Recent results for the 3D Quasi-Geostrophic Equation

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Physical model

- The Quasi-Geostrophic system of equations models the evolution of the temperature in the atmosphere.
- It can be rigorously derived from the Primitive Equations (Euler equation with Coriolis force and Boussinesq approximation, see Bourgeois Beale (94) and Desjardins Grenier 98)
- At large scale, this Rossby effect is very important. Asymptotically, this leads to the so-called geostrophic balance which enforces the wind velocity to be orthogonal to the gradient of the pressure in the atmosphere (see Pedlosky).
- This model is extensively used in computations of oceanic and atmospheric circulation, for instance, to simulate global warming.

The unknown and parameters

- The dynamic is encoded in Ψ, the stream function for the geostrophic flow.
- That is, the 3D velocity $(w, U) = (0, u, v)$ has its horizontal component verifying

$$(u, v) = (-\partial_{x_2}\Psi, \partial_{x_1}\Psi), \quad \text{or in short : } U = \overline{\nabla}^{\perp}\Psi,$$

  where we denote

$$\overline{\nabla}\Psi = (0, \partial_{x_1}\Psi, \partial_{x_2}\Psi).$$

- From the model, the buoyancy is given by

$$\Theta = \partial_z\Psi.$$

- We denote

$$\nabla_\lambda\phi = (\lambda\partial_z\phi, \partial_{x_1}\phi, \partial_{x_2}\phi), \qquad L_\lambda\phi = \mathrm{div}\,(\nabla_\lambda\phi).$$

  where $\lambda = -1/\Theta_z^0$, is a given function, of z only, associated to the buoyancy of a reference state.

The equation

The function Ψ is solution to the following Initial Boundary value problem:

$$(\partial_t + \overline{\nabla}^\perp \Psi \cdot \nabla)(L_\lambda \Psi + \beta_0 x_2) = 0, \qquad t > 0, \ \ z > 0, \ \ x \in \mathbb{R}^2,$$
$$(\partial_t + \overline{\nabla}^\perp \Psi \cdot \nabla)\gamma_\nu(\nabla_\lambda \Psi) = \nu \overline{\Delta} \Psi, \qquad t > 0, \ \ z = 0, \ \ x \in \mathbb{R}^2,$$
$$\Psi(0, z, x) = \Psi^0(z, x). \qquad t = 0, \ \ z > 0, \ \ x \in \mathbb{R}^2.$$

The parameter β_0 comes from the usual β-plane approximation. The term $\gamma_\nu(\nabla_\lambda \Psi)$ stands for the Neumann condition at $z = 0$ associated to the operator $L_\lambda \Psi$. If λ is regular, this coincides with $-\lambda(0)\partial_z \Psi(0, \cdot)$. The ν term is due to the Ekman pumping. $\nu = 0$ corresponds to the inviscid case.

- Both, the value of the elliptic operator $L_\lambda \Psi$, and the Neumann condition $\gamma_\nu(\nabla_\lambda \Psi)$ at the boundary $z = 0$, are advected by the stratified flow with velocity $U = \overline{\nabla}^\perp \Psi$. At each time, Ψ can be recovered, solving the boundary value elliptic equation.
- Main difficulty: treatment of the boundary condition.

The inviscid case

We assume that $\nu = 0$, and that there exists $\Lambda > 0$ such that

$$\frac{1}{\Lambda} \leq \lambda(z) \leq \Lambda, \qquad \text{for} \;\; z \in \mathbb{R}^+.$$

Theorem (Puel-V.)

Consider an initial value Ψ^0 such that

$$L_\lambda \Psi^0 \text{ and } \nabla_\lambda \Psi^0 \text{ are in } L^2(\mathbb{R}^+ \times \mathbb{R}^2), \qquad \gamma_\nu(\nabla_\lambda \Psi^0) \in L^2(\mathbb{R}^2).$$

Then, there exists Ψ weak solution to the Quasi-Geostrophic equation on $(0, \infty) \times \mathbb{R}^+ \times \mathbb{R}^2$, such that for every $T > 0$,
$\nabla_\lambda \Psi \in L^\infty(0, T; L^2(\mathbb{R}^+ \times \mathbb{R}^2)) \cap C^0(0, T; L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2))$.

Novack recently extended the theory to general L^p.

Lateral boundary conditions

We consider a domain of the form $\mathbb{R}^+ \times \Omega$, where $\Omega \subset \mathbb{R}^2$ is bounded. Think, for instance about a rotating box filled with a fluid.

Theorem (Novack-V.)

The natural lateral boundary conditions on $\mathbb{R}^+ \times \partial\Omega$ are

$$\Psi \text{ depends only on } z \text{ on } \mathbb{R}^+ \times \partial\Omega,$$
$$\frac{d}{dt} \int_{\partial\Omega} \partial_\nu \Psi \, d\hat{x} = 0.$$

We can also construct global weak solutions of QG with the addition of these boundary conditions.

This corresponds to a partial Dirichlet condition (up to the dependency on z), together with a mean value of Neumann condition on $\partial\Omega$.

The case with Ekman pumping

We assume that $\lambda(z) = 1$, and $\nu > 0$.

Theorem (Novack-V.)

Consider an initial value $\nabla\Psi^0 \in L^2(\mathbb{R}^3_+) \cap H^p((0,\infty) \times \mathbb{R}^2)$ with $p \geq 3$. Then, there exists a unique global solution $\nabla\Psi$ to the Quasigeostrophic equation on $(0,\infty) \times \mathbb{R}^+ \times \mathbb{R}^2$, such that for every $T > 0$, $\nabla_\lambda\Psi \in C^0(0,T; H^p(\mathbb{R}^+ \times \mathbb{R}^2))$.

Especially, if the initial is smooth (C^∞), then the unique solution is also smooth.

Main difficulty

To simplify the exposition, let us consider the case with out forcing with $\beta = 0$, and $\lambda = 1$.

$$
\begin{aligned}
&(\partial_t + \overline{\nabla}^\perp \Psi \cdot \nabla)(\Delta \Psi) = 0, \qquad \text{for } \; z > 0, \\
&(\partial_t + \overline{\nabla}^\perp \Psi \cdot \nabla)(\partial_z \Psi) = 0, \qquad \text{for } \; z = 0, \\
&\Psi(0, z, x) = \Psi^0(z, x). \qquad t = 0.
\end{aligned}
$$

- A priori estimates: for any $1 \leq p \leq \infty$:

$$
\begin{aligned}
&\|\Delta \Psi(t)\|_{L^p(\mathbb{R}^+ \times \mathbb{R}^2)} \leq \|\Delta \Psi(0)\|_{L^p(\mathbb{R}^+ \times \mathbb{R}^2)}, \\
&\|\partial_z \Psi(t, 0)\|_{L^p(\mathbb{R}^2)} \leq \|\partial_z \Psi(0, 0)\|_{L^p(\times \mathbb{R}^2)},
\end{aligned}
$$

- No compactness on the trace of $\partial_z \Psi$ at $z = 0$!

A special case: the Surface Quasi Geostrophic Equation

- If $\Delta\Psi(0) = 0$, then $\Delta\Psi(t) = 0$ for all $t \geq 0$.
- Denote $\theta = \partial_z \Psi$ defined at $z = 0$. Then θ is solution to

$$\partial_t \theta + U.\nabla\theta = 0, \qquad t > 0, (x, y) \in \mathbb{R}^2, \tag{1}$$

$$\theta = \theta_0, \qquad t = 0, (x, y) \in \mathbb{R}^2, \tag{2}$$

  and the velocity U can be expressed in $\mathbb{R}^2$, via a nonlocal operator, as

$$U = \nabla^{\perp} \Delta^{-1/2} \theta.$$

- This model has been popularized as a toy problem for 3D fluid mechanics (see Constantin, Majda, Held, Pierrehumbert, Garner, Swanson ...).
- Our theorem extends to QG the result of Resnick for SQG, using different techniques.

A new formulation (1)

- The proof does NOT use (and does not show) compactness on the trace of $\partial_z \Psi$ at $z = 0$.
- It is based on a reformulation of the problem into a system of equations (without equation on the trace).
- The stability (and compactness) for this problem is pretty simple.
- We then have to show the equivalence between the two formulations.

A new formulation (2)

- Consider the Hodge decomposition in $L^2(\mathbb{R}^+ \times \mathbb{R}^2)$:

$$u = \nabla_\lambda \phi + \mathrm{curl} v = \mathbb{P}_\lambda u + \mathbb{P}_{\mathrm{curl}} u,$$

  with $\mathrm{curl} v \cdot \nu = 0$ at $z = 0$.
- The QG problem can be reformulated as

$$\partial_t \nabla_\lambda \Psi + \mathbb{P}_\lambda (\bar{\nabla} \Psi^\perp \cdot \bar{\nabla} \nabla_\lambda \Psi) = 0, \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+.$$

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- Taking the div of the equation gives the first QG equation, thanks to

$$\mathrm{div}\,(\mathbb{P}_\lambda \cdot) = \mathrm{div}\,(\cdot), \qquad \partial_i (\bar{\nabla} \Psi)^\perp \cdot \bar{\nabla} \partial_i \Psi = 0.$$

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- Taking the trace of the system a $z = 0$ gives (formally) the trace condition of QG, since formally, at $z = 0$

$$\mathbb{P}_\lambda (f) \cdot \nu = f \cdot \nu.$$

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- Euler Equation:

$$\partial_t \text{curl} v + \mathbb{P}_{\text{curl}}[\text{curl} v \cdot \nabla \text{curl} v] = 0, \qquad (t, x, z) \in \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+.$$

  with $\mathbb{P}_\lambda(\text{curl} v) = 0$ (that is $\text{curl} v \cdot \nu = 0$ at $z = 0$).

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with $\mathbb{P}_\lambda(\text{curl} v) = 0$ (that is $\text{curl} v \cdot \nu = 0$ at $z = 0$).

- The first equation of QG is equivalent to the vorticity equation of Euler:
  - QG:

$$\partial_t \text{div} \nabla_\lambda \Psi + \bar{\nabla}\Psi^\perp \cdot \bar{\nabla}(\text{div}\nabla_\lambda \Psi) = 0$$

  - Euler:

$$\partial_t \text{curlcurl} v + \text{curl} v \cdot \nabla(\text{curlcurl} v) = 0.$$

Proof of the Theorem

- Compactness holds for the reformulated problem.

  Note that $\mathbb{P}_\lambda$ commutes with $\bar{\nabla}$, and is continuous in L^p.

- The two formulation of QG are equivalent.

A special case: the Surface Quasi Geostrophic Equation

- If $\Delta\Psi(0) = 0$, then $\Delta\Psi(t) = 0$ for all $t \geq 0$.
- Denote $\theta = \partial_z \Psi$ defined at $z = 0$. Then θ is solution to

$$\partial_t \theta + U.\nabla\theta = \nu\overline{\Delta}\Psi, \qquad t > 0, (x, y) \in \mathbb{R}^2, \tag{3}$$

$$\theta = \theta_0, \qquad t = 0, (x, y) \in \mathbb{R}^2, \tag{4}$$

and the velocity U and the Ekman pumping term $\nu\overline{\Delta}\Psi$ can be expressed in $\mathbb{R}^2$, via a nonlocal operator, as

$$U = \nabla^{\perp}\Delta^{-1/2}\theta, \qquad \nu\overline{\Delta}\Psi = \nu\Delta^{1/2}\theta.$$

- The propagation of regularity for this equation has first been proved by Kiselev, Nazarov and Volberg. The global regularity of solutions with initial values in L^2 has been proved first by Caffarelli V. Several other proofs has been proposed by Kiselev and Volberg, and Constantin and Vicol.

The 3D case

- In the 3D case, the equation in $z > 0$ is hyperbolic. We can have only propagation of regularity.
- We need the propagation of almost Lipschitz norm (possible log Lipschitz).
- The regularization effects on the boundary are only C^α.

Sketch of the proof (1)

We decompose the solution $\Psi = \Psi_1 + \Psi_2$ into two components as follows:

$$\begin{cases} \Delta\Psi_1 = 0 \\ \partial_\nu \Psi_1 = \partial_\nu \Psi \end{cases} \qquad \begin{cases} \Delta\Psi_2 = \Delta\Psi \\ \partial_\nu \Psi_2 = 0. \end{cases}$$

- The bulk of the proof is centered around verifying a version of the Beale-Kato-Majda criterion.

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- The bulk of the proof is centered around verifying a version of the Beale-Kato-Majda criterion.
- The equation on the boundary of $\theta = \partial_\nu \Psi_1$ is of the form

  $$\partial_t \theta + u \cdot \overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = f,$$

  with $f = \overline{\Delta}\Psi_2$.
- The natural a priori bound for f is in $B^0_{\infty,\infty}$.
- Using De Giorgi techniques, we get θ bounded in C^α.

Sketch of the proof (2)

- Bootstrapping an increase of regularity on the C^α on the drift-diffusion equation on the boundary gives that $\partial_\nu \Psi \in L^\infty(0, T; B^1_{\infty,\infty})$ on the boundary.
- Using that the flow is stratified, this gives the "almost Lipschitz" bound needed on the velocity in $z > 0$ generated by the boundary.

Remark on the lateral boundary conditions

- In the case of the inviscid SQG, defined on a Bounded domain $\Omega \subset \mathbb{R}^2$, we need to define the velocity U.
- Constantin and Ignatova (17) (see also Constantin and Nguyen) proposed to define it through the Operator $\bar{\Delta}_D^{-1/2}$ with Dirichlet boundary condition 0 on $\partial\Omega$:

$$U = \bar{\nabla}^{\perp} \bar{\Delta}_D^{-1/2} \theta.$$

- This corresponds to a Dirichlet condition $\Psi = 0$ on $\mathbb{R}^+ \times \partial\Omega$ for the 3D QG.
- This is not the boundary condition derived from the primitive equation.
- The corresponding boundary condition for SQG can be retrieved using the Extension Operator of Caffarelli-Silvestre.

Thank You !!