Processes with reinforcement and approximation of Quasi-Stationary Distributions

Michel Benaim
Neuchâtel University

Luminy, September 2018
Talk based on recent collaborations with

Bertrand Cloez
(Montpellier)

Fabien Panloup
(Anger)

Stochastic Approximation of Quasi-Stationary Distributions on Compact Spaces and Applications

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*Stochastic Approximation on non-compact measure spaces and application to measure-valued Polya Processes* arXiv September 6, 2018
A metric space, $\partial$ a cemetery point

$(X_t)_{t \in \mathbb{Z}^+}$ a Markov chain on $E \cup \partial$ eventually absorbed by $\partial$:

(i) $\tau_\partial = \inf \{ t \geq 0 : X_t = \partial \} < \infty$ a.s

(ii) $X_t = \partial \Rightarrow X_{t+s} = \partial$.

Definition

A Quasi-Stationary Distribution for $X$ is a probability $\mu$ on $E$ such that

$P_\mu(X_t \in \cdot \mid t < \tau_\delta) = \mu(\cdot)$.

Under appropriate assumptions, such a QSD exists and

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Natural object in Population Dynamics because eventually everyone gets killed...
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How to compute/estimate such a QSD?

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⇝ Idea explored by Burdzy, Holyst & March (2000); Del Moral & Miclo (2000); Villemonais (2014); Cloez & Thai (2016)
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\(\Rightarrow\) This original idea goes back to Aldous, Flannery & Palacios (1988) and their approach relies on branching processes.

\(\Rightarrow\) Here we will revisit it using tools from stochastic approximation, self-reinforced processes combined with recent ideas & results due to Champagnat and Villemonais (2015)
• \( K \) the Sub-Markov Kernel of \( X \) on \( \mathcal{E} \):

\[
K(x, \cdot) = \mathbb{P}_x(X_1 \in \cdot)
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Notation & Hypotheses

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Remark: $H_1 \land H_2 \Rightarrow$ Existence of (at least) one QSD
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- \( H1 \) \( \mathcal{E} \) is compact and \( K \) Feller;

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Remark

$H1 \& H2 \implies$ Existence of (at least) one QSD
• H3 There exists an open set $U \subset \partial$ which is accessible:
• **H3** There exists an open set $U \subset \partial$ which is *accessible*:

$$\forall x \sum_{n} K^n(x, U) > 0.$$
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and petite :

rem A

If $K$ were Markov (i.e. $K(x, E) = 1$), $H_1$, $H_2$, $H_3$ would ensure the uniqueness of an invariant measure $\mu$. But, this is not sufficient to ensure uniqueness of a QSD!
• **H3** There exists an open set $U \subset \partial$ which is accessible:

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where $\Psi$ is a probability on $\mathcal{E}$
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**Remark**

*If $K$ were Markov (i.e. $K(x, \mathcal{E}) = 1$), $H1, H2, H3$ would ensure the uniqueness of an invariant measure $\mu$. But, this is not sufficient to ensure uniqueness of a QSD!*
• **H4** There exists a non increasing convex function \( C : \mathbb{R}^+ \mapsto \mathbb{R}^+ \) satisfying

\[
\int_0^\infty C(s)ds = \infty
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such that

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\frac{\Psi(K^n1)}{\sup_{x \in \mathcal{E}} K^n1(x)} \geq C(n)
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The Champagnat-Villemonais condition

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• **When** $C(n) = C$, **this condition is due to Champagnat and Villemonais (2015)**
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- *When $C(n) = C$, this condition is due to Champagnat and Villemonais (2015)*
- *It ensures the uniqueness of the QSD.*
Example

\[ a, b \in (0, 1) \]
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- If \( a < b \), \( C(t) = C \) and there is a unique QSD \( \mu = \delta_2 \)

[Diagram of two states with transitions labeled by a, b, and 1-a]
Example

$a, b \in (0, 1)$

- If $a < b$, $C(t) = C$ and there is a unique QSD $\mu = \delta_2$
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- If \(a > b\) there is another QSD \(\mu^* = \frac{a-b}{1-b} \delta_1 + \frac{1-a}{1-b} \delta_2\).
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Figure: $b = 1/3; a \mapsto \mu(1), \mu^*(1)$
• For each \( \mu \) probability on \( \mathcal{E} \)

\[
K_\mu(x, dy) = K(x, dy) + (1 - K(x, \mathcal{E})) \mu(dy)
\]
Results

• For each $\mu$ probability on $\mathcal{E}$

$$K_\mu(x, dy) = K(x, dy) + (1 - K(x, \mathcal{E}))\mu(dy)$$

= kernel of a chain which behaves like $(X_t)$ until it dies and, then is redistributed according to $\mu$. 

$\Pi(\mu)$ the invariant probability measure of $K_\mu$.

$\Pi(\mu) = \mu G\mu = \sum_{n \geq 0} K^n$. 

$\{\Phi_t\}_{t \in \mathbb{R}^+}$ the deterministic semiflow induced by the ODE $\dot{\mu} = -\mu + \Pi(\mu)$ (in a weak sense).
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(in a weak sense)
Theorem

Under hypotheses H1 (Feller) and H2 (\(\partial\) accessible), the limit set of \((\mu_n)\) is almost surely a Attractor Free set of \(\Phi\)
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Set $L$ is \textit{Attractor free} means:

(i) $L$ is compact,
(ii) connected,
(iii) invariant: $\Phi_t(L) = L$ and
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Example

Set $\mu = x \delta_1 + (1-x) \delta_2$.

The ODE writes $\dot{x} = -x + (1-b)x(1-a) + (1-b)x$.
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$$\dot{x} = -x + \frac{(1 - b)x}{(1 - a) + (1 - b)x}.$$
Under the additional assumptions $H_3$ and $H_4$ (The Champagnat Villemonais condition); $\Phi$ has a global attractor given as the unique QSD $\{\mu\}$. Hence, there is only one attractor free set $\{\mu\}$ and

$$\mu_n \to \mu.$$
Strategy of proof

(i) Show that \( t \mapsto \hat{\mu}_t := \mu e^t \) is an Asymptotic Trajectory of \( \Phi \).

\[ \lim_{t \to \infty} \sup_{0 \leq s \leq T} \text{dist}(\hat{\mu}_t + s, \Phi_s(\hat{\mu}_t)) = 0. \]

(ii) Use old results from the late 90s on the dynamics of APT (B, B & Hirsch):

- The limit set of an APT is attractor free
  \[ \Rightarrow \text{If } L \text{ is attractor free and meets the basin of attraction of attractor } A; \text{ Then } L \subset A. \]

- If \( \Phi \) has a global attractor \( A \);
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(iii) Show that under \( H_4 \), \( \Phi \) has the unique QSD of \( K \) as global attractor.
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Strategy recently applied by

\[ \frac{dX}{dt} = \nabla V(X(t)) dt + dW_t \]

where

\[ \tau = \inf \left\{ t \geq 0 : \int_0^t \kappa(X_s) ds > \xi \right\} \]

\[ \xi \text{ is an independent random variable with exponential distribution;} \]

\[ \kappa > 0 \] and smooth

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• Wang, Roberts and Steinsaltz \((\text{arXiv August 23, 2018})\) to diffusions on a compact manifold with soft killing:

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dX_t = \nabla V(X_t) dt + dW_t
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\tau_\partial = \inf\{t \geq 0 : \int_0^t \kappa(X_s) ds > \xi\}
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where \(\xi\) is an independent random variable with exponential distribution; \(\kappa > 0\) and smooth

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