Dirichlet problem for supercritical non-local operators

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(A joint work with Guohuan Zhao)

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Introduction

- Consider the following classical Dirichlet problem in a bounded C^2-domain $D \subset \mathbb{R}^d$:

$$\Delta u + f = 0 \;\text{ in } D \text{ and } u|_{\partial D} = 0,$$

where f is a continuous function on $\bar{D}$.

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- Microscopic interpretation for u?

- Let W be a d-dimensional standard Brownian motion on some complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbf{P})$.

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- Let $X_t^x := \sqrt{2}W_t + x$ and define the exiting time from domain D by

$$\tau_D^x := \inf\{t > 0 : X_t^x \notin D\} = \inf\{t > 0 : X_t^x \in \partial D\}.$$

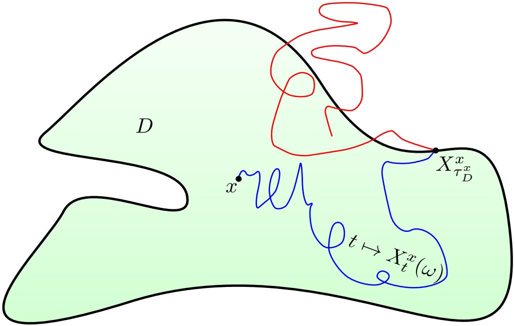

Figure 1 : Exiting time

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- Indeed, letting $D_n \uparrow\uparrow D$, by Itô's formula, we have

$$u(X^x_{\tau^x_{D_n}}) = u(x) + \int_0^{\tau^x_{D_n}} \nabla u(X_s^x)\mathrm{d}W_s + \int_0^{\tau^x_{D_n}} \Delta u(X_s^x)\mathrm{d}s.$$

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- Taking expectations and limits $n \to \infty$, we obtain

$$0 = \mathbb{E}u(X^x_{\tau^x_D}) = u(x) - \mathbb{E}\left(\int_0^{\tau_D^x} f(X_s^x)\mathrm{d}s\right).$$

- Consider the following SDE with jumps:

$$\mathrm{d}X_t^x = b(X_t^x)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)}, \quad X_0^x = x, \ \alpha \in (0, 2),$$

where $L_t^{(\alpha)}$ is a rotationally invariant and symmetric α-stable process, and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz vector field.

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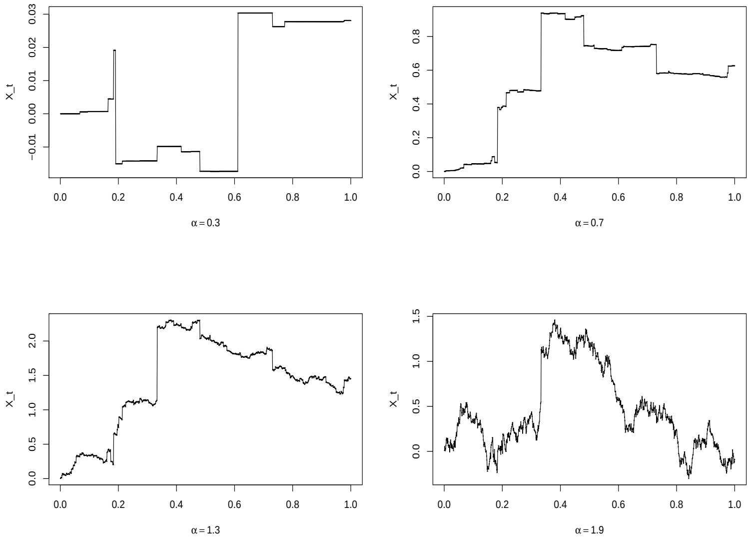
- The generator of this SDE is given by

$$\mathrm{d}\mathbb{E}f(X_t^x)/\mathrm{d}t|_{t=0} = \Delta^{\frac{\alpha}{2}} f(x) + b \cdot \nabla f(x) =: \mathscr{L}_b^{(\alpha)} f(x),$$

where $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$ is the usual fractional Laplacian operator.

- Notice that the set $\{t > 0 : X_t^x \neq X_{t-}^x\}$ is a countable dense subset of $(0, \infty)$.



- Let $\tau_D^x := \inf\{t > 0 : X_t^x \notin D\} = \inf\{t > 0 : X_t^x \in D^c\}$.

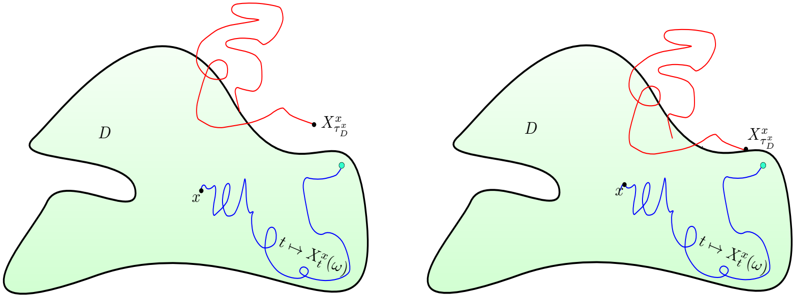


Figure 2 : Exiting time

We are interesting in the following problem:

- Let $f \in C_b^0(\bar{D})$. Is it possible to show that

$$u(x) := \mathbb{E}\left(\int_0^{\tau_D^x} f(X_s^x)\mathrm{d}s\right)$$

uniquely solves the following nonlocal elliptic Dirichlet problem?

$$\Delta^{\frac{\alpha}{2}} u + b \cdot \nabla u + f = 0 \ \text{ in } D \text{ and } u = 0 \text{ in } D^c.$$

- By the scaling property of $\Delta^{\frac{\alpha}{2}}$, it is easy to see that $u_R(x) := R^{-\alpha} u(Rx + x_0)$

$$\Delta^{\frac{\alpha}{2}} u_R + R^{\alpha-1} b_R \cdot \nabla u_R = -f_R \text{ in } B_R(x_0),$$

where $B_R(x_0) := \{x \in \mathbb{R}^d : |x - x_0| < R\}$.

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- For $\alpha \in (0, 1)$, if $R \to 0$, then the drift term will blow up, so roughly speaking, the first order term plays a dominant role. In this sense we call $L_b^{(\alpha)}$ with $\alpha \in (0, 1)$ the supercritical nonlocal operator.

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- For $\alpha = 1$, since $\Delta^{\frac{1}{2}}$ has the same order as $b \cdot \nabla$, we shall call $L_b^{(1)}$ the critical operator.

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- For $\alpha \in (1, 2)$, if $R \to 0$, then the drift term will go to zero and $\Delta^{\frac{\alpha}{2}}$ plays a dominant role, it is naturally called subcritical operator.

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- Ros Oton-Serra (2014, 2016): Optimal boundary regularity for $\Delta^{\frac{\alpha}{2}}|_D$.
- $\cdots\cdots$

Difficulty in the supercritical case

Proposition 1 ($X^x_t := x + L^{(\alpha)}_t + t$)

Let $D := (0, 1)$ and $\alpha \in (0, 2)$. It holds that for $\alpha \in [1, 2)$,

$$\text{(i) } \mathbb{P}(X^x_{\tau^x_D} = 0 \textit{ or } 1) = 0, \ \text{(ii) } \mathbb{E}\tau^x_D \leqslant c_\alpha d_x^{\alpha/2}, \quad x \in D,$$

where $d_x := (x \wedge (1 - x))_+$ is the distance of x to D^c; and for $\alpha \in (0, 1)$,

$$\text{(iii) } \sup_{x\in D} \mathbb{P}(X^x_{\tau_D} = 1) > 0, \ \text{(iv) } \sup_{x\in D} \mathbb{P}(X^x_{\tau^x_D} = 0) = 0, \ \text{(v) } \inf_{x\in(0,1/4)} \mathbb{E}\tau^x_D > 0.$$

Main results

- Consider the following nonlocal parabolic Dirichlet problem:

$$\begin{cases} \partial_t u = \mathscr{L}_\kappa^{(\alpha)} u + b \cdot \nabla u + f \ \text{ on } \ \mathbb{R}_+ \times D, \\ u = 0 \ \text{ on } \ \mathbb{R}_+ \times D^c, \quad u(0,\cdot) = \varphi \text{ on } D, \end{cases} \tag{2.1}$$

where $\mathscr{L}_\kappa^{(\alpha)}$ is the nonlocal operator defined by

$$\mathscr{L}_\kappa^{(\alpha)} u(x) := \int_{\mathbb{R}^d} \Big(u(x+z) - u(x) - z^{(\alpha)} \cdot \nabla u(x) \Big) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \mathrm{d}z.$$

Here $\mathbb{R}_+ := [0,\infty)$, $\alpha \in (0,2)$ and

$$z^{(\alpha)} := \mathbf{1}_{\alpha=1} z \mathbf{1}_{\{|z|\leqslant 1\}} + \mathbf{1}_{\alpha \in (1,2)} z.$$

- We always assume $\kappa(x, z) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ satisfies that for some $\kappa_0 > 0$ and $\beta \in (0, 1)$,

$$\left[\begin{array}{c} \kappa_0^{-1} \leqslant \kappa(x, z) \leqslant \kappa_0,\ |\kappa(x, z) - \kappa(x', z)| \leqslant \kappa_0 |x - x'|^\beta \\ \mathbf{1}_{\alpha=1} \int_{r<|z|<R} z \cdot \kappa(x, z) \mathrm{d}z = 0,\ 0 < r < R < \infty \end{array}\right]. \qquad (\mathbf{H}_\kappa^\beta)$$

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$$\begin{bmatrix} \kappa_0^{-1}\leqslant\kappa(x,z)\leqslant\kappa_0,\ |\kappa(x,z)-\kappa(x',z)|\leqslant\kappa_0|x-x'|^\beta \\ \mathbf{1}_{\alpha=1}\int_{r<|z|<R} z\cdot\kappa(x,z)\mathrm{d}z=0,\ 0<r<R<\infty \end{bmatrix}. \qquad (\mathbf{H}_\kappa^\beta)$$

Definition 2

We call a function $u\in L^\infty_{loc}(\mathbb{R}_+;L^\infty(\mathbb{R}^d))\cap C(\mathbb{R}_+\times D)$ a classical solution of Dirichlet problem (2.1) if $u|_{\mathbb{R}_+\times D^c}=0$ and it satisfies

$$u(t,x)=\varphi(x)+\int_0^t\left(\mathscr{L}^{(\alpha)}_{\kappa,b}u+f\right)(s,x)\mathrm{d}s \text{ in } \mathbb{R}_+\times D$$

in the pointwise sense, where $\mathscr{L}^{(\alpha)}_{\kappa,b}:=\mathscr{L}^{(\alpha)}_\kappa+b\cdot\nabla$.

- Let D be a bounded domain. For $x, y \in D$, define

$$d_x := \mathrm{dist}(x, \partial D), \quad d_{x,y} := \min\{d_x, d_y\}.$$

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- For $\theta \in \mathbb{R}$, $k \in \{0\} \cup \mathbb{N}$ and $0 < \gamma \notin \mathbb{N}$, define

$$[u]^{(\theta)}_{k;D} := \sup_{x \in D} d_x^{k+\theta} |\nabla^k u(x)|$$

and

$$[u]^{(\theta)}_{\gamma;D} := \sup_{x,y \in D} \left(d_{x,y}^{\gamma+\theta} \frac{|\nabla^{[\gamma]} u(x) - \nabla^{[\gamma]} u(y)|}{|x-y|^{\gamma-[\gamma]}} \right).$$

- For general $\gamma \geqslant 0$ with $\gamma + \theta \geqslant 0$, we introduce the Banach space:

$$\mathcal{C}^{(\theta)}_{\gamma}(D) := \left\{ u \in C^{\gamma}(D) \cap L^{\infty}(\mathbb{R}^d) : \|u\|^{(\theta)}_{\gamma;D} < \infty, \ u|_{D^c} = 0 \right\},$$

where

$$\|u\|^{(\theta)}_{\gamma;D} := [u]^{(\theta)}_{0;D} + [u]^{(\theta)}_{\gamma;D}.$$

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where

$$\|u\|^{(\theta)}_{\gamma;D} := [u]^{(\theta)}_{0;D} + [u]^{(\theta)}_{\gamma;D}.$$

- For $T > 0$,

$$\mathbb{B}^{(\theta)}_{\gamma;T}(D) := L^{\infty}([0,T]; \mathcal{C}^{(\theta)}_{\gamma}(D)),\ \ \mathbb{B}^{(\theta)}_{\gamma}(D) := \cap_{T>0} \mathbb{B}^{(\theta)}_{\gamma;T}(D).$$

Theorem 3 (Sub and critical cases)

Let D be a bounded C^2-domain and $\alpha \in (0,2)$, $\beta \in (0,1)$. Suppose ($\mathbf{H}_\kappa^\beta$) *and*

$$b \in C^\beta \text{ if } \alpha \in [1,2), \text{ and } b = 0 \text{ if } \alpha \in (0,1).$$

Then there exists $\theta_0 \in (0, \frac{\alpha}{2})$ such that for any $\theta \in (0, \theta_0]$ and $\gamma \in (0, \beta]$ with $\alpha + \gamma \notin \mathbb{N}$, if one of the following two conditions holds:

$$(i)\ \theta \geqslant \gamma; \quad (ii)\ \theta < \gamma \ \text{ and } \ |\kappa(x,z) - \kappa(x,z')| \leqslant \kappa_1 |z - z'|^\gamma,$$

then for all $f \in \mathbb{B}_\gamma^{(\alpha-\theta)}(D)$ and $\varphi \in \mathcal{C}_{\alpha+\gamma}^{(-\theta)}(D)$, there is a unique classical solution $u \in \mathbb{B}_{\alpha+\gamma}^{(-\theta)}(D)$ to equation (2.1), and there is a constant $c = c(\kappa_0, \kappa_1, \gamma, \theta, \alpha, \beta, d, \|b\|_{C^\beta}) > 0$ such that for all $T > 0$,

$$\|u\|_{\mathbb{B}^{(-\theta)}_{\alpha+\gamma;T}(D)} + \|\partial_t u\|_{\mathbb{B}^{(\alpha-\theta)}_{\gamma;T}(D)} \leqslant c\|\varphi\|^{(-\theta)}_{\alpha+\gamma} + c(1+T)\|f\|_{\mathbb{B}^{(\alpha-\theta)}_{\gamma;T}(D)}.$$

(Sub and critical cases, Ctd.)

Moreover, the unique solution u has the following probabilistic representation:

$$u(t,x) = \mathbb{E}_x\Big(\varphi(X_t)\mathbf{1}_{\{\tau_D>t\}}\Big) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_D} f(t-s,X_s)\mathrm{d}s\right), \tag{2.2}$$

where $(X,\mathbb{P}_x; x\in\mathbb{R}^d)$ is the Markov process associated with $\mathscr{L}^{(\alpha)}_{\kappa,b}$ and $\tau_D := \inf\{t\geqslant 0 : X_t \notin D\}$ is the first exit time of X from D. We also have the following estimate:

$$\mathbb{E}_x\left(\int_0^{t\wedge\tau_D} |f(t-s,X_s)|\mathrm{d}s\right) \leqslant c d_x^\theta \|f\|_{L^\infty_t(\mathcal{C}_0^{(\alpha-\theta)}(D))}. \tag{2.3}$$

Remark: Notice that in the estimate (2.3), f is allowed to be explosive near the boundary.

Theorem 4 (Supercritical Case)

Let $\alpha, \beta \in (0,1)$ *with* $\alpha + \beta > 1$ *and* $\gamma \in (1-\alpha, \beta]$. *Suppose* $(\mathbf{H}^\beta_\kappa)$, $b \in C^\beta$ *and*

$$|\kappa(x,z) - \kappa(x,z')| \leqslant \kappa_1 |z - z'|^\gamma.$$

Let $\varphi \in \mathcal{C}^{(0)}_{\alpha+\gamma}(D)$ *and* $f \in \mathbb{B}^{(0)}_\gamma(D)$. *We have the following conclusions:*

(A) *Suppose that* $b(z_0) \cdot \vec{n}(z_0) < 0$ *for each* $z_0 \in \partial D$. *Equation* (2.1) *admits a unique solution*

$$u \in L^\infty_{loc}(\mathbb{R}_+; C^{\alpha+\gamma}_{loc}(D) \cap L^\infty(\mathbb{R}^d)) \cap C(\mathbb{R}_+; C^0(D)).$$

The unique solution u still has the probabilistic representation (2.2).

(Supercritical case, Ctd.)

(B) Suppose that $b(z_0) \cdot \vec{n}(z_0) = 0$ for each $z_0 \in \partial D$. Equation (2.1) admits a unique solution

$$u \in L^\infty_{loc}(\mathbb{R}_+; C^{\alpha+\gamma}_{loc}(D) \cap L^\infty(\mathbb{R}^d)) \cap C((0,\infty) \times \bar{D}).$$

Moreover, we also have the following boundary decay estimate: for some $\theta \in (0,1)$,

$$|u(t,x)| \leqslant c\Big(\|f\|_\infty + \|\varphi\|_\infty / t\Big) d_x^\theta, \quad t > 0, \; x \in D.$$

The unique solution u still has the probabilistic representation (2.2).

(Supercritical case, Ctd.)

(C) Suppose that $b(z_0) \cdot \vec{n}(z_0) > 0$ for each $z_0 \in \partial D$. Equation (2.1) admits a unique solution

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Moreover, we also have the following boundary decay estimate:

$$|u(t,x)| \leqslant c\Big(\|f\|_\infty + \|\varphi\|_\infty / t\Big) d_x, \quad t > 0, \ x \in D.$$

The unique solution u still has the probabilistic representation (2.2).

Theorem 5 (Mixed Case: $\alpha \in [\frac{1}{2}, 1)$)

Let $\beta \in [2(1-\alpha), 1]$ *and* $\gamma \in (1-\alpha, \beta]$. *Suppose* $(\mathbf{H}^{\beta}_{\kappa})$, $b \in C^{\beta}$ *and*

$$|\kappa(x,z) - \kappa(x,z')| \leqslant \kappa_1 |z - z'|^{\gamma}.$$

Let D *be a bounded* C^2*-domain and define*

$$\begin{aligned} \Gamma_{>} &:= \{z \in \partial D : b(z) \cdot \vec{n}(z) > 0\}, \\ \Gamma_{=} &:= \{z \in \partial D : b(z) \cdot \vec{n}(z) = 0\}, \\ \Gamma_{<} &:= \{z \in \partial D : b(z) \cdot \vec{n}(z) < 0\}. \end{aligned}$$

For any $\varphi \in \mathcal{C}^{(0)}_{\alpha+\gamma}(D)$ *and* $f \in \mathbb{B}^{(0)}_{\gamma}(D)$, *there is a unique solution* u *to* (2.1) *in the class that*

$$u \in L^{\infty}_{loc}(\mathbb{R}_+; C^{\alpha+\gamma}_{loc}(D) \cap L^{\infty}(\mathbb{R}^d)) \cap C([0,\infty) \times D) \cap C((0,\infty) \times (D \cup \Gamma_{\geqslant})),$$

where $\Gamma_{\geqslant} := \Gamma_{=} \cup \Gamma_{>}$.

Ctd.

Moreover, the probabilistic representation (2.2) holds and

(i) For each $z \in \Gamma_>$, there are $\delta, c > 0$ such that

$$\sup_{x\in D\cap B_\delta(z)} (d_x^{-1}|u(t,x)|) \leqslant c\Big(\|f\|_\infty + \|\varphi\|_\infty/t\Big), \quad t > 0.$$

(ii) For each $z \in \Gamma_=^o$ (the interior of $\Gamma_=$), there are $\theta, \delta, c > 0$ such that

$$\sup_{x\in D\cap B_\delta(z)} (d_x^{-\theta}|u(t,x)|) \leqslant c\Big(\|f\|_\infty + \|\varphi\|_\infty/t\Big), \quad t > 0.$$

(iii) For each $x \in D$, it holds that

$$\mathbb{P}_x(X_{\tau_D} \in \Gamma_<) = 0,$$

where $(X, \mathbb{P}_x; x \in \mathbb{R}^d)$ is the Markov process associated with $\mathscr{L}^{(\alpha)}_{\kappa,b}$ and $\tau_D := \inf\{t \geqslant 0 : X_t \notin D\}$.

Example

Let $d = 1$ and $D = (0,1)$. Let Z_t be an one-dimensional symmetric α-stable process with $\alpha \in (0,1)$. For each $t > 0$ and $x \in D$, define

$$X_t^x := x\mathrm{e}^t + Z_t + \tfrac{\mathrm{e}^t - 1}{2} + \int_0^t Z_s \mathrm{e}^{t-s} \mathrm{d}s.$$

Notice that $X_t = X_t^x$ solves the following SDE:

$$\mathrm{d}X_t = (X_t - \tfrac{1}{2})\mathrm{d}t + \mathrm{d}Z_t, \quad X_0 = x.$$

Example

Define

$$u(t,x) := \mathbb{E}\Big(\sin\big(X_t^x\big)\mathbf{1}_{\{\tau_D^x>t\}}\Big),$$

where $\tau_D^x := \inf\{t \geqslant 0 : X_t^x \notin D\}$. By Theorem 4, for some $\eta > 1$ and any $0 < a < b < 1$, we have

$$u \in L^\infty_{loc}(\mathbb{R}_+; C^\eta((a,b))) \cap C((0,\infty) \times [0,1]),$$

and for any $t > 0$ and $x \in D$, it holds that

$$u(t,x) = \sin(x) + \int_0^t \Big(\Delta^{\frac{\alpha}{2}} u(s,x) + (x - \tfrac{1}{2})\partial_x u(s,x)\Big)\mathrm{d}s,$$

and

$$|u(t,x)| \leqslant c(x \wedge (1-x))_+/t, \ \ x \in D, t > 0.$$

Numerical Experimental Results: $D = (0, 1)$, $\alpha = 0.5$.

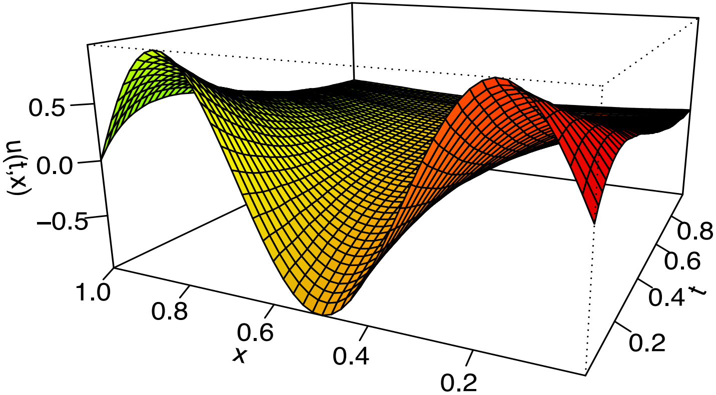

Figure 3 : Case A: $b(x) = \frac{1}{2} - x$, $\varphi(x) = \sin(3\pi x)$

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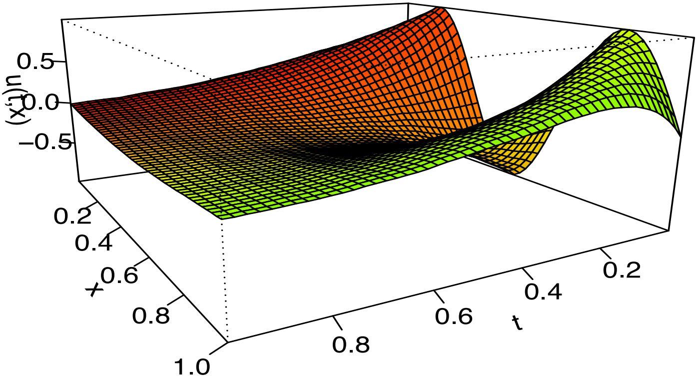

Figure 4 : Case A: $b(x) = \frac{1}{2} - x$, $\varphi(x) = \sin(3\pi x)$

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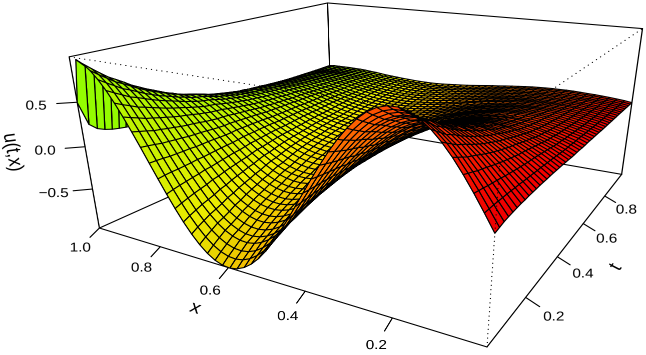

Figure 5 : Case B: $b(x) = 0$, $\varphi(x) = \sin(5\pi x/2)$

Numerical Experimental Results: $D = (0, 1)$, $\alpha = 0.5$.

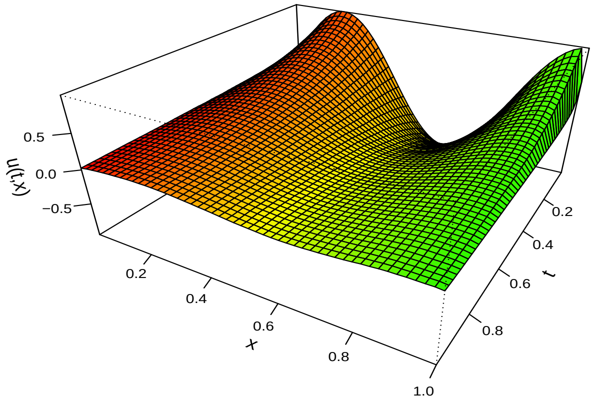

Figure 6 : Case B: $b(x) = 0$, $\varphi(x) = \sin(5\pi x/2)$

Numerical Experimental Results: $D = (0, 1)$, $\alpha = 0.5$.

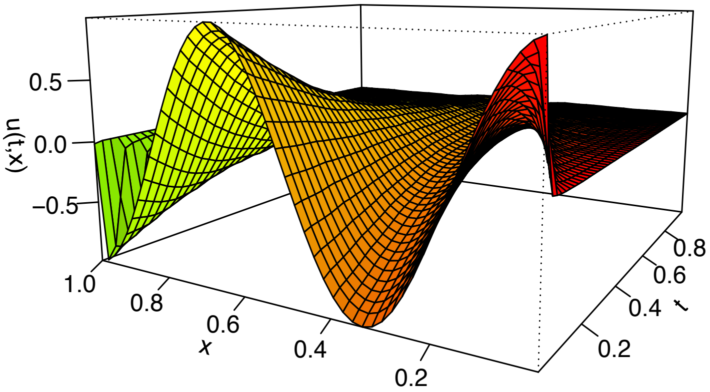

Figure 7 : Case C: $b(x) = x - \frac{1}{2}$, $\varphi(x) = \sin(3\pi x + \pi/2)$

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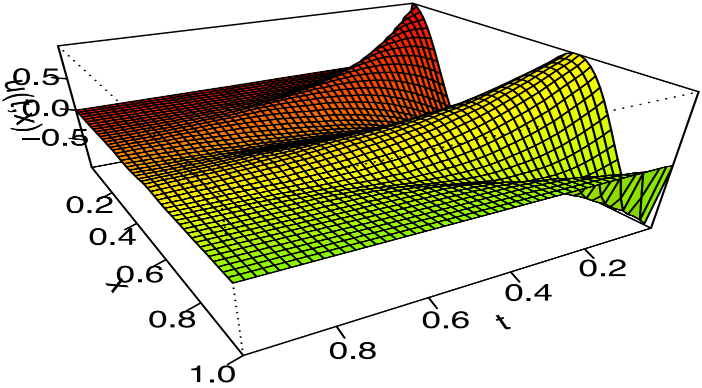

Figure 8 : Case C: $b(x) = x - \frac{1}{2}$, $\varphi(x) = \sin(3\pi x + \pi/2)$

Schauder's estimates

Theorem 6 (Global Schauder's estimates)

Let $\beta \in (0,1)$ *with* $\alpha+\beta>1$ *and* $\gamma \in (0,\beta]$ *with* $\alpha+\gamma \notin \mathbb{N}$. *Under* $(\mathbf{H}^{\beta}_{\kappa})$ *and* $b \in C^{\beta}$, *there are constants* $c = c(\kappa_0,\alpha,\beta,\gamma,d) > 0$ *and* $m = m(\alpha,\beta,\gamma) > 0$ *such that for all* $T>0$ *and* $u \in \mathbf{A}^{\alpha,\gamma}_T$,

$$\|u\|_{\mathbf{B}^{\alpha+\gamma}_T} \leqslant c\Big(\|u(0)\|_{C^{\alpha+\gamma}} + (1+\|b\|^m_{C^\beta})\|u\|_{\mathbf{B}^0_T} + \|f\|_{\mathbf{B}^\gamma_T}\Big), \tag{3.1}$$

where $f := \partial_t u - \mathscr{L}^{(\alpha)}_{\kappa} u - b\cdot\nabla u$ *and*

$$\mathbf{B}^\gamma_T := L^\infty_T(C^\gamma), \quad \mathbf{A}^{\alpha,\gamma}_T := \Big\{u \in \mathbf{B}^{\alpha+\gamma}_T, \partial_t u \in \mathbf{B}^\gamma_T\Big\}.$$

Theorem 6 (Global Schauder's estimates)

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where $f := \partial_t u - \mathscr{L}^{(\alpha)}_\kappa u - b\cdot\nabla u$ and

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Remark: When $\mathscr{L}^{(\alpha)}_\kappa = \Delta^{\alpha/2}$, the above Schauder estimate was established by Priola (2009) for $\alpha \in [1,2)$ and Silvestre (2012) for $\alpha \in (0,1)$.

Lemma 7 (Key tool for proving Schauder's estimate.)

Under $(\mathbf{H}^{\beta}_{\kappa})$ *with* $\kappa(x,z)=\kappa(z)$, *there is a constant* $c_0=c_0(\kappa_0,\alpha,d)>0$ *such that for all* $p\in[2,\infty)$ *and* $f\in C_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}|\Delta_j f|^{p-2}\Delta_j f\cdot\mathscr{L}^{(\alpha)}_{\kappa}\Delta_j f\mathrm{d}x\leqslant -c_0 2^{\alpha j}\|\Delta_j f\|_p^p,\quad j=0,1,2,\cdots,\qquad(3.2)$$

where Δ_j *is the block operator in Littlewood-Paley decomposition.*

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where Δ_j *is the block operator in Littlewood-Paley decomposition.*

Remark: Estimate (3.2) with constant c_0 depending on p was proved in [1] by using Bernstein's inequality established in [2].

[1] Chen, Z.Q., Zhang, X. and Zhao, G.: Well-posedness of supercritical SDE driven by Lévy processes with irregular drifts. *arXiv:1709.04632*, (2017).

[2] Chen, Q., Miao, C. and Zhang, Z.: A New Bernstein's Inequality and the 2D Dissipative Quasi-Geostrophic Equation. *Commun. Math. Phys.*, 271, pp821-838, (2007).

Theorem 8 (Schauder's interior estimates)

Let D be a bounded domain and $\alpha \in (0,2)$, $\beta \in (0,1)$. Suppose $(\mathbf{H}_\kappa^\beta)$ and $b \in C^\beta$. For given $\gamma \in (0,\beta]$ with $\alpha+\gamma \notin \mathbb{N}$ and $\theta \in [0, \alpha \wedge 1)$, let $u \in \mathbb{B}^{(-\theta)}_{\alpha+\gamma}(D)$ satisfy

$$\partial_t u = \mathscr{L}_\kappa^{(\alpha)} u + \mathbf{1}_{\alpha\in[1,2)} b \cdot \nabla u + f \text{ in } \mathbb{R}_+ \times D. \tag{3.3}$$

If one of the following two conditions holds:

$$(i)\ \theta \geqslant \gamma; \quad (ii)\ \ \theta < \gamma \text{ and } |\kappa(x,z) - \kappa(x,z')| \leqslant \kappa_1 |z-z'|^\gamma,$$

then there is a constant $c = c(d, \kappa_0, \kappa_1, \alpha, \beta, \gamma, \theta, \lambda_D) > 0$ such that

$$\|u\|_{\mathbb{B}^{(-\theta)}_{\alpha+\gamma;T}(D)} \leqslant c \left(\|u(0)\|_{\mathcal{C}^{(-\theta)}_{\alpha+\gamma}(D)} + \|f\|_{\mathbb{B}^{(\alpha-\theta)}_{\gamma;T}(D)} + \|u\|_{\mathbb{B}^{(-\theta)}_{0;T}(D)} \right), \tag{3.4}$$

provided that the right hand side is finite.

Theorem 9 (Viscosity approximation)

Let D be a bounded domain and $\alpha, \beta \in (0,1)$ with $\alpha + \beta > 1$. Suppose $(\mathbf{H}_\kappa^\beta)$ and $b \in C^\beta$. For $\gamma \in (1-\alpha, \beta]$, $\theta \in [0,1)$ and $\nu > 0$, let $u \in \mathbb{B}^{(-\theta)}_{1+\gamma}(D)$ satisfy

$$\partial_t u = \nu \Delta^{1/2} u + \mathscr{L}_\kappa^{(\alpha)} u + b \cdot \nabla u + f \text{ in } \mathbb{R}_+ \times D.$$

If in addition for some $\kappa_1 > 0$, $|\kappa(x,z) - \kappa(x,z')| \leqslant \kappa_1 |z - z'|^\gamma$, then there are $\theta_0 = \theta_0(\alpha, \beta, \gamma) > 0$ and $c = c(d, \kappa_0, \kappa_1, \alpha, \beta, \gamma, \theta_0, \lambda_D) > 0$ such that for all $T > 0$ and $\nu > 0$,

$$\|u\|_{\mathbb{B}^{(\theta_0)}_{\alpha+\gamma;T}(D)} \leqslant c \left(\|u(0)\|_{\mathcal{C}^{(\theta_0)}_{\alpha+\gamma}(D)} + \|f\|_{\mathbb{B}^{(\alpha+\theta_0)}_{\gamma;T}(D)} + \|u\|_{\mathbb{B}^{(0)}_{0;T}(D)} \right).$$

To present the probabilistic representation for nonlocal Dirichlet problem, we introduce the following class of functions pair: for $\gamma > 0$,

$$\mathbf{H}^\gamma(D) := \left\{ (u, f) \,\middle|\, \begin{array}{c} u \in L^\infty_{loc}(\mathbb{R}_+; C^\gamma_{loc}(D) \cap L^\infty(\mathbb{R}^d)) \cap C(\mathbb{R}_+ \times (\partial D)^c) \\ \partial_t u \in L^\infty_{loc}(\mathbb{R}_+; C_{loc}(D)), \quad f \in L^\infty_{loc}(\mathbb{R}_+ \times D) \end{array} \right\}.$$

Theorem 10 (Probabilistic representation)

Let D be a bounded domain and $(u, f) \in \mathbf{H}^{(\alpha+\varepsilon)\vee 1}(D)$ satisfy

$$\partial_t u = \mathscr{L}_\kappa^{(\alpha)} u + b \cdot \nabla u + f \ \text{ on } \ \mathbb{R}_+ \times D,$$

where κ satisfies $(\mathbf{H}_\kappa^\beta)$ and $b \in C^{\beta'}$ with $\alpha + \beta' > 1$. Suppose that $\partial D = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are two disjoint measurable sets, and

$$\mathbb{P}_x(X_{\tau_D} \in \Gamma_0) = 0, \quad x \in D,$$

and

$$u \in C((0,\infty) \times (D \cup \Gamma_1)) \text{ with } u|_{(0,\infty)\times\Gamma_1} = 0.$$

(Probabilistic representation, Ctd.)

Then for all $x \in \mathbb{R}^d$ and $t > 0$, it holds that

$$u(t,x) = \underbrace{\mathbb{E}_x\Big(u(0,X_t); t < \tau_D\Big)}_{\text{Initial value}} + \underbrace{\mathbb{E}_x\left(\int_0^{t\wedge\tau_D} f(t-s,X_s)\mathrm{d}s\right)}_{\text{Non-homeogenous term}}$$
$$+ \underbrace{\mathbb{E}_x\Big(u(t-\tau_D, X_{\tau_D}); t \geqslant \tau_D\Big)}_{\text{Exterior boundary term}}.$$

In particular, we have the following maximal principle:

$$\|u\|_{L^\infty_T(C^0(D))} \leqslant \|u(0)\|_{C^0(D)} + T\|f\|_{L^\infty_T(C^0(D))} + \|u\|_{L^\infty_T(C^0(\bar{D}^c))}.$$

Lemma 11 ($\alpha \in (0, 1)$: b towards the inside along the boundary)

Let $z_0 \in \partial D$. If $b(z_0) \cdot \vec{n}(z_0) < 0$, then there is a neighborhood $\Gamma \subset \partial D$ of z_0 such that for each $x \in D$,

$$\mathbb{P}_x(X_{\tau_D} \in \Gamma) = 0.$$

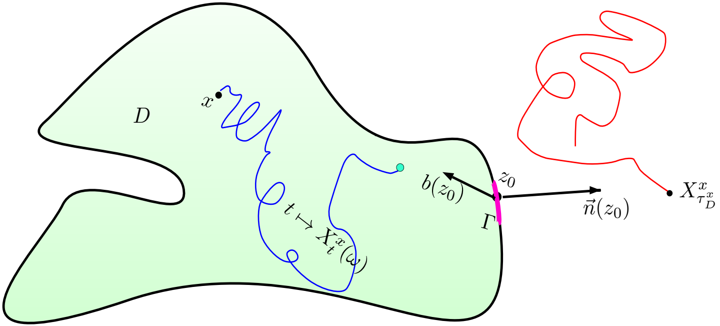

Figure 9 : Exiting time

Lemma 12 (*b* towards the outside along the boundary)

Let $z_0 \in \partial D$. *If* $b(z_0) \cdot \vec{n}(z_0) > 0$, *then there are* $\delta, C > 0$ *such that for all* $x \in D \cap B_\delta(z_0)$,

$$\mathbb{E}_x \tau_D \leqslant C d_{D^c}^{-1}(x).$$

Lemma 13 ($\alpha \in (0,1)$: b tangent to the boundary)

Let $z_0 \in \partial D$ and $\delta_0 > 0$. Suppose that

$$b(z) \cdot \vec{n}(z) = 0 \text{ for each } z \in \partial D \cap B_{\delta_0}(z_0)$$

Then there are $\delta, \theta, C > 0$ such that for all $x \in D \cap B_\delta(z_0)$,

$$\mathbb{E}_x \tau_D \leqslant C d_{D^c}^{-\theta}(x).$$

Remark: Let $\Gamma := \{z \in \partial D : b(z) \cdot \vec{n}(z) = 0\}$. For any interior point z_0 of closed set Γ, we have

$$\lim_{D \ni x \to z_0} \mathbb{E}_x \tau_D = 0.$$

For general $z_0 \in \partial\Gamma$, it is not known whether we have the above limit.

Lemma 14 ($\alpha \in (0,1)$: Continuous up to the boundary)

Let $\frac{1}{2} \leqslant \alpha < 1$ *and* $z_0 \in \partial D$. *Assume* $b(z_0) \cdot \vec{n}(z_0) = 0$ *and* $b \in C^\beta$ *with* $\beta \in [2(1-\alpha), 1]$. *Then* $\lim_{D \ni x \to z_0} \mathbb{E}_x \tau_D = 0$.

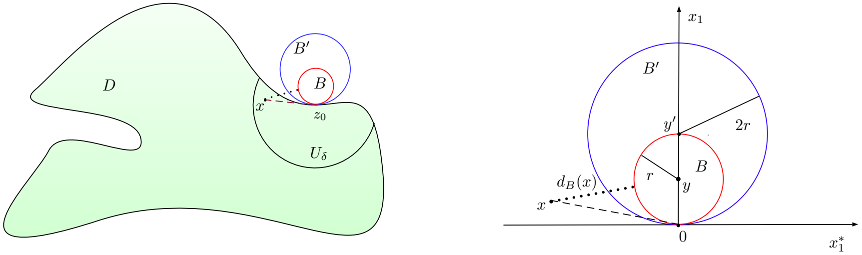

Figure 10 : Distance function to exterior tangent ball

$$|x - z_0|^2 \leqslant (9r) d_B(x), \; x \in U_\delta.$$

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- Existence-uniqueness of classical solutions for nonlocal parabolic Dirichlet problem.
- Probabilistic representation for nonlocal Dirichlet problem.
- Question: Is it possible to drop $\alpha \geqslant \frac{1}{2}$ in the mixed case by showing

$$\mathbb{P}_x(X_{\tau_D} \in \partial\Gamma) = 0, \quad x \in D \ ???$$

Here $\Gamma := \{z \in \partial D : b(z) \cdot \vec{n}(z) = 0\}$ is a closed set.

Thank you very much for your attention!