Homogenization of stable-like operators

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Non Standard Diffusions in Fluids, Kinetic Equations and Probability

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Outline

1. **Aim**

2. **Symmetric setting: random medium**
   - Framework: Dirichlet form
   - Main results

3. **Non-symmetric case: periodic coefficient**
   - Framework: operator
   - Main result

Homogenization

(1)

$$L = \sum_{1 \leqslant i,j \leqslant d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

(2) Oscillating coefficients

$$L^{\varepsilon} = \sum_{1 \leqslant i,j \leqslant d} \frac{\partial}{\partial x_i} \left(a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} \right), \quad \varepsilon > 0.$$

(3) Homogenization

$$L^{\varepsilon} \to \bar{L}, \quad \varepsilon \to 0,$$

where $\bar{L}$ is with constant coefficient.

(i) Periodic homogenization: $a_{ij}(x)$ is a periodic function.

(ii) Stochastic homogenization (in a stationary, ergodic random media): $a_{ij}(x;\omega) = a_{ij}(\tau_x \omega)$, where $\{\tau_x\}_{x \in \mathbb{R}^d}$ is a measurable group of transformations defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\{\tau_x\}_{x \in \mathbb{R}^d}$ is stationary and ergodic.

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$$\mathcal{E}^\varepsilon(f,g) = \sum_{1\leqslant i,j\leqslant d} \int a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial f(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_j}\,dx.$$

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(iii) Non-divergence form; perturbation; $\cdots$

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Homogenization of stable-like operators

Question: Homogenization problem for stable-like operators

(1) What kind of stable-like operator L we will consider?

(2) How can we do the homogenization? What kind of scaling we will choose?

(3) What expression of the limiting operator $\bar{L}$?

What is the limiting operator $\bar{L}$?

Limiting operator (the generator of symmetric α-stable Lévy process)

$$\bar{L}f(x) = \int \big(f(x+z) - f(x) - \langle \nabla f(x), z\rangle 1_{\{|z|\leqslant 1\}}\big) \frac{\bar{k}(z)}{|z|^{d+\alpha}}\, dz$$

$$= p.v. \int \big(f(x+z) - f(x)\big) \frac{\bar{k}(z)}{|z|^{d+\alpha}}\, dz,$$

where $\alpha \in (0,2)$, $0 < k_1 \leqslant \bar{k}(z) \leqslant k_2 < \infty$, and $\bar{k}(z) = \bar{k}(-z)$ and $\bar{k}(z/\varepsilon) = \bar{k}(z)$ for all $z \in \mathbb{R}^d$ and $\varepsilon > 0$. With $\bar{k}(z) = c_{d,\alpha}$, $\bar{L} = -(-\Delta)^{\alpha/2}$ (rotationally invariant α-stable Lévy process).

(1) Stationary increments: $\bar{X}_{r+t} - \bar{X}_{r+s}$ has the same distribution as that of $\bar{X}_t - \bar{X}_s$ for all $r > 0$ and $0 < s < t$.

(2) Scaling property: For any $\varepsilon > 0$, $(\varepsilon \bar{X}_{\varepsilon^{-\alpha} t})_{t\geqslant 0}$ has the same distribution as that of $(\bar{X}_t)_{t\geqslant 0}$.

(3) Transition density function (fundamental solution):

$$\bar{p}(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}, \quad t > 0, x, y \in \mathbb{R}^d.$$

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What kind of stable-like operator L under scaling with the limiting operator $\bar{L}$?

Let $(X_t)_{t\geqslant 0}$ be a α-stable-like process (not only α-stable Lévy process and, in general, not having the scaling property) with generator as follow

- Symmetric setting:

$$Lf(x) = p.v. \int (f(y) - f(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}} \, dy$$

where $c(x,y) = c(y,x)$ for all $x, y \in \mathbb{R}^d$.

- Non-symmetric setting:

$$Lf(x) = p.v. \int (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}} \, dz,$$

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What kind of homogenization: For any $\varepsilon > 0$ and $t > 0$, let $X_t^{(\varepsilon)} := \varepsilon X_{\varepsilon^{-\alpha}t}$.

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Known results

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- R.L. Schilling and T. Uemura: Homogenization of symmetric Lévy processes on $\mathbb{R}^d$, arXiv:1808.01667
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   - Framework: Dirichlet form
   - Main results
3. Non-symmetric case: periodic coefficient
   - Framework: operator
   - Main result

Symmetric stable-like operator L in random medium

Let $(X_t^\omega)_{t\geqslant 0}$ be a symmetric α-stable-like process with generator as follow

- $$L^\omega f(x) = p.v. \int (f(y) - f(x)) \frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\, dy$$

  where $c(x,y;\omega) = c(y,x;\omega)$ for all $x, y \in \mathbb{R}^d$.

- Non-local Dirichlet form:

  $$\begin{aligned}\mathcal{E}^\omega(f,g) &= -\int f(x) L^\omega g(x)\, dx \\ &= \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f(x)-f(y))(g(x)-g(y)) \frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\, dx\, dy\end{aligned}$$

  on $L^2(\mathbb{R}^d; dx)$.

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Non-local symmetric Dirichlet form: starting point

- A little more general, allowing the degenerate reference measure:

$$\mathcal{E}^\omega(f,g) := \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} (f(x)-f(y))(g(x)-g(y))\frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\,dx\,dy$$

  on $L^2(\mathbb{R}^d;\mu(x;\omega)\,dx)$.

- The corresponding operator on $L^2(\mathbb{R}^d;\mu(x;\omega)\,dx)$:

$$L^\omega f(x) = \frac{1}{\mu(x;\omega)}\int_{\mathbb{R}^d} (f(y)-f(x))\,\frac{c(x,y;\omega)}{|x-y|^{d+\alpha}}\,dy.$$

Motivation of assumptions: scaling processes

For any $\varepsilon > 0$, set $X^{\varepsilon,\omega} = (X_t^{\varepsilon,\omega})_{t\geqslant 0} := (\varepsilon X^{\omega}_{\varepsilon^{-\alpha}t})_{t\geqslant 0}$.

Lemma

The process $X^{\varepsilon,\omega}$ enjoys a symmetric measure $\mu^{\varepsilon,\omega}(dx) = \mu\left(\frac{x}{\varepsilon};\omega\right) dx$, and the associated regular Dirichlet form $(\mathcal{E}^{\varepsilon,\omega}, \mathcal{F}^{\varepsilon,\omega})$ on $L^2(\mathbb{R}^d;\mu^{\varepsilon,\omega}(dx))$ is given by

$$\mathcal{E}^{\varepsilon,\omega}(f,g) = \frac{1}{2}\iint (f(x)-f(y))(g(x)-g(y))\frac{c\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon};\omega\right)}{|x-y|^{d+\alpha}}\,dx\,dy.$$

Limiting Dirichlet form:

$$\bar{\mathcal{E}}(f,g) = \frac{1}{2}\iint (f(y)-f(x))(g(y)-g(x))\frac{\bar{k}(x-y)}{|x-y|^{d+\alpha}}\,dx\,dy,$$

where $\bar{k}(z) = \bar{k}(-z)$ for all $z \in \mathbb{R}^d$.

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Assumption on measure

- $$\mathcal{E}^{\varepsilon,\omega}(f,g) = \frac{1}{2}\iint (f(x)-f(y))(g(x)-g(y))\frac{c\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon};\omega\right)}{|x-y|^{d+\alpha}}\,dx\,dy.$$

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- Random medium: A probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which a measurable group of transformations $\{\tau_x\}_{x\in\mathbb{R}^d}$ is defined. We assume that under $(\Omega,\mathcal{F},\mathbb{P})$, $\{\tau_x\}_{x\in\mathbb{R}^d}$ is stationary and ergodic.

- Assumption $(A\text{-}\mu)$ Let $\mu:\mathbb{R}^d\times\Omega\to[0,\infty)$ satisfy that $\mu(x+y;\omega)=\mu(x;\tau_y\omega)$ for any $x,y\in\mathbb{R}^d$ and $\omega\in\Omega$, and $\mathbb{E}\,\mu(0;\omega)=1$. *(Note that, by the ergodic theorem, as $\varepsilon\to 0$,*

$$\int f(x)\mu(x/\varepsilon;\omega)\,dx = \int f(x)\mu(0;\tau_{x/\varepsilon}\omega)\,dx$$

$$\to \int f(x)\mathbb{E}\mu(0;\omega)\,dx = \int f(x)\,dx.)$$

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Outline

Assumptions on the coefficient $c(x, y; \omega)$

Assumption $(A\text{-}c)$:

(i) Suppose that $d \geqslant 2$ and $c(x+z, y+z; \omega) = c(x, y; \tau_z\omega)$ for all $x, y, z \in \mathbb{R}^d$ and $\omega \in \Omega$, such that for all $z \in \mathbb{R}^d$ and a.s. $\omega \in \Omega$,

$$\underline{\Lambda}_1(\omega)\underline{\Lambda}_2(\tau_z\omega) \leqslant c(0, z; \omega) \leqslant \overline{\Lambda}_1(\omega)\overline{\Lambda}_2(\tau_z\omega),$$

and

$$\lim_{\varepsilon \to 0} \sup_{|z| \leqslant r} \left| \mathbb{E}\, c\left(0, \frac{z}{\varepsilon}; \omega\right) - \bar{c}(z) \right| = 0, \quad r > 0$$

for nonnegative locally bounded function $\bar{c}(z)$.

(ii) The random media $(\Omega; \mathbb{P})$ satisfies a space-mixing condition in the sense that there exist $l > d$ and $C_0 > 0$ such that for any $\psi_1, \psi_2 \in L^2(\Omega; \mathbb{P})$

$$\Big| \mathbb{E}\big(\psi_1(\omega)\psi_2(\tau_x\omega)\big) - \mathbb{E}\psi_1(\omega) \cdot \mathbb{E}\psi_2(\tau_x\omega) \Big| \leqslant C_0 \|\psi_1\|_{L^2(\Omega;\mathbb{P})} \|\psi_2\|_{L^2(\Omega;\mathbb{P})} \left(1 \wedge |x|^{-l}\right);$$

see Andres (14), $\cdots$ (RCM with dynamic bounded conductances).

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Result 1

Theorem

Under assumptions (A-μ) and (A-c), Dirichlet forms $(\mathcal{E}^{n,\omega}, \mathcal{F}^{n,\omega})$ (with $\varepsilon = 1/n$) converges to $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ in the sense of Mosco, if

$$\mathbb{E}\Big(\overline{\Lambda}_1^p + \overline{\Lambda}_2^p + \mu(0)^p + \underline{\Lambda}_1^{-q} + \underline{\Lambda}_2^{-q} + \mu(0)^{-q}\Big) < \infty,$$

where $p \geqslant 4$ and $q > 2d/\alpha$ with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2} + \frac{\alpha}{2d}$. $\bar{k}(z) = \frac{1}{2}(\bar{c}(z) + \bar{c}(-z))$.

Mosco convergence with changing measures (Kuwae-Shioya (03')):

(1) for every sequence $\{f_n\}_{n\geqslant 1}$ on $L^2(\mathbb{R}^d; \mu_n(dx))$ converging weakly to $f \in L^2(\mathbb{R}^d; dx)$,

$$\liminf_{n\to\infty} \mathcal{E}^{(n,\omega)}(f_n, f_n) \geqslant \bar{\mathcal{E}}(f, f).$$

(2) for any $f \in L^2(\mathbb{R}^d; dx)$, there is $\{f_n\}_{\geqslant 1} \subset L^2(\mathbb{R}^d; \mu_n(dx))$ converging strongly to f such that

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Idea of the proof:

(1) The embedding theorem for fractional Laplacian on $\mathbb{R}^d$, combined with the Hölder inequality.

(2) The ergodic theorem (maximal ergodic theorem), the Borel-Cantelli lemma.

Remark 1: space-mixing condition

Theorem

Under assumptions $(A\text{-}\mu)$ *and* $(A\text{-}c)$, *Dirichlet forms* $(\mathcal{E}^{n,\omega}, \mathcal{F}^{n,\omega})$ *(with* $\varepsilon = 1/n$*) converges to* $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ *in the sense of Mosco, if*

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$$\Big|\mathbb{E}\big(\psi_1(\omega)\psi_2(\tau_x\omega)\big) - \mathbb{E}\psi_1(\omega)\cdot\mathbb{E}\psi_2(\tau_x\omega)\Big| \leqslant C_0\|\psi_1\|_{L^2(\Omega;\mathbb{P})}\|\psi_2\|_{L^2(\Omega;\mathbb{P})}\left(1 \wedge |x|^{-l}\right).$$

Note: assumption on $\mathbb{E}c(0, z/\varepsilon; \omega)$:

$$c\left(0, \frac{y-x}{\varepsilon}; \tau_{\frac{x}{\varepsilon}}\omega\right) \xrightarrow{(A\text{–}c)(ii):\ mixing} \mathbb{E}c\left(0, \frac{y-x}{\varepsilon}; \tau_{\frac{x}{\varepsilon}}\omega\right) \xrightarrow{(A\text{–}c)(i)+ergodic} \bar{k}(x-y).$$

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Remarks 2 / 3: product form / another convergence

Theorem

Consider the case that $c(x, y; \omega) = a_0(x; \omega) a_0(y; \omega)$ *with* $a_0(x + y; \omega) = a_0(x; \tau_y \omega)$ *for all* $x, y \in \mathbb{R}^d$ *and* $\omega \in \Omega$. *Under assumption (A-μ), Dirichlet forms* $(\mathcal{E}^{n,\omega}, \mathcal{F}^{n,\omega})$ *(with* $\varepsilon = 1/n$*) converges to* $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ *in the sense of Mosco, if*

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(1) Product form: Kassmann-Piatnitski-Zhizhina (18')

(2) Convergence in the sense that for a.s. $\omega \in \Omega$ and $f, g \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{\varepsilon \to 0} \mathcal{E}^{\varepsilon,\omega}(U_\lambda^{\varepsilon,\omega} f, g) = \bar{\mathcal{E}}(\bar{U}_\lambda f, g)$$

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Result 2: Assumption $(A\text{-}c^*)$

(1) Result 1: Assumption $(A\text{-}c)$(i):

$$\lim_{\varepsilon\to 0}\sup_{|z|\leqslant r}\left|\mathbb{E}\, c\left(0,\frac{z}{\varepsilon};\omega\right)-\bar{c}(z)\right|=0,\quad r>0.$$

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- Roughly speaking, Assumption $(A\text{-}c^*)$ is stronger than Assumption $(A\text{-}c)$(i). Example: scaling property $c\left(x,\frac{z}{\varepsilon};\omega\right)=c\left(x,z;\omega\right)$, see Schwab (10',13').
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$$c\Big(0, \frac{y-x}{\varepsilon}; \tau_{\frac{x}{\varepsilon}}\omega\Big) \xrightarrow{(A-c^*)} \bar{c}\Big(0, y-x; \tau_{\frac{x}{\varepsilon}}\omega\Big) \xrightarrow{ergodic} \bar{k}(x-y).$$

Assume that, there are $\underline{\Lambda}(\omega)$ and $\overline{\Lambda}(\omega)$ such that

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Under assumptions (A-μ) and (A-c^*), Dirichlet forms $(\mathcal{E}^{n,\omega}, \mathcal{F}^{n,\omega})$ (with $\varepsilon = 1/n$) converges to $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ in the sense of Mosco, if

$$\mathbb{E}\Big(\overline{\Lambda}^p + \mu(0)^p + \underline{\Lambda}^{-q} + \mu(0)^{-q}\Big) < \infty,$$

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Outline

1. Aim
2. Symmetric setting: random medium
   - Framework: Dirichlet form
   - Main results
3. **Non-symmetric case: periodic coefficient**
   - **Framework: operator**
   - Main result

Non-symmetric setting from Kassmann et al. (18')

- Let $\alpha \in (0,1)$. Consider the following operator acting on $C_b^2(\mathbb{R}^d)$:

$$Lf(x) = \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}}\, dz.$$

(Note that, $c(x,y)$ is not symmetric with respect to (x,y).)

- Coefficients: Let $c(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ be periodic with respect to both variables such that

  (i) $0 < C_1 \leqslant c(x,y) \leqslant C_2 < \infty$ for all $x, y \in \mathbb{R}^d$.
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Settings: periodic homogenization

- Let $\alpha \in (1,2)$. Consider the following operator acting on $C_b^2(\mathbb{R}^d)$:

$$Lf(x) = p.v. \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{k(x,z)}{|z|^{d+\alpha}}\, dz$$
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where

$$b_0(x) := \frac{1}{2}\int z \frac{(k(x,z) - k(x,-z))}{|z|^{d+\alpha}}\, dz, \quad x \in \mathbb{R}^d.$$

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$$\bar{L}f(x) = \int (f(x+z) - f(x) - \langle \nabla f(x), z\rangle) \frac{\bar{k}_0}{|z|^{d+\alpha}}\, dz.$$

Additionally, when $b_0(x) \equiv 0$ *for all* $x \in \mathbb{R}^d$ *(in particular, in balanced case:* $k(x,z) = k(x,-z)$ *for all* $x, z \in \mathbb{R}^d$*), then* $\bar{b}_0 = 0$*.*

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Approach

- In general case: Limit theory for semimartingale (Jacod-Shiryaev, 03'), Feller processes (Schilling, 98'). See Tomisaki (92'); Fujiwara-Tomisaki (94').

- In balanced case, i.e. $k(x,z) = k(x,-z)$: By the corrector method, we can prove that for every $u \in C_c^\infty(\mathbb{R}^d)$, there exists a class of functions $\{v^\varepsilon\}_{\varepsilon>0}$ such that

$$\lim_{\varepsilon\to 0} [\|u - v^\varepsilon\|_\infty + \|\bar{L}u - L^\varepsilon v^\varepsilon\|_\infty] = 0,$$

where

$$v^\varepsilon(x) = u(x) + \varepsilon^\alpha \bar{L}_0 u(x)\psi_1\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d.$$

See Arisawa (09', 13'); Schwab (10', 13').

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Thank you for your attention!