TECHNISCHE UNIVERSITÄT WIEN

Large-time behavior in (hypo)coercive ODE-systems and kinetic models

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Goals & strategies

- Given an evolution eq: $\frac{\mathrm{d}}{\mathrm{d}t} f = L\, f, \; t \geq 0;$ $\quad L$ ... const in t operator
- Assume L has a unique steady state: $L\, f_\infty = 0$

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- Goal: find an estimate on $\|f(t) - f_\infty\|$
  - possibly exponential decay: $\|f(t) - f_\infty\| \le c e^{-\mu t} \|f(0) - f_\infty\|$
  - possibly with sharp (= maximum) rate $\mu > 0$
    and minimal c [uniform for all $f(0)$]

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  - possibly with sharp ($=$ maximum) rate $\mu > 0$ and minimal c [uniform for all $f(0)$]

- Strategy: construct an "appropriate" Lyapunov functional $e(f, f_\infty)$ (e.g. norm^2, energy, entropy) with

$$\frac{\mathrm{d}}{\mathrm{d}t} e(f(t), f_\infty) \leq -2\mu\, e(f(t), f_\infty)\,.$$

$\Rightarrow$ Gronwall lemma: $e(f(t), f_\infty) \leq e^{-2\mu t} e(f(0), f_\infty),\ t \geq 0$

Decay for nonsymmetric ODEs: find Lyapunov functionals!

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{R}^n \tag{1}$$

Definition: $\mathbf{C}$ is *coercive* if $x^T\mathbf{C}x \geq \kappa\|x\|^2\ \forall x$ (for some $\kappa > 0$).

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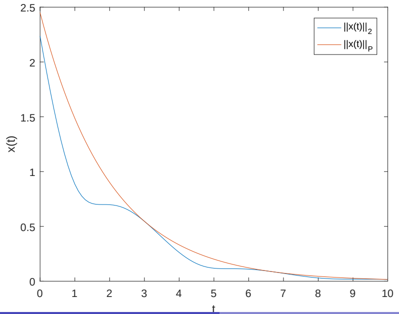
ex: $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_{\mathbf{C}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$ decay rate $= \frac{1}{2}$ for (1).

- $\mathbf{C}$ not coercive $\Rightarrow$ no decay of $\|x(t)\|_2$ by trivial energy method!
- But decay of modified norm $\|x(t)\|_{\mathbf{P}} := \sqrt{x^T\,\mathbf{P}\,x}$; $\mathbf{P} := [2\ -1; -1\ 2]$



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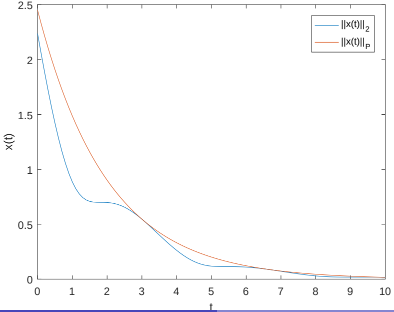

Main theme:
How to find $\mathbf{P}$ / the Lyapunov functional?

Outline:

1. hypocoercive ODEs
2. kinetic relaxation equations (BGK-type)
3. entropy methods for (non-)degenerate Fokker-Planck equations

hypocoercive ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{R}^n$$

Definition: $\mathbf{C}$ is *hypocoercive* (= positive stable) if $\exists\, \mu > 0$ such that:

$$\Re(\lambda_j) \geq \mu, \qquad j = 1, ..., n.$$

If all eigenvalues of $\mathbf{C}$ are non-defective:

$$\exists c \geq 1 : \qquad \|x(t)\|_2 \leq c\|x(0)\|_2 e^{-\mu t}, \quad t \geq 0.$$

• always: $\mu \geq \kappa := \max_x \dfrac{x^T\mathbf{C}x}{\|x\|^2}$

Choice of $\mathbf{P}$ for $\|x\|_{\mathbf{P}}$ / Lyapunov's direct method

Lemma 1

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ *be positive stable, i.e.* $\mu := \min\{\Re\,\lambda_{\mathbf{C}}\} > 0$.

1. *If all* $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective*
   (i.e. geometric = algebraic multiplicity)
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P} \geq 2\mu \mathbf{P}$.

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2. *If (at least) one* $\lambda_{\mathbf{C}}^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0\ : \quad \mathbf{PC} + \mathbf{C}^\top \mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}$.

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Proof: $\mathbf{P}$ can be constructed explicitly; e.g. for $\mathbf{C}$ non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes \bar{z}_j^\top\ ; \qquad z_j \ \dots \text{ eigenvectors of } \mathbf{C}^\top$$

□

- $\mathbf{P}$ not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of $\mathbf{P}$.
- For complex $\mathbf{C}$: $\mathbf{P} > 0$ Hermitian with $\mathbf{PC} + \mathbf{C}^*\mathbf{P} \geq 2\mu\mathbf{P}$.

Decay of $\mathbf{P}$–norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case):

  Let $\|x\|_P^2 := x^T \, \mathbf{P} \, x$.

$$\frac{d}{dt}\|x\|_{\mathbf{P}}^2 = -x^T \big(\underbrace{\mathbf{P}\mathbf{C} + \mathbf{C}^T\mathbf{P}}_{\geq 2\mu \mathbf{P}}\big) x \leq -2\mu \|x\|_{\mathbf{P}}^2$$

$$\Rightarrow \quad \|x(t)\|_{\mathbf{P}} \leq \|x(0)\|_{\mathbf{P}} \, e^{-\mu t}, \quad t \geq 0.$$

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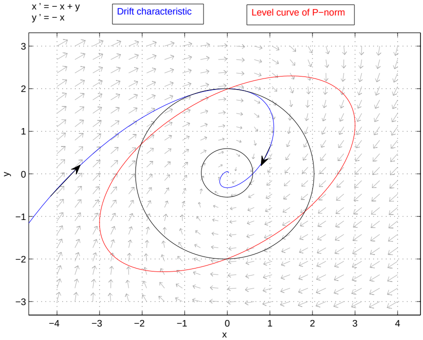
- Comparison to strategy for C_0-semigroups [Pazy book]:

  $T(t)$ is contraction with norm $|x| := \sup_{t\geq 0} \| T(t)x\|$,
  but it does not yield the sharp decay rate.

Decay of $\mathbf{P}$–norm (continued)

ex: $\dot{x} = -\mathbf{C}x$ with $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

- At x_2-axis: trajectory $x(t)$ tangent to level curve of $|x|$:



- level curve of "distorted" vector norm $\sqrt{x^T \mathbf{P}\, x}$; $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

  $\rightarrow$ uniform decay with sharp rate $\frac{1}{2}$

Hypocoercivity index

Conservative-dissipative system:

$$\dot{x} = -(i\mathbf{C}_1 + \mathbf{C}_2)x\,, \qquad \mathbf{C}_1 \in \mathbb{C}^{n\times n} \text{... Hermitian, } \mathbf{C}_2 \geq 0 \qquad (2)$$

Definition 1 (Achleitner-AA-Carlen 2018)

The *hypocoercivity index of* $\mathbf{C} = i\mathbf{C}_1 + \mathbf{C}_2$ *is the smallest integer* $m_0 \in \mathbb{N}_0 \cup \{\infty\}$, *such that* $\sum_{j=0}^{m_0} \mathbf{C}_1^j \mathbf{C}_2 \mathbf{C}_1^j > 0$.

- $\mathbf{C}$ is coercive $\Leftrightarrow$ $\mathbf{C}_2 > 0 \Leftrightarrow m_0 = 0$
- $\mathbf{C}$ is hypocoercive $\Leftrightarrow$ $m_0 < \infty$
- If $\mathbf{C}$ is hypocoercive: $\frac{n-\text{rank}\mathbf{C}_2}{\text{rank}\mathbf{C}_2} \leq m_0 \leq n - \text{rank}\mathbf{C}_2$
- m_0 describes the structural complexity of (2).

Hypocoercivity index for $\dot{x} = -(i\mathbf{C}_1 + \mathbf{C}_2)x$

Lemma 2 (Achleitner-AA-Carlen 2018)

$\mathbf{C}$ *is hypocoercive iff no (non-trivial) subspace of* $\ker \mathbf{C}_2$ *is invariant under* $\mathbf{C}_1$.

ex: $\mathbf{C}_2 = \mathrm{diag}(0, 0, 1, 1)$

(a) $\mathbf{C}_1 = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$, hypocoercivity index $= 1$

(b) $\mathbf{C}_1 = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, hypocoercivity index $= 2$

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Space-inhomogen. Bhatnagar-Gross-Krook model (1954)

- application: gas dynamics, computational fluid dynamics
- (vs. Boltzmann eq.) simplified kinetic model; relaxation operator:

$$f_t + v \cdot \nabla_x f = M_f(x,v,t) - f(x,v,t)\ , \quad t \geq 0\ ,\ x \in \mathbb{T}^d\ ,\ v \in \mathbb{R}^d$$

- relaxation towards local Maxwellian M_f with same hydrodynamic moments as f $\rightarrow$ local-in-x dissipation:

$$M_f(x,v) = \frac{\rho(x)}{(2\pi T(x))^{\frac{d}{2}}}\, e^{-\frac{|v-u(x)|^2}{2T(x)}}\ ,$$

density $\rho(x) := \int_{\mathbb{R}^d} f(x,v)\, dv\ ,$

mean velocity $u(x) := \frac{1}{\rho(x)} \int_{\mathbb{R}^d} v f(x,v)\, dv\ ,$

temperature $T(x) := \frac{1}{d\rho(x)} \int_{\mathbb{R}^d} |v-u(x)|^2 f(x,v)\, dv\ .$

- transport operator $-v \cdot \nabla_x$ leads to uniformity in x $\quad\rightarrow\quad$ hypocoercive

1) linear 2-velocity BGK model, 1D (Goldstein-Taylor model)

for $f(x,t) = \binom{f_+(x,t)}{f_-(x,t)}$ corresponding to $v = \pm 1$:

$$\partial_t f_\pm \pm \partial_x f_\pm = \pm \frac{1}{2}(f_- - f_+), \quad t \geq 0, \quad 2\pi\text{–periodic in } x$$

$f^\infty(x) = \binom{f_+^\infty}{f_-^\infty} = \binom{\frac{1}{2}}{\frac{1}{2}}$

x-Fourier modes; discrete velocity basis $\{\binom{1}{1}, \binom{1}{-1}\}$

$\Rightarrow \quad u_k(t) \in \mathbb{C}^2, \quad k \in \mathbb{Z}$

<ins>Ref's:</ins>

[Dolbeault-Mouhot-Schmeiser '15] exp. decay (only 1 conserved quantity)

[Achleitner-AA-Carlen '16] sharp decay rate

2-velocity BGK model: decay of Fourier modes

$$\frac{\mathrm{d}}{\mathrm{d}t} u_k = -\mathbf{C}_k\, u_k, \quad \mathbf{C}_k = \begin{pmatrix} 0 & ik \\ ik & 1 \end{pmatrix}, \quad k \in \mathbb{Z}$$

$$\lambda_{\mathbf{C}_0} = 0,\ 1; \qquad \lambda_{\mathbf{C}_k} = \frac{1}{2} \pm i\sqrt{k^2 - \frac{1}{4}}, \quad k \neq 0$$

$$u_0^\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad u_k^\infty = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad k \neq 0$$

Decay of each mode in $\mathbf{P}_k$-norm:

$$\begin{aligned} \|u_0(t) - u_0^\infty\|_2 &\leq e^{-t}\|u_0(0) - u_0^\infty\|_2 \\ \|u_k(t)\|_{\mathbf{P}_k} &\leq e^{-t/2}\|u_k(0)\|_{\mathbf{P}_k} \quad k \neq 0 \end{aligned}$$

Lemma 1 $\quad \Rightarrow \quad \mathbf{P}_k\mathbf{C}_k + \mathbf{C}_k^*\mathbf{P}_k \geq 2 \cdot \frac{1}{2}\mathbf{P}_k\,, \quad k \neq 0$

2-velocity BGK model: decay of sequence of modes

$$k \neq 0: \quad \mathbf{P}_k := \begin{pmatrix} 1 & \frac{-i}{2k} \\ \frac{i}{2k} & 1 \end{pmatrix} \xrightarrow{|k|\to\infty} \mathbf{I}$$

$\Rightarrow$ "generic" Lyapunov functional:

$$e(\{u_k\}) := \sqrt{\sum_{k\in\mathbb{Z}} \|u_k\|^2_{\mathbf{P}_k}} \quad \sim \quad \left\| \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix} \right\|_{L^2(0,2\pi;\mathbb{R}^2)}$$

Theorem 2 (Achleitner-AA-Carlen 2016)

Let $\int_0^{2\pi} [f_+^I(x) + f_-^I(x)]\, dx = 2\pi.$

$$\Rightarrow \quad \|f(t) - f^\infty\|_{L^2(0,2\pi;\mathbb{R}^2)} \le \sqrt{3}\, e^{-t/2} \|f^I - f^\infty\|_{L^2(0,2\pi;\mathbb{R}^2)}, \quad t \ge 0.$$

- *Decay rate* $\frac{1}{2}$ *sharp.*

- $\sqrt{3} = \mathrm{cond}(\mathbf{P}_1)$, due to equivalence of $\|.\|_{\mathbf{P}_1}$ and $\|.\|_2$

2) Linear continuous velocity BGK model: Fourier modes

$$f_t + v f_x = M_T(v) \int_{\mathbb{R}} f(x,v,t)\,dv - f(x,v,t), \quad t \geq 0$$

• x-Fourier modes $k \in \mathbb{Z}$ decouple; Hermite function basis in v

$\Rightarrow$ consider "infinite vector" $\hat{\mathbf{f}}_k(t) \in \ell^2(\mathbb{N}_0)$ for each fixed k:

$$\partial_t \hat{\mathbf{f}}_k + ik\sqrt{T}\,\mathbf{L}_1 \hat{\mathbf{f}}_k = \mathbf{L}_2 \hat{\mathbf{f}}_k\,, \quad t \geq 0;\ k \in \mathbb{Z}\,,$$

$$\mathbf{L}_1 = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & \ddots \end{pmatrix}, \quad \mathbf{L}_2 = \mathrm{diag}(0, -1, -1, \cdots)$$

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- need: spectral gap of the "infinite matrix" $\mathbf{C}_k := ik\sqrt{T}\,\mathbf{L}_1 - \mathbf{L}_2$, uniform in $k \in \mathbb{Z}$.
- difficult for ∞-dimensional case

Continuous velocity BGK model: decay of Fourier modes

$\rightarrow$ approximate matrices $\mathbf{P}_k$: ansatz for upper left 2×2 block:

$$\mathbf{P}_k^{2\times 2} = \begin{pmatrix} 1 & -i\alpha/k \\ i\alpha/k & 1 \end{pmatrix}, \quad \text{remaining diagonal} = 1.$$

Motivation: mix 0^{th}, 1^{st} mode of $\mathbf{L}_2 = \mathrm{diag}(0, -1, \cdots)$

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Lyapunov fct: $\quad e(f) := \sum_{k\in\mathbb{Z}} \left\langle (f_k(v) - M_T(v)), \tilde{\mathbf{P}}_k (f_k(v) - M_T(v)) \right\rangle_{L^2(M_T^{-1})}$

$\tilde{\mathbf{P}}_k$ ... bounded operator on $L^2(M_T^{-1})$, represented by matrix $\mathbf{P}_k$.

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Theorem 3

Let $T = 1$. $\Rightarrow \|f(t) - f_\infty\|_{L^2((0,2\pi)\times\mathbb{R};M_1(v)^{-1})}$ decays with rate = 0.273...

This estimated rate is off by factor 2.5 (compared to numerical result).

3) True BGK model, linearized around f_∞

Let $T = 1$, $d = 1$:

$$\partial_t \hat{\mathbf{h}}_k + ik\, \mathbf{L}_1 \hat{\mathbf{h}}_k = \mathbf{L}_4 \hat{\mathbf{h}}_k\,, \quad k \in \mathbb{Z}; \tag{3}$$

$$\mathbf{L}_4 = \operatorname{diag}(0, 0, 0, -1, -1, \cdots)$$

- 3 conserved quantities (mass, mean velocity, temperature)
- iterated hypocoercive structure: $\circ\!\longrightarrow\!\circ\!\longrightarrow\!\circ\!\longrightarrow\!\bullet$ (index $= 3$): Tridiagonal matrix $\mathbf{L}_1$ couples mass-mode to velocity-mode to energy-mode to the dissipative mode 3.

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$\Rightarrow$ simplified ansatz for $\mathbf{P}_k$ (with some α, β, $\gamma \in \mathbb{R}$) - in analogy to $\mathbf{L}_1$:

$$\begin{pmatrix} 1 & -i\alpha/k & 0 & 0 \\ i\alpha/k & 1 & -i\beta/k & 0 \\ 0 & i\beta/k & 1 & -i\gamma/k \\ 0 & 0 & i\gamma/k & 1 \end{pmatrix}$$

as its upper-left 4×4 block; rest is $\mathbf{I}$.

Theorem: exponential decay for (3) [also for $d = 2, 3$: index $= 2$]

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review of entropy method / exponential convergence to f_∞: 1) for linear symmetric Fokker-Planck equations

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$
\begin{aligned}
f_t &= \operatorname{div}\Big(\mathbf{D} \cdot [\nabla f + f \nabla A(x)]\Big) =: Lf \\
f(x,0) &= f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n), \int_{\mathbb{R}^n} f_0 \, dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0
\end{aligned}
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$$\begin{aligned} f_\infty(x) &= e^{-A(x)} \ \ldots \text{ (unique) normalized steady state} \\ Lf &= \operatorname{div}\Big(f_\infty \mathbf{D} \nabla \frac{f}{f_\infty}\Big) \ \ldots \text{ symmetric in } L^2(\mathbb{R}^n, f_\infty^{-1}) \\ \mathbf{D} > 0 &\ \ldots \text{ positive definite matrix} \\ A(x) &\ \ldots \text{ scalar confinement potential, i.e. } A(x) \to \infty \text{ as } |x| \to \infty; \\ & \qquad \textit{idea}: A(x) \gtrsim c|x|^2 \end{aligned}$$

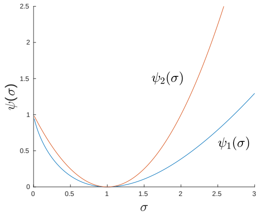
admissible relative entropies (for entropy method)

"nonsymmetric distance" of probability densities $f_{1,2}$:

$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^n} \psi\left(\frac{f_1}{f_2}\right) f_2 \,\mathrm{d}x \geq 0 \qquad \ldots \text{ relative entropy}$$

$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ ... entropy generators

$$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \tfrac{1}{2}\psi''\psi^{\mathrm{IV}}$$



$$e_\psi(f_1|f_2) = 0 \iff f_1 = f_2$$

admissible relative entropies (for entropy method)

"nonsymmetric distance" of probability densities $f_{1,2}$:

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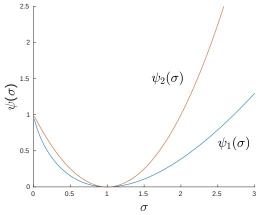

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examples: 1) $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$ (Boltzmann entropy)

2) $\psi_p(\sigma) = \sigma^p - 1 - p\,(\sigma - 1), \quad 1 < p \leq 2$

entropy dissipation

Lemma 3

Let $f(t)$ solve Fokker-Planck equation $f_t = div\Big(\mathbf{D} \cdot [\nabla f + f \nabla A(x)]\Big)$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|f_\infty) \quad = \quad -\int_{\mathbb{R}^n} \psi''\left(\frac{f(t)}{f_\infty}\right) \nabla^\top \frac{f(t)}{f_\infty} \cdot \mathbf{D} \cdot \nabla \frac{f(t)}{f_\infty} f_\infty \, \mathrm{d}x$$

$$=: \quad -I_\psi(f(t)|f_\infty) \leq 0 \text{ ... (negative) Fisher information}$$

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Goal: quantify exponential decay rate

Step 1: exponent. decay of entropy dissipation for $\mathbf{D} \equiv$const

Let $\mathbf{D}$ = const. in x matrix. $\quad f_\infty(x) = e^{-A(x)}$

Theorem 4

Let $I_\psi(f_0|f_\infty) < \infty$. Let $\mathbf{D}, A$ satisfy a

$$\textcolor{red}{\textit{Bakry - Emery condition}} \qquad \frac{\partial^2 A(x)}{\partial x^2} \geq \underbrace{\lambda_1}_{>0} \mathbf{D}^{-1} \qquad \forall x \in \mathbb{R}^n$$

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} I_\psi(f_0|f_\infty), \qquad t \geq 0$$

A ... uniformly convex if $\mathbf{D} = \mathbf{I}$

Ref's: [Bakry-Emery] 1984/85;
[Arnold-Markowich-Toscani-Unterreiter] Comm. PDE 2001

- robust w.r.t. many nonlinear perturbations

Step 2: exponential decay of relative entropy for $\mathbf{D} \equiv$const

Theorem 5

Let $\mathbf{D}$, A *satisfy Bakry - Emery condition:* $\frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

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Proof: from proof of Theorem 4 :

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leq -2\lambda_1 \underbrace{I(t)}_{=-e'(t)} \qquad \Big| \int_t^\infty \ldots \,\mathrm{d}t$$

Since $I(t)$, $e(t) \xrightarrow{t\to\infty} 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) \leq -2\lambda_1 e(t)$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) \leq -2\lambda_1 e(t) \qquad (= \text{Log. Sobolev-type inequality}) \quad \square \tag{4}$$

Cor: same for $f_t = \operatorname{div}\Big(\mathbf{D} \cdot [\nabla f + f\{\nabla A(x) + \vec{F}(x)\}]\Big)$ with $\operatorname{div}(\mathbf{D} \cdot \vec{F} f_\infty) = 0$

functional inequalities – as by-product of entropy method

consider entropy inequality (4) for log. entropy at t=0 : $e(0) \leq \frac{1}{2\lambda_1} \underbrace{I(0)}_{=-e'(0)}$

i.e.
$$\int_{\mathbb{R}^n} \frac{f_0}{f_\infty} \ln \frac{f_0}{f_\infty} f_\infty \, \mathrm{d}x \leq \frac{4}{2\lambda_1} \int_{\mathbb{R}^n} \left| \nabla \sqrt{\frac{f_0}{f_\infty}} \right|^2 f_\infty \, \mathrm{d}x$$

$\forall$ probability densities f_0, (log-concave) f_∞ . Evolution no longer needed !

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with $f^2 := \frac{f_0}{f_\infty} \Rightarrow$

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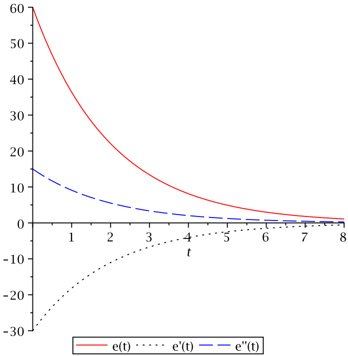
logarithmic Sobolev inequality

ex: $A(x) = c + \frac{|x|^2}{2a} \;\Rightarrow\; f_\infty(x) = (2\pi a)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x|^2}{2a}} =: M_a(x), \;\; \lambda_1 = \frac{1}{a}$

Ref: [Federbush 1969], [Gross 1975]

problem of entropy decay (cp. standard entropy method)

decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|^2_{L^2(f_\infty^{-1})}$:



standard Fokker-Planck equation:
non-degenerate $\rightarrow$ $e(t)$ is convex;
entropy dissip. $e'(t) < 0 \ \forall\, f \neq f_\infty$;
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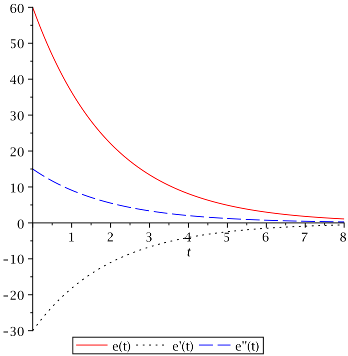

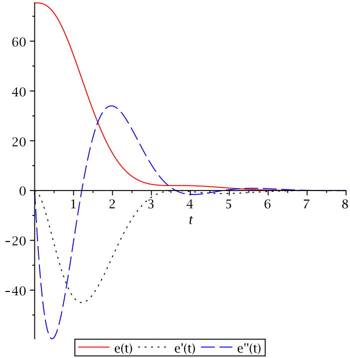

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non-degenerate → $e(t)$ is convex;
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$e' \leq -\mu e$ possible (with $\mu > 0$)

degenerate FP eq. with $\mathbf{D} \geq 0$:
→ $e(t)$ is not convex;
$e'(t) = 0$ for some $f \neq f_\infty$;
$e' \leq -\mu e$ wrong (in general)

2) degenerate Fokker-Planck equation: conditions

$$f_t = \text{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: Lf\,, \quad 0 \le \mathbf{D} \in \mathbb{R}^{n\times n} \text{ ... degenerate FP}$$

Condition A: No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
(equivalent: L is hypoelliptic.)

Proposition 1

Let Condition A hold.

a) *Let* $f_0 \in L^1(\mathbb{R}^d) \quad \Rightarrow \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. *[Hörmander 1969]*

b) *Let* $f_0 \in L^1_+(\mathbb{R}^d) \quad \Rightarrow \quad f(x,t) > 0,\ \forall t > 0$. *(Green's fct > 0)*

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Condition B: Cond. A + let $\mathbf{C} \in \mathbb{R}^{n\times n}$ be positively stable (i.e. $\Re\lambda_C > 0$)
$\to \exists$ confinement potential; drift towards $x = 0$.

- hypoelliptic + confinement = hypocoercive (for FP eq.)

steady state

$$f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \tag{5}$$

Theorem 6

(5) *has a unique (normalized) steady state* $f_\infty \in L^1(\mathbb{R}^n)$ *iff Condition B holds.*

Then: $f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$ ... *non-isotropic Gaussian*

$0 < \mathbf{K} \in \mathbb{R}^{n\times n}$ ... *unique solution of* $2\mathbf{D} = \mathbf{C}\mathbf{K} + \mathbf{K}\mathbf{C}^\top$

(continuous Lyapunov equation)

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• via a coordinate transformation, $\mathbf{K}$ can be chosen as $\mathbf{I}$. Then $\mathbf{D} = \mathbf{C}_s$.
(assumed from now on)

new entropy method for degen. FP: $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f)$

- $e'(t) = 0$ for some $f \neq f_\infty \quad \Rightarrow \quad$ entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi = -\int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \,\mathrm{d}x =: -I_\psi(f) \leq 0$$

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$\Rightarrow$ define modified "entropy dissipation" as auxiliary functional:

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{P}}_{>0} \cdot \nabla \frac{f}{f_\infty}\, f_\infty \,\mathrm{d}x \geq 0$$

goal: estimate between $S(f(t))$, $\frac{\mathrm{d}}{\mathrm{d}t} S(f(t))$ for "good" choice of $\mathbf{P} > 0$.
Then:

$$\mathbf{P} \geq c_P\, \mathbf{D} \quad \Rightarrow \quad S_\psi(f) \geq c_P\, I_\psi(f) \quad \searrow 0$$

Step 1: exponential decay of auxiliary functional $S_\psi(f)$

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \mathbf{P} \cdot \nabla \frac{f}{f_\infty}\, f_\infty \, \mathrm{d}x \geq 0$$

with **P** from Lemma 1: $\mathbf{PC} + \mathbf{C}^\top \mathbf{P} \geq 2\mu \mathbf{P}$

Proposition 2

$\mu := \min\{\Re \lambda_C\}$. *Let f_0 satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) \left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \, \mathrm{d}x < \infty \quad (\sim \textit{ weighted } H^1\textit{–seminorm})$$

1. *If all $\lambda_C^{\min}$ are non-defective* $\Rightarrow \quad S(f(t)) \leq e^{-2\mu t} S(f_0)\,, \quad t \geq 0;$

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2. *If one $\lambda_C^{\min}$ is defective* $\Rightarrow \quad S(f(t),\varepsilon) \leq e^{-2(\mu-\varepsilon)t} S(f_0,\varepsilon)\,, \quad t \geq 0.$

- spectral gap of **C** $\to$ decay rate $= \mu$

Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t}S(f(t)) = -\int \psi''\Big(\frac{f}{f_\infty}\Big)u^\top \underbrace{\overbrace{\big[\,\mathbf{PC} + \mathbf{C}^\top\mathbf{P}\,\big]}^{\text{replaces BEC}}}_{\geq 2\mu\mathbf{P}\ \ldots\ \text{by Lemma 1}} u f_\infty \, \mathrm{d}x$$

$$-2\int \underbrace{\mathrm{Tr}\,(\mathbf{XY})}_{\geq 0} f_\infty \, \mathrm{d}x \leq -2\mu\, S(f(t))\,; \quad u := \nabla \frac{f}{f_\infty}$$

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with Cauchy-Schwarz

Step 2: exponential decay of relative entropy

Theorem 7

Let f_0 satisfy:

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) |u_0|^2 \, f_\infty \, \mathrm{d}x < \infty \, .$$

$$\Rightarrow \quad \boxed{e(f(t)|f_\infty) \leq c\, S(f(t)) \leq c\, e^{-2\mu t} S(f_0)\,, \quad t \geq 0}$$

(reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$)

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Proof: Consider non-degenerate (auxiliary) symmetric FP equation:

$$g_t = \operatorname{div}\Big(\underbrace{\mathbf{P}}_{>0}(\nabla \frac{g}{f_\infty}) f_\infty\Big)\,; \quad g_\infty = f_\infty = c\, e^{-|x|^2/2} = c\, e^{-A(x)} \qquad (6)$$

It satisfies the Bakry-Emery condition $\frac{\partial^2 A}{\partial x^2} = \mathbf{I} \geq \lambda_P \mathbf{P}^{-1}$.

$$\Rightarrow \quad \text{convex Sobolev inequality:} \quad e_\psi(g|f_\infty) \leq \frac{1}{2\lambda_P} S_\psi(g) \quad \forall\, g$$

Remark: $S_\psi(g)$ is the true entropy dissipation for (6) !

exp. decay of rel. entropy for $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of Th.7 with parabolic regularization for initial time $\Rightarrow$

Theorem 8 (Arnold-Erb 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\,\lambda_C\}$. $\Rightarrow \exists\, c > 0$:

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Proof:

$$e(t) \overset{\text{CSI}}{\le} \frac{1}{2\lambda_P} S(f(t)) \overset{\text{decay}}{\le} \frac{1}{2\lambda_P} \mathrm{e}^{-2\mu(t-\delta)} S(f(\delta)) \overset{\text{regularization}}{\le} c(\delta)\, e^{-2\mu t} \mathrm{e}\,(0)$$

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Remark: Rate μ is sharp, but constant c is not.

3) kinetic Fokker-Planck eq. with non-quadratic potential

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \sigma \Delta_v f + \nu \operatorname{div}_v(vf)\,; \quad x, v \in \mathbb{R}^n$$

steady state factors in x, v: $\quad f_\infty(x,v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2} + V(x)\right]}$

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Theorem 9 (Arnold-Erb, Achleitner-Arnold 2015)

Let $n=1$; $V(x) = \frac{\omega_0^2}{2}|x|^2 + \tilde V(x)$ *... given confinement potential with*
$\sqrt{\max V''(x)} - \sqrt{\min V''(x)} \le \nu$

$$\Rightarrow \qquad e_\psi(f(t)|f_\infty) \le c\, S_\psi(f_0) e^{-2\tilde\mu t}, \qquad t \ge 0 \qquad \text{[explicit rate } \tilde\mu\text{]}$$

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- $\tilde V$ ... $\mathcal{O}(1)$ perturbation
- Greens function not explicit

4) local vs. global decay rate in non-symmetric FP eq's

decay of logarithmic entropy for the non-symmetric FP equation

$$f_t \;=\; \operatorname{div}\Big[\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} \nabla f + \begin{pmatrix} 1/4\, x_1 - 4\, x_2 \\ 4\, x_1 + x_2 \end{pmatrix} f \Big], \quad \mathbf{D} = \mathbf{C}_s$$

with $f_\infty(x) = c\, \mathrm{e}^{-|x|^2/2} = \mathrm{e}^{-A(x)}$: find $e(f(t)|f_\infty) \le c\, e(0)\, \mathrm{e}^{-\lambda t}$

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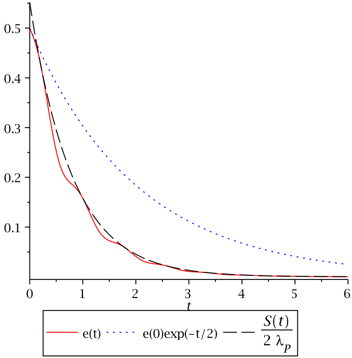

standard entropy method:
BEC $\frac{\partial^2 A}{\partial x^2} = \mathbf{I} \ge \lambda \mathbf{D}^{-1}$ yields sharp
local decay rate: $\lambda_l = \frac{1}{4}; \quad c = 1$

"Hypocoercive method" yields sharp
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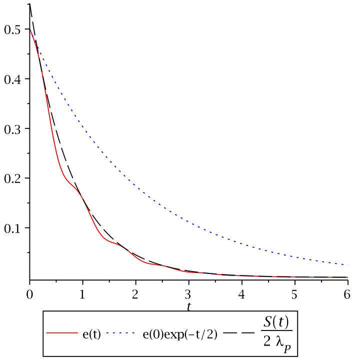

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Lemma: 2D, ∀ admissible $\mathbf{P}$:
multiplicative constant $\frac{S(0)}{2\lambda_P}$ is sharp.

Why does the "hypocoercive method" work?

- $Lf := \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}xf)$ is NOT symmetric, but has a complete sequence of mutually orthogonal eigenspaces (in $L^2(f_\infty^{-1})$).

- evolution in the first subspace is given by $\dot{y}_1 = -\mathbf{C}y_1$, $y_1 \in \mathbb{R}^n$
- evolution in the second subspace is given by $\dot{Y}_2 = -\mathbf{C}Y_2 - Y_2\mathbf{C}^T$
  $+$ tensored generalizations for higher subspaces

- Sharp decay of $\dot{x} = -\mathbf{C}x$ carries over to sharp L^2-decay of FP eq.
  – when using the $\|.\|_{\mathbf{P}}$.

Conclusion

- Algebraic lemma $\mathbf{PC} + \mathbf{C}^*\mathbf{P} \geq 2\mu\mathbf{P}$ yields the generic decay norm $\|.\|_{\mathbf{P}}$ for linear ODEs.
- exact / approximate generalizations to kinetic BGK-models (by modal decomposition):
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