Processes with reinforcement and approximation of Quasi-Stationary Distributions

Michel Benaim
Neuchâtel University

Luminy, September 2018

Talk based on recent collaborations with

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Bertrand Cloez (Montpellier)

and **Fabien Panloup** (Anger)

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Stochastic Approximation of Quasi-Stationary Distributions on Compact Spaces and Applications Annals of Applied Probability, 2018

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• A. Wang (Oxford) , G. O Roberts (Warwick) and D. Steinsaltz (Oxford)

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Stochastic Approximation on non-compact measure spaces and application to measure-valued Polya Processes arXiv September 6, 2018

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Natural object in Population Dynamics because eventually everyone gets killed...

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$\rightsquigarrow$ Idea explored by Burdzy, Holyst & March (2000); Del Moral & Miclo (2000); Villemonais (2014); Cloez & Thai (2016)...

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⇝ Here we will revisit it using tools from *stochastic approximation, self-reinforced processes* combined with recent ideas & results due to Champagnat and Villemonais (2015)

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Remark

H1&H2 $\Rightarrow$ *Existence of (at least) one QSD*

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Remark

If K were Markov (i.e $K(x, \mathcal{E}) = 1$), $H1, H2, H3$ would ensure the uniqueness of an invariant measure μ But, this is not sufficient to ensure uniqueness of a QSD !

The Champagnat-Villemonais condition

• **H4** There exists a non increasing convex function $C : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfying

$$\int_0^\infty C(s)ds = \infty$$

such that

$$\frac{\Psi(K^n\mathbf{1})}{\sup_{x\in\mathcal{E}} K^n\mathbf{1}(x)} \geq C(n)$$

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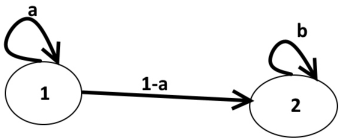
• *It ensures the uniqueness of the QSD.*

Example

$a, b \in (0, 1)$



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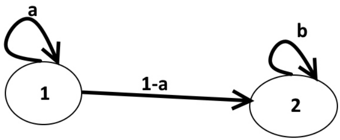

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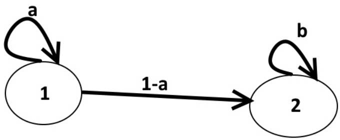

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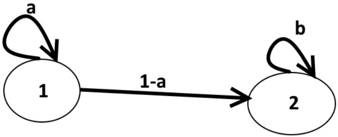

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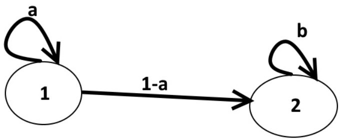

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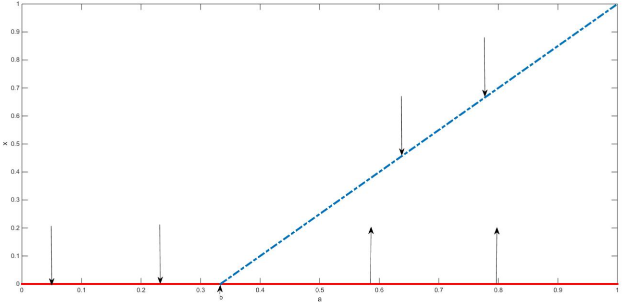

Figure: $b = 1/3; a \mapsto \mu(1), \mu^*(1)$

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- $\{\Phi_t\}_{t \in \mathbb{R}^+}$ the deterministic semiflow induced by the ODE

$$\dot{\mu} = -\mu + \Pi(\mu)$$

(in a weak sense)

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(iv) $\Phi|_L$ has no proper attractor

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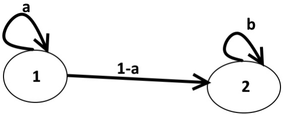

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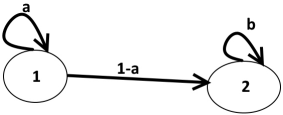

Set $\mu = x\delta_1 + (1-x)\delta_2$. The ODE writes

$$\dot{x} = -x + \frac{(1-b)x}{(1-a) + (1-b)x}.$$

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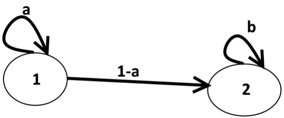

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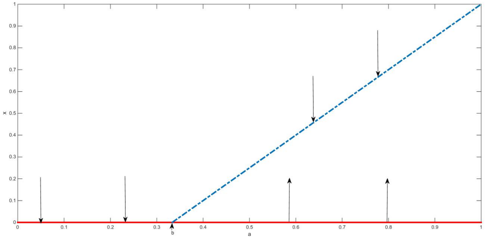

Theorem

Under the additional assumptions H3 and H4 (The Champagnat Villemonais condition); Φ *has a global attractor given as the unique QSD* $\{\mu\}$*. Hence, there is only one attractor free set* $\{\mu\}$ *and*

$$\mu_n \to \mu.$$

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(iii) Show that under $H4$, Φ has the unique QSD of K as global attractor

Recent results

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