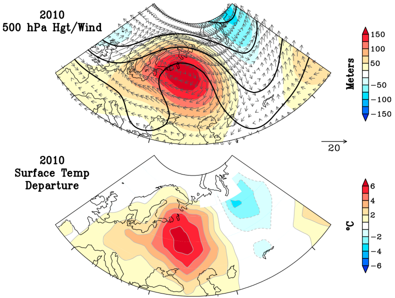


2010 Heat Wave over Eastern Europe (Dole and col., 2011)

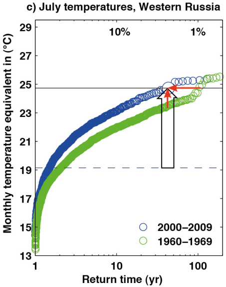
The Jet Stream, Rossby Waves, and Blocking Events

Higher troposphere winds (NASA)

Anthropogenic Causes of the 2010 Heat Wave



(Dole et al., 2011)



Return time of monthly temperature

(Otto et al., 2012)

- A clear anthropogenic impact.
- What are the dynamical mechanisms for such extreme events?

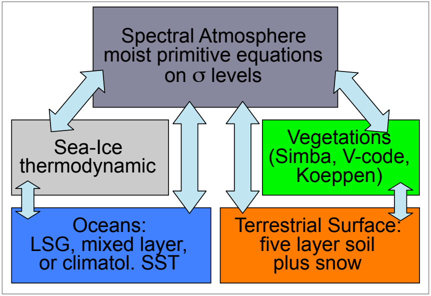
How to Study 10 000 Year Heat Waves with a 200 Year Computation?

Sampling rare events in dynamical systems

The scientific questions:

- What is the probability and the dynamics of those rare events?
- How to sample rare events?
- Are direct numerical simulations a reasonable approach?

Model: the Planetary Simulator (Plasim) - Hamburg



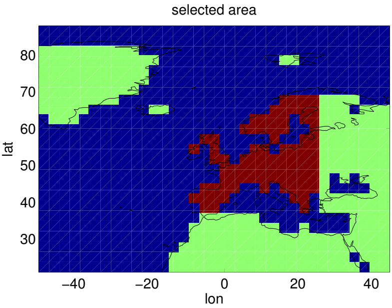
Plasim main properties

- We use the Planet Simulator (PlaSim) model (an Earth system model of intermediate complexity) developed at Hamburg.
- T42 horizontal resolution (64x128 grid points), 10 vertical layers, about $10^5 - 10^6$ degrees of freedom.

Plasim Northern Hemisphere Dynamics

500 HPa geopotential height and temperature anomalies

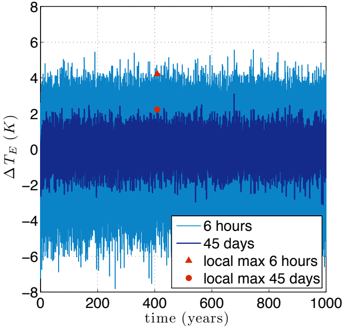
Observable: Averaged Surface Temperature



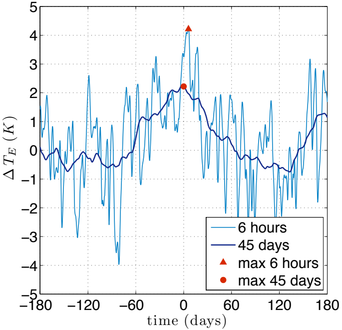
The observable will be Europe averaged surface temperature

$$a = \frac{1}{T}\int_0^T \mathrm{d}t \left\langle Temp \right\rangle_{\mathrm{Europe}}(t)$$

45-Day Averaged Temperature over Europe (Plasim Model)



Full timeseries



Close to a heat wave

$$a = \frac{1}{T}\int_0^T \mathrm{d}t \, \langle Temp \rangle_{\text{Europe}}(t) \text{ with T=45 days}$$

Outline

Large Variances for Estimators of Rare Event Probabilities

- Monte Carlo sampling of small probabilities (sampling the probability from iid random variables)

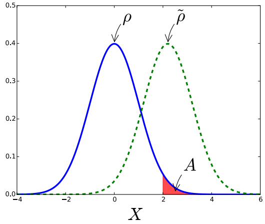
$$\gamma_A = \int \mathrm{d}x\, \rho(x) 1_A(x) = \mathbb{E}\left(1_A\right). \text{ Estimator: } \hat{\gamma}_A = \frac{1}{N}\sum_{n=1}^{N} 1_A(X_n).$$

- The variance of $\hat{\gamma}_A$ is $\mathrm{Var}\left(1_A\right)/N = \left(\gamma_A - \gamma_A^2\right)/N$. The relative error is

$$\mathrm{Er} \simeq \frac{1}{\sqrt{\gamma_A N}}.$$

- The number of observations has to grow at least as fast as the probability decreases to keep the relative error constant.

Importance Sampling



Tilted probability: $\rho(x) = L(x)\tilde{\rho}(x)$

- We sample a tilted probability with PDF $\tilde{\rho}(x)$.

$$\gamma_A = \int_A \rho(x)\,\mathrm{d}x = \int_A L(x)\tilde{\rho}(x)\,\mathrm{d}x. \text{ Estimator } \hat{\gamma}_A = \frac{1}{N}\sum_{n=1}^{N} L(X_n) 1_A(X_n).$$

- If L is well chosen, rare events for ρ are common for $\tilde{\rho}$, and the variance is much lower.
- How to perform importance sampling for a climate model?

Donsker–Varadhan Large deviations for time averaged observables

- Donsker–Varadhan: large time asymptotics for time averaged observables.
- Time averaged observables

$$P\left[\frac{1}{T}\int_0^T \langle Temp\rangle_{\mathrm{Europe}}\,\mathrm{d}t = a\right] \underset{T\to\infty}{\asymp} C\mathrm{e}^{-TI[a]}.$$

- $I(a)$ is the large deviation rate function. It has a minimum for the most probable value a_*, its second derivative at a_* describes the Gaussian fluctuations, but it describes also much rarer fluctuations.

Numerical Computation of Donsker–Varadhan Large Deviations

- Importance sampling: how to sample efficiently the tilted distribution

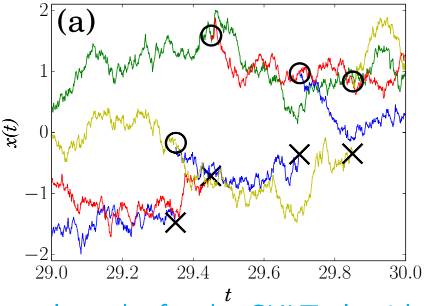
$$\tilde{P}_k\left(\{X(t)\}_{0\leq t\leq T}\right) = \frac{1}{\exp(T\lambda(k))} P_0\left(\{X(t)\}_{0\leq t\leq T}\right) \exp\left[k\int_0^T A(X(t))\,\mathrm{d}t\right] ?$$

- We use the Giardina–Kurchan–Leconte–Tailleur algorithm (Giardina et al 2006).
- We consider an ensemble of N trajectories $\{x_n(t)\}$. At each time $t_i = i\tau$, each trajectory may be killed or cloned according to the weights

$$\frac{1}{W_i(k)} \exp\left(k\int_{t_{i-1}}^{t_i} A(x_n(t))\,\mathrm{d}t\right) \text{ with } W_i(k) = \sum_{n=1}^{N} \exp\left(k\int_{t_{i-1}}^{t_i} A(x_n(t))\,\mathrm{d}t\right).$$

Genealogical Algorithm: Selecting and Cloning Trajectories

The trajectory statistics is tilted towards the events of interest.



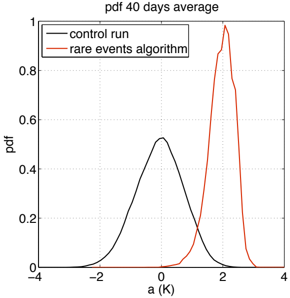
Sample paths for the GKLT algorithm

(from Bouchet, Jack, Lecomte, Nemoto, 2016)

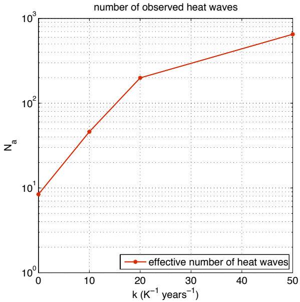
- Computing numerically Donsker–Varadhan large deviations through a genealogical algorithm.

Outline

Importance Sampling of Extreme Heat Waves in a Climate Model



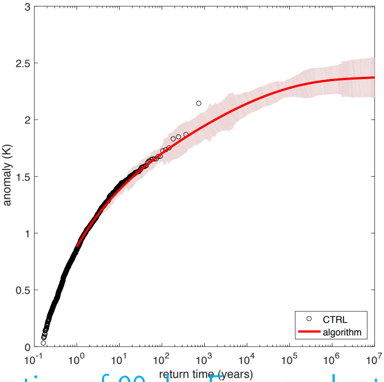
PDF of time averaged temperature



Heat wave number

- At a fixed numerical cost, we get hundreds more heat waves with the large deviation algorithm than with the control run.
- We can consider interesting dynamical studies.

The Return Times of Extreme Heat Waves



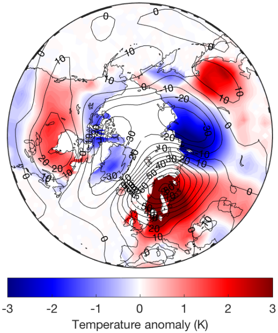
Return time of 90 day European heat waves

- At a fixed numerical cost, with the large deviation algorithm, we can study events which are several orders of magnitude rarer than the ones we could study with the control run.

A Typical Heat Wave

500 HPa geopotential height and temperature anomalies

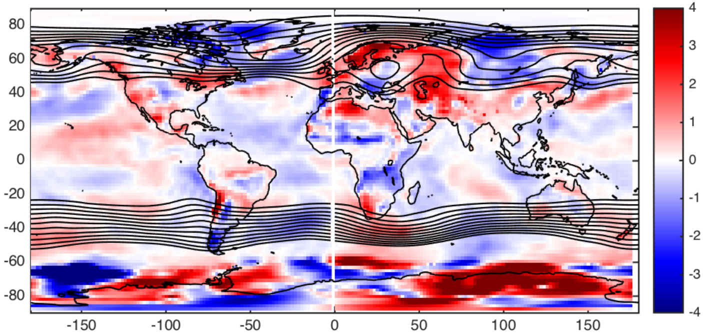
Heat Wave Conditional Statistics and Teleconnection Patterns



500 HPa geopotential height anomalies and temperature anomalies

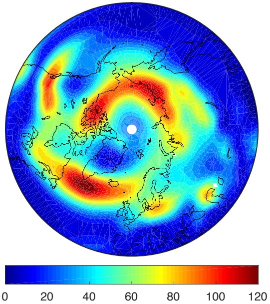
Heat wave statistics defined as statistics conditioned with $\frac{1}{T}\int_0^T \left\langle Temp \right\rangle_{\text{Europe}}(t)\,\mathrm{d}t > 2^{\circ}\mathrm{C}$, with $T = 40\,\text{days}$.

July 2018 Heat Waves

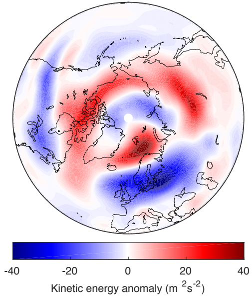


Oberved (reanalysis) July 2018 500 HPa geopotential height and temperature anomalies with respect to the last ten years).

Heat Waves and Shift of the Jet Stream



Northern hemisphere mean kinetic energy



Kinetic energy anomaly during the heat waves

- The european heat waves are associated with a northward shift of the jet stream over Europe and a southward shift over Asia.

Extreme Heat Waves: Conclusions

Conclusions:

- Large deviation algorithms provide a wonderful tool to sample rare events, for instance heat waves.
- It should open a new range of dynamical studies in GCM, even the more complex ones.

Work in progress:

- A dynamical study of heat waves based on hundreds of sampled heat waves.
- Relation with blocking events? Are they different types of dynamics leading to heats waves? Which physical processes? Relation between heat waves precursors and instantons?

F. Ragone, J. Wouters, and F. Bouchet, PNAS, 2018